CHAPTER 12

# Limits and Derivatives

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## Then

In **Chapter 1,** you learned about limits and rates of change.

## Now

In Chapter 12, you will:

- Evaluate limits of polynomial and rational functions.
- Find instantaneous rates of change.
- Find and evaluate derivatives of polynomial functions.
- Approximate the area under a curve.
- Find antiderivatives, and use the Fundamental Theorem of Calculus.

## Why? ▲

**BUNGEE JUMPING** The basic tools of calculus, derivatives and integrals, are very useful when working with rates that are not constant. A bungee jumper experiences varying rates of descent and ascent, as well as changing acceleration, depending on her position during the jump.

**PREREAD** Use the Mid-Chapter Quiz to write two or three questions about the first three lessons that will help you to predict the organization of the first half of Chapter 12.

# Get Ready for the Chapter

**Diagnose Readiness** You have two options for checking Prerequisite Skills.

## 1 Textbook Option

Take the Quick Check below.

## QuickCheck

**Use the graph of each function to describe its end behavior.** (Lesson 1-3)

**1.** $q(x) = -\frac{2}{x}$

**2.** $f(x) = \frac{7}{x}$

**3.** $p(x) = \frac{x+5}{x-4}$

**4.** $m(x) = \frac{7-10x}{2x+7}$

**5.** **MUSIC** The average cost to produce $x$ CDs can be represented by $A(x) = \frac{1700}{x} + 1200$. Find the limit as $x$ approaches positive infinity. (Lesson 1-3)

**Find the average rate of change of each function on the given interval.** (Lesson 1-4)

**6.** $g(x) = 2x^2 + 4x - 1$; $[-2, 1]$

**7.** $f(x) = -2x^3 - 5x^2 + 6$; $[-4, -1]$

**8.** $f(x) = 4x^3 - x^2 + 9x - 1$; $[-2, 4]$

**9.** **BOOKS** The profit associated with producing $x$ books per week can be represented by $C(x) = -2x^2 + 140x + 25$. Find the average rate of change of the cost if 50 books are produced instead of 25 books. (Lesson 1-4)

**Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.** (Lesson 2-5)

**10.** $f(x) = \frac{4x^2}{2x^2+1}$

**11.** $h(x) = \frac{2x^2-8}{x-10}$

**12.** $f(x) = \frac{(x-1)(x+5)}{(x+2)(x-4)}$

**13.** $g(x) = \frac{x^2-16}{(x-2)(x+4)}$

**Find the next four terms of each arithmetic or geometric sequence.** (Lesson 10-2 and 10-3)

**14.** 3, 7, 11, 15, …

**15.** 8, 3, −2, −7, …

**16.** 5, −1, −7, −13, …

**17.** −4, 12, −36, 108, …

**18.** 5, −10, 20, −40, …

**19.** −28, −21, −14, −7, …

## 2 Online Option

Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

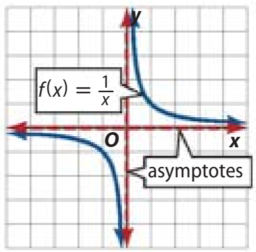
## NewVocabulary

| English | | Español |
|---|---|---|
| one-sided limit | p. 738 | límite unilateral |
| two-sided limit | p. 738 | límite bilateral |
| direct substitution | p. 748 | sustitucíon directa |
| indeterminate form | p. 749 | forma indeterminada |
| tangent line | p. 758 | tangente |
| instantaneous rate of change | p. 758 | tasa instantánea de cambio |
| instantaneous velocity | p. 760 | velocidad instantánea |
| derivative | p. 766 | derivada |
| differentiation | p. 766 | diferenciacíon |
| differential equation | p. 766 | ecuacíon diferencial |
| differential operator | p. 766 | operador diferencial |
| regular partition | p. 777 | particíon regular |
| definite integral | p. 777 | integral definida |
| lower limit | p. 777 | límite inferior |
| upper limit | p. 777 | límite superior |
| right Riemann sum | p. 777 | suma de Riemann por la derecha |
| integration | p. 777 | integracíon |
| antiderivative | p. 784 | antiderivada |
| indefinite integral | p. 785 | integral indefinida |
| Fundamental Theorem of Calculus | p. 786 | teorema fundamental del cálculo |

## ReviewVocabulary

limit p. 24 límite the unique value that a function approaches

asymptotes p. 130 asíntota a line or curve that a graph approaches




holes p. 135 agujeros removable discontinuities on the graph of a function that occur when the numerator and denominator of the function have common factors

LESSON 12-1

# Estimating Limits Graphically

| Then | Now | Why? |
|---|---|---|
| You estimated limits to determine the continuity and end behavior of functions. (Lesson 1-3) | **1** Estimate limits of functions at a point. **2** Estimate limits of functions at infinity. | Are there limits to world records set by athletes? At the 2008 Bejing Olympics, Elena Isinbaeva of Russia won the pole vault, setting a new world record by vaulting 5.05 meters. The logistic function $f(x) = \frac{5.334}{1 + 62548.213e^{-0.129x}}$, where $x$ is the number of years since 1900, models the world record heights in women's pole vaulting from 1996 to 2008. You can use the limit of this function as $x$ goes to infinity to predict a limiting height for this track and field event. |


Gabriel Bouys/AFP/Getty Images

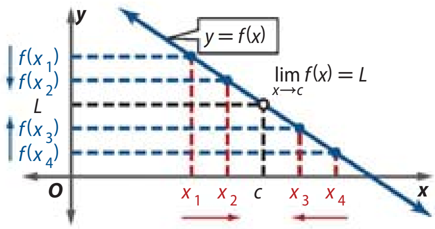
**NewVocabulary**
one-sided limit
two-sided limit

## 1 Estimate Limits at a Point

Calculus centers around two fundamental problems:

- finding the equation of a line tangent to the graph of a function at a point, and
- finding the area between the graph of a function and the $x$-axis.

Fundamental to the solutions of each of these problems is an understanding of the concept of a limit. Recall from Lesson 1-3 that if $f(x)$ approaches a unique value $L$ as $x$ approaches $c$ from either side, then the limit of $f(x)$ as $x$ approaches $c$ is $L$, written $\lim_{x \to c} f(x) = L$.



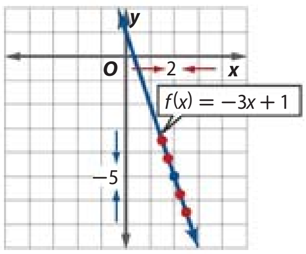

You can apply this description to estimate the limit of a function $f(x)$ as $x$ approaches a fixed value $c$ or $\lim_{x \to c} f(x)$ by using a graph or making a table of values.

### Example 1 Estimate a Limit = $f(c)$

**Estimate $\lim_{x \to 2} (-3x + 1)$ using a graph. Support your conjecture using a table of values.**

**Analyze Graphically**

The graph of $f(x) = -3x + 1$ suggests that as $x$ gets closer to 2, the corresponding function value gets closer to $-5$. Therefore, we can estimate that $\lim_{x \to 2} (-3x + 1)$ is $-5$.




**Support Numerically**

Make a table of values for $f$, choosing $x$-values that approach 2 by using some values slightly less than 2 and some values slightly greater than 2.

$x$ approaches 2 → ← $x$ approaches 2

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −4.7 | −4.97 | −4.997 | | −5.003 | −5.03 | −5.3 |

The pattern of outputs suggests that as $x$ gets closer to 2 from the left or from the right, $f(x)$ gets closer to $-5$. This supports our graphical analysis.

**GuidedPractice**

**Estimate each limit using a graph. Support your conjecture using a table of values.**

**1A.** $\lim_{x \to -3} (1 - 5x)$

**1B.** $\lim_{x \to 1} (x^2 - 1)$

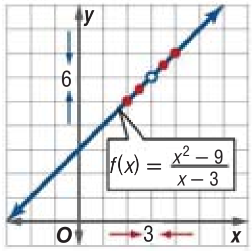
In Example 1, $\lim_{x \to 2} (-3x + 1)$ is the same as the value of $f(2)$. However, the limit of a function is not always equal to a function value.

## Example 2 Estimate a Limit $\neq f(c)$

**Estimate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$ using a graph. Support your conjecture using a table of values.**

### Analyze Graphically

The graph of $f(x) = \frac{x^2 - 9}{x - 3}$ suggests that as $x$ gets closer to 3, the corresponding function value approaches 6. Therefore, we can estimate that $\lim_{x \to 2} \frac{x^2 - 9}{x - 3}$ is 6.




**TechnologyTip**

**Tables** To help create a table using a graphing calculator, enter the function using the [Y=] menu. Then use the Table function by pressing [2nd] [TABLE]. To approach a specific value, change the starting point and interval for $x$ in your table by pressing [2nd] [TBLSET] and adjusting the TBLSET options.

### Support Numerically

Make a table of values, choosing $x$-values that approach 3 from either side.

$x$ approaches 3 → ← $x$ approaches 3

| $x$ | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 5.9 | 5.99 | 5.999 | | 6.001 | 6.01 | 6.1 |

The pattern of outputs suggests that as $x$ gets closer to 3, $f(x)$ gets closer to 6. This supports our graphical analysis.

### GuidedPractice

**Estimate each limit using a graph. Support your conjecture using a table of values.**

**2A.** $\lim_{x \to -2} \frac{x + 2}{x^2 - 4}$

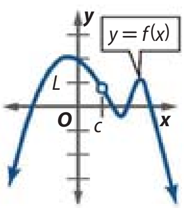
**2B.** $\lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5}$

In Example 2, notice that the limit as $x$ approaches 3 is 6, even though $f(3) \neq 6$. In fact, $f(3)$ does not even exist, because the expression $\frac{x^2 - 9}{x - 3}$ is not defined when $x = 3$. This illustrates an important point about limits.

## KeyConcept Independence of Limit from Function Value at a Point

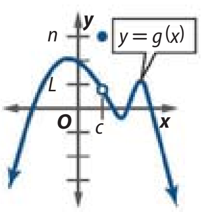
**Words** The limit of a function $f(x)$ as $x$ approaches $c$ *does not* depend on the value of the function *at* point $c$.

**Symbols**



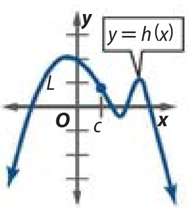

$\lim_{x \to c} f(x) = L$

$f(c)$ is undefined.




$\lim_{x \to c} g(x) = L$

$g(c) = n$




$\lim_{x \to c} h(x) = L$

$h(c) = L$

It is important to understand that a limit is not about what happens at the number that $x$ is approaching. Instead, a limit is about what happens *near* or *close to* that number.

In finding limits using a table or a graph, we have looked at the value of $f(x)$ as $x$ approaches $c$ from *each* side. We can describe the behavior of a graph from the left and right of $x$ more concisely in terms of **one-sided limits**.

**ReadingMath**

**One-Sided Limits** The notation $\lim_{x \to c^-} f(x)$ can also be read as *the limit of f(x) as x approaches c from below.* The notation $\lim_{x \to c^+} f(x)$ can also be read as *the limit of f(x) as x approaches c from above.*

## KeyConcept One-Sided Limits

**Left-Hand Limit**

If the value of $f(x)$ approaches a unique number $L_1$ as $x$ approaches $c$ from the left, then

$\lim_{x \to c^-} f(x) = L_1$, which is read

*The limit of f(x) as x approaches c from the left is $L_1$.*

**Right-Hand Limit**

If the value of $f(x)$ approaches a unique number $L_2$ as $x$ approaches $c$ from the right, then

$\lim_{x \to c^+} f(x) = L_2$, which is read

*The limit of f(x) as x approaches c from the right is $L_2$*

Using these definitions, we can state more concisely what it means for a **two-sided limit** to exist.

## KeyConcept Existence of a Limit at a Point

The limit of a function $f(x)$ as $x$ approaches $c$ exists if and only if both one-sided limits exist *and* are equal. That is, if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L, \text{ then } \lim_{x \to c} f(x) = L.$$

## Example 3 Estimate One-Sided and Two-Sided Limits

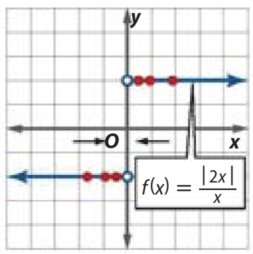
**Estimate each one-sided or two-sided limit, if it exists.**

**a.** $\lim_{x \to 0^-} \frac{|2x|}{x}$, $\lim_{x \to 0^+} \frac{|2x|}{x}$, and $\lim_{x \to 0} \frac{|2x|}{x}$

The graph of $f(x) = \frac{|2x|}{x}$ suggests that

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$

Because the left- and right-hand limits of $f(x)$ as $x$ approaches 0 are not the same, $\lim_{x \to 0} \frac{|2x|}{x}$ does not exist.

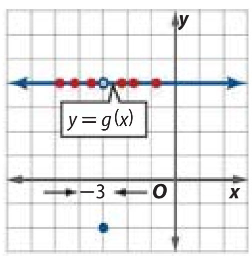



**b.** $\lim_{x \to -3^-} g(x)$, $\lim_{x \to -3^+} g(x)$, and $\lim_{x \to -3} g(x)$, where $g(x) = \begin{cases} 4 & \text{if } x \neq -3 \\ -2 & \text{if } x = -3 \end{cases}$

The graph of $g(x)$ suggests that

$$\lim_{x \to -3^-} g(x) = 4 \quad \text{and} \quad \lim_{x \to -3^+} g(x) = 4.$$

Because the left- and right-hand limits of $g(x)$ as $x$ approaches 3 are the same, $\lim_{x \to -3} g(x)$ exists and is 4.




**GuidedPractice**

**3A.** $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$,

where $f(x) = \begin{cases} x^3 + 2 & \text{if } x < 1 \\ 2x + 1 & \text{if } x \geq 1 \end{cases}$

**3B.** $\lim_{x \to -2^-} g(x)$, $\lim_{x \to -2^+} g(x)$, and $\lim_{x \to -2} g(x)$,

where $g(x) = \begin{cases} -0.5x + 2 & \text{if } x < -2 \\ -x^2 & \text{if } x \geq -2 \end{cases}$

Another way a limit can fail to exist is when the value of $f(x)$ as $x$ approaches $c$ does not approach a fixed finite value. Instead, the value of $f(x)$ increases without bound, indicated by $\infty$, or decreases without bound, indicated by $-\infty$.

## Example 4 Limits and Unbounded Behavior

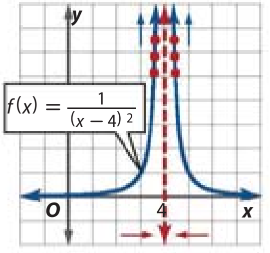
**Estimate each limit, if it exists.**

**a.** $\lim_{x \to 4} \frac{1}{(x-4)^2}$

**Analyze Graphically** The graph of $f(x) = \frac{1}{(x-4)^2}$ suggests that

$$\lim_{x \to 4^-} \frac{1}{(x-4)^2} = \infty \quad \text{and} \quad \lim_{x \to 4^+} \frac{1}{(x-4)^2} = \infty,$$

because as $x$ gets closer to 4, the function values of the graph increase.




Neither one-sided limit at $x = 4$ exists; therefore, we can conclude that $\lim_{x \to 4} \frac{1}{(x-4)^2}$ does not exist. However, because both sides agree (both tend to $\infty$), we describe the behavior of $f(x)$ at 4 by writing $\lim_{x \to 4} \frac{1}{(x-4)^2} = \infty$.

**Support Numerically**

*x* approaches 4 → ← *x* approaches 4

| $x$ | 3.9 | 3.99 | 3.999 | 4 | 4.001 | 4.01 | 4.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 100 | 10,000 | 1,000,000 | | 1,000,000 | 10,000 | 100 |

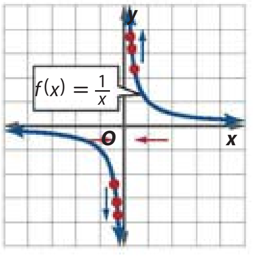
The pattern of outputs suggests that as $x$ gets closer to 4 from the left and the right, $f(x)$ grows without bound. This supports our graphical analysis.

**b.** $\lim_{x \to 0} \frac{1}{x}$

**Analyze Graphically** The graph of $f(x) = \frac{1}{x}$ suggests that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = \infty,$$

because as $x$ gets closer to 0, the function values from the left decrease and the function values from the right increase.




Neither one-sided limit at $x = 0$ exists; therefore, $\lim_{x \to 0} \frac{1}{x}$ does not exist. In this case, we cannot describe the behavior of $f(x)$ at 0 using a single expression because the unbounded behaviors from the left and right differ.

**Support Numerically**

*x* approaches 0 → ← *x* approaches 0

| $x$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −10 | −100 | −1000 | | 1000 | 100 | 10 |

The pattern of outputs suggests that as $x$ gets closer to 0 from the left and the right, $f(x)$ decreases and increases without bound, respectively. This supports our graphical analysis.

### GuidedPractice

**4A.** $\lim_{x \to 3} \frac{x^2 - 4}{x - 3}$

**4B.** $\lim_{x \to 0} -\frac{2}{x^4}$

### ReadingMath

**Without Bound** For $f(x)$ to increase or decrease *without bound* as $x \to c$ means that by choosing an $x$-value arbitrarily close to $c$, you can obtain a function value with an absolute value that is as great as you want. The closer to $c$ this $x$-value is chosen, the greater $|f(x)|$ is.

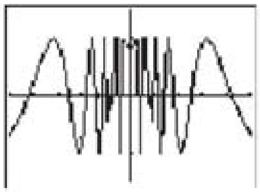
### WatchOut!

**Infinite Limits** It is important to understand that the expressions $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0} f(x) = \infty$ are descriptions of why these limits fail to exist. The symbols $\infty$ and $-\infty$ do not represent real numbers.

A limit can also fail to exist if instead of approaching a fixed value, $f(x)$ *oscillates* or bounces back and forth between two values.

**Technology Tip**

**Infinite Oscillations** The TRACE feature on a graphing calculator can be useful in estimating limits. However, you cannot always trust what a graphing calculator tells you. In the case of the function in Example 5, the calculator uses a *finite* number of points to produce the graph, but close to 0 this function has *infinite* oscillations.

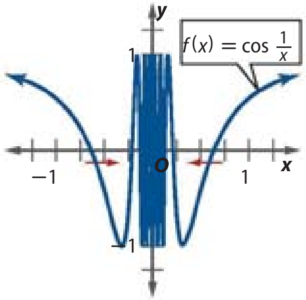


[−0.25, 0.25] scl: 0.05 by [−1.5, 1.5] scl: 1

## Example 5 Limits and Oscillating Behavior

**Estimate $\lim_{x \to 0} \cos \frac{1}{x}$, if it exists.**

The graph of $f(x) = \cos \frac{1}{x}$ suggests that as $x$ gets closer to 0, the corresponding function values oscillate between −1 and 1. This means that for an $x_1$-value close to 0 such that $f(x_1) = 1$, you can always find an $x_2$-value closer to 0 such that $f(x_2) = -1$. Likewise, for an $x_3$-value close to 0 such that $f(x_3) = -1$, you can always find an $x_4$-value closer to 0 such that $f(x_4) = 1$.




Therefore, $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

**Guided Practice**

**Estimate each limit, if it exists.**

**5A.** $\lim_{x \to 0} \sin \frac{1}{x}$

**5B.** $\lim_{x \to 0} (x^2 \sin x)$

The three most common reasons why the limit of a function fails to exist at a point are summarized below.

### Concept Summary Why Limits at a Point Do Not Exist

The limit of $f(x)$ as $x$ approaches $c$ does not exist if:

- $f(x)$ approaches a different value from the left of $c$ than from the right,
- $f(x)$ increases or decreases without bound from the left and/or the right of $c$, or
- $f(x)$ oscillates between two fixed values.

## 2 Estimate Limits at Infinity

To this point, limits have been used to describe how a function $f(x)$ behaves as $x$ approaches a fixed value $c$. In Lesson 1-3, you learned that limits can also be used to describe the end behavior of a function, that is, how a function behaves as $x$ increases or decreases without bound. The notation for such limits is summarized below.

### Key Concept Limits at Infinity

- If the value of $f(x)$ approaches a unique number $L_1$ as $x$ increases, then $\lim_{x \to \infty} f(x) = L_1$, which is read *the limit of f(x) as x approaches infinity is $L_1$*.
- If the value of $f(x)$ approaches a unique number $L_2$ as $x$ decreases, then $\lim_{x \to -\infty} f(x) = L_2$, which is read *the limit of f(x) as x approaches negative infinity is $L_2$*.

In Lesson 2-4, you learned that unbounded behavior that can be described by ∞ or −∞ indicates the location of a vertical asymptote. You also learned that the existence of a limit at infinity indicates the location of a horizontal asymptote. That is,

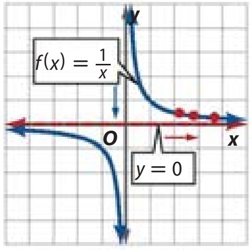
- $x = c$ is a *vertical asymptote* of the graph of $f(x)$ if $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty$, and
- $y = c$ is a *horizontal asymptote* of the graph of $f(x)$ if $\lim_{x \to -\infty} f(x) = c$ or $\lim_{x \to \infty} f(x) = c$.

## Example 6 Estimate Limits at Infinity

**Estimate each limit, if it exists.**

**a.** $\lim_{x \to \infty} \frac{1}{x}$

**Analyze Graphically** The graph of $f(x) = \frac{1}{x}$ suggests that $\lim_{x \to \infty} \frac{1}{x} = 0$. As $x$ increases, $f(x)$ gets closer to 0.




**Support Numerically**

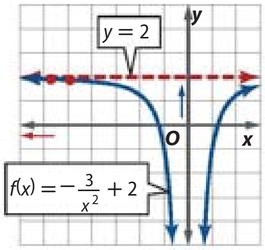
*x* approaches ∞ →

| $x$ | 10 | 100 | 1000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $f(x)$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 |

The pattern of outputs suggests that as $x$ grows increasingly larger, $f(x)$ approaches 0. ✔

**b.** $\lim_{x \to -\infty} \left(-\frac{3}{x^2} + 2\right)$

**Analyze Graphically** The graph of $f(x) = -\frac{3}{x^2} + 2$ suggests that $\lim_{x \to -\infty} \left(-\frac{3}{x^2} + 2\right) = 2$. As $x$ decreases, $f(x)$ gets closer to 2.




**Support Numerically**

← *x* approaches −∞

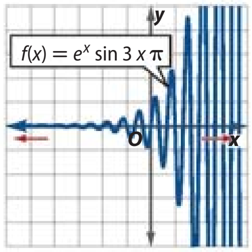
| $x$ | −100,000 | −10,000 | −1000 | −100 | −10 |
|---|---|---|---|---|---|
| $f(x)$ | 1.99999 | 1.99999 | 1.99999 | 1.9997 | 1.97 |

The pattern of outputs suggests that as $x$ decreases, $f(x)$ approaches 2.

**c.** $\lim_{x \to -\infty} e^x \sin 3\pi x$ **and** $\lim_{x \to \infty} e^x \sin 3\pi x$

**Analyze Graphically** The graph of $f(x) = e^x \sin 3x\pi$ suggests that $\lim_{x \to -\infty} e^x \sin 3\pi x = 0$. As $x$ decreases, $f(x)$ oscillates but tends toward 0.

The graph suggests that $\lim_{x \to \infty} e^x \sin 3\pi x$ does not exist. As $x$ increases, $f(x)$ oscillates between ever increasing values.




**Support Numerically**

← *x* approaches −∞ *x* approaches ∞ →

| $x$ | −100 | −50 | −10 | 0 | 10 | 50 | 100 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $3 \times 10^{-44}$ | $-2.0 \times 10^{-22}$ | −0.00005 | 0 | 21966 | $4.8 \times 10^{21}$ | $-2.0 \times 10^{43}$ |

The pattern of outputs suggests that as $x$ decreases, $f(x)$ approaches 0 and as $x$ increases, $f(x)$ oscillates.

### GuidedPractice

**6A.** $\lim_{x \to \infty} \left(\frac{1}{x^4} - 3\right)$

**6B.** $\lim_{x \to -\infty} e^x$

**6C.** $\lim_{x \to \infty} \sin x$

### StudyTip

**Asymptotes** The limit in Example 6a indicates that there is an asymptote at $y = 0$, while the limit in Example 6b indicates that there is an asymptote at $y = 2$.

### WatchOut!

**Oscillating Behavior**
Do not assume that because a function $f(x)$ exhibits oscillating behavior, it has no limit as $x$ approaches either ∞ or −∞. If the oscillations fluctuate between two fixed values or are unbounded, then the limit does not exist. If, however, the oscillations diminish and approach a fixed value, then the limit does exist.

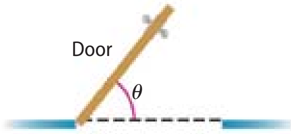
You can use graphical and numerical approaches to estimate limits at infinity in many real-world situations.

**Real-WorldLink**

A swing door operator is a device that opens and closes a swing door at a reduced speed to assist people who use wheelchairs.

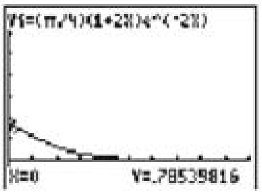
## Real-World Example 7 Estimate Limits at Infinity

**a. HYDRAULICS A low-energy swing door uses a spring to close the door and a hydraulic mechanism to dampen or slow down the door's movement. If the door is opened to an angle of $\frac{\pi}{4}$ and then released from this resting position, the angle $\theta$ of the door $t$ seconds after it is released is given by $\theta(t) = \frac{\pi}{4}(1 + 2t)e^{-2t}$. Estimate $\lim_{t \to \infty} \theta(t)$, if it exists, and interpret your result.**




**Estimate the Limit**

Graph $\theta(t) = \frac{\pi}{4}(1 + 2t)e^{-2t}$ using a graphing calculator. The graph shows that when $t = 0$, $\theta(t) \approx 0.785$ or about $\frac{\pi}{4}$. Notice that as $t$ increases, the function values of the graph tend toward 0. So, we can estimate that $\lim_{t \to \infty} \theta(t)$ is 0.




[0, 5] scl: 0.5 by [−0.5, 3.5] scl: 0.5

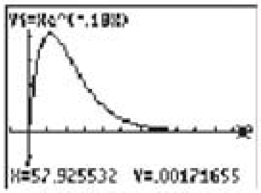
**Interpret the Result**

A limit of 0 in this situation indicates that the angle the door makes with its closed position tends toward a measure of 0 radians. That is, as the number of seconds after the door has been released increases, the door comes closer and closer to being completely closed.

**b. MEDICINE The concentration in milligrams per milliliter of a certain medicine in a patient's bloodstream $t$ hours after the medicine has been administered is given by $C(t) = Ate^{-0.18t}$, where $A$ is a positive constant. Estimate $\lim_{t \to \infty} C(t)$, if it exists, and interpret your result.**

**Estimate the Limit**

Graph the related function $C_1(t) = te^{-0.18t}$ using a graphing calculator. The graph shows that as $t$ increases, the function values of the graph tend toward 0. So, we can estimate that $\lim_{t \to \infty} C_1(t)$ is 0.

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Because $A$ is a positive constant, the graph of $C(t) = Ate^{-0.18t}$ will be the graph of $C_1(t) = te^{-0.18t}$ expanded vertically by a factor of $A$ but will still tend to 0 as $t$ increases without bound. So, we can estimate that $\lim_{t \to \infty} C(t)$ is also 0.

**Interpret the Result**

A limit of 0 in this situation indicates that all of the medicine will eventually be eliminated from the patient's bloodstream.

### GuidedPractice

**7A. ELECTRICITY** The typical voltage $V$ supplied by an electrical outlet in the United States can be modeled by the function $V(t) = 165 \sin 120\pi t$, where $t$ is the time in seconds. Estimate $\lim_{t \to \infty} V(t)$, if it exists, and interpret your result.

**7B. BIOLOGY** Fruit flies are placed in a half-pint milk bottle with a piece of fruit and a yeast plant. The fruit fly population after $t$ days is given by $P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$. Estimate $\lim_{t \to \infty} P(t)$, if it exists, and interpret your result.

## Exercises

= Step-by-Step Solutions begin on page R29.

**Estimate each limit using a graph. Support your conjecture using a table of values.** (Examples 1 and 2)

**1.** $\lim_{x \to 5} (4x - 10)$

**2.** $\lim_{x \to 2} \left(\frac{1}{2}x^5 - 2x^3 + 3x^2\right)$

**3.** $\lim_{x \to -2} (x^2 + 2x - 15)$

**4.** $\lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4}$

**5.** $\lim_{x \to 3} (2x^3 - 10x + 1)$

**6.** $\lim_{x \to 0} \frac{x \cos x}{x^2 + x}$

**7.** $\lim_{x \to 0} [5(\cos^2 x - \cos x)]$

**8.** $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$

**9.** $\lim_{x \to 6} (x + \sin x)$

**10.** $\lim_{x \to -5} \frac{x^2 + x - 20}{x + 5}$

**Estimate each one-sided or two-sided limit, if it exists.** (Example 3)

**11.** $\lim_{x \to 0^+} \frac{\sin x - x}{x}$

**12.** $\lim_{x \to 0^-} \frac{|4x|}{x}$

**13.** $\lim_{x \to 0} \frac{2x^2}{|x|}$

**14.** $\lim_{x \to 9^+} \frac{3 - \sqrt{x}}{x - 9}$

**15.** $\lim_{x \to 3^-} \frac{x^2 - 5x + 6}{x - 3}$

**16.** $\lim_{x \to -\frac{1}{2}^-} \frac{|2x + 1|}{x}$

**17.** $\lim_{x \to -2} \frac{x^2 + 5x + 6}{|x + 2|}$

**18.** $\lim_{x \to -7} \frac{x^2 - x - 56}{x + 7}$

**19.** $\lim_{x \to 0^-} (\sqrt{-x} - 7)$

**20.** $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$

**21.** $\lim_{x \to 4^+} \frac{x^2 - x - 12}{|x - 4|}$

**22.** $\lim_{x \to 0^+} (\sqrt{x} + 2x + 3)$

**23.** $\lim_{x \to 0} \frac{|3x|}{2x}$

**24.** $\lim_{x \to -1} \frac{|x + 1|}{x^2 - 1}$

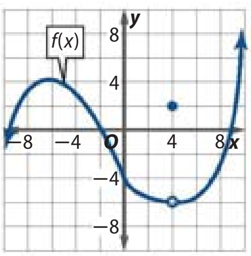
**25.** $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

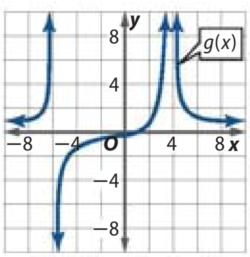
**26.** $\lim_{x \to 3} f(x)$ where $f(x) = \begin{cases} 3x & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$

**27.** $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x - 5 & \text{if } x < 0 \\ x^2 + 5 & \text{if } x \geq 0 \end{cases}$

**28.** $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} -x^2 + 2 & \text{if } x < 0 \\ \frac{2x}{x} & \text{if } x \geq 0 \end{cases}$

**For each function below, estimate each limit if it exists.** (Examples 1–4)




**29.** $\lim_{x \to -6} f(x)$

**31.** $\lim_{x \to 4} g(x)$

**30.** $\lim_{x \to 4} f(x)$

**32.** $\lim_{x \to -6} g(x)$

**Estimate each limit, if it exists.** (Examples 4–6)

**33.** $\lim_{x \to -4} \frac{-17}{x^2 + 8x + 16}$

**34.** $\lim_{x \to 5} \frac{x^2}{x^2 - 10x + 25}$

**35.** $\lim_{x \to 4} \frac{|x|}{x - 4}$

**36.** $\lim_{x \to \infty} e^{2x - 5}$

**37.** $\lim_{x \to 6} \frac{5}{(x - 6)^2}$

**38.** $\lim_{x \to -\infty} (x^5 - 7x^4 - 4x + 1)$

**39.** $\lim_{x \to \infty} \frac{x^2 + x - 22}{4x^3 - 13}$

**40.** $\lim_{x \to -3} \frac{x^2 + 9x + 20}{x + 3}$

**41.** $\lim_{x \to \infty} \frac{3x - 4}{9x + 3}$

**42.** $\lim_{x \to \infty} x \cos x$

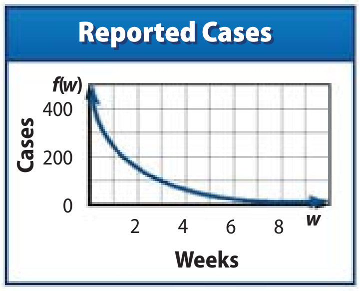
**43.** $\lim_{x \to -\infty} \frac{3^x + 3^{-x}}{3^x - 3^{-x}}$

**44.** $\lim_{x \to 0} \frac{\sin |x|}{x}$

**45.** $\lim_{x \to 0} x^2 \cos \frac{1}{x}$

**46.** $\lim_{x \to -\infty} \frac{4x - 13}{2x + 8}$

**47.** **MEDICINE** A vaccine was quickly administered to combat a small outbreak of a minor infection. The number of reported cases of the infection $w$ weeks since the vaccine was administered is shown. (Example 7)




**a.** Use the graph to estimate $\lim_{w \to 1} f(w)$ and $\lim_{w \to 3} f(w)$.

**b.** Use the graph to estimate $\lim_{w \to \infty} f(w)$ if it exists, and interpret your results.

**48.** **TRACK AND FIELD** The logistic function $f(x) = \frac{5.334}{1 + 62548.213e^{-0.129x}}$, where $x$ is the number of years since 1900, models the world record heights in meters for women's pole vaulting from 1996 to 2008. (Example 7)

**a.** Graph the function for $96 \leq x \leq 196$.

**b.** Estimate $\lim_{x \to \infty} \frac{5.334}{1 + 62548.213e^{-0.129x}}$, if it exists.

**c.** Explain the relationship between the limit of the function and the world record heights.

**49.** **INTERNET VIDEO** A group of friends created a video parody of several popular songs and posted it online. As word of the video spread, interest grew. A model that can be used to estimate the number of people $p$ that viewed the video is $p(d) = 12(1.25012)^d - 12$, where $d$ is days since the video was originally posted. (Example 7)

**a.** Graph the function for $0 \leq d \leq 20$.

**b.** Estimate the number of people who viewed the video by the ends of the 5th, 10th, and 20th days. How many people will have viewed it after 2 months? (Use $d = 60$.)

**c.** Estimate $\lim_{d \to \infty} p(d)$, if it exists, and interpret your results.

**50. TECHNOLOGY** The number of cell phone owners between the ages of 18 and 25 has steadily increased since the 1990s. A sequence model that can estimate the number of people ages 18–25 per cell phone is $a_n = 64.39(0.82605)^n + 1$, where $n$ represents years since 1993. (Example 7)

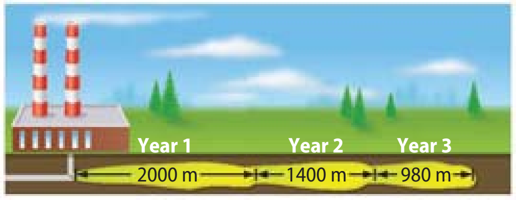
a. Graph the function for the years 1993 to 2011.

b. Use the graph to estimate the amount of people per cell phone for 1998, 2007, and 2011.

c. Use your graph to estimate $\lim_{n \to \infty} a_n$.

d. Explain the relationship between the limit of the function and the number of people per cell phone.

**51. CHEMICALS** An underground pipeline is leaking a toxic chemical. After the leak began, it spread as shown below. The distance the chemical spreads each year can be defined as $d(t) = 2000(0.7)^{t-1}$, for $t \geq 1$, where $t$ is years since the leak began. (Example 7)




a. Graph the function for $1 \leq t \leq 15$.

b. Use your graph to find values of $d$ for $t = 5$, 10, and 15 years.

c. Use your graph to estimate $\lim_{t \to \infty} d(t)$.

d. Will the chemical ever spread to a hospital that is located 7000 meters away from the leak? Recall that the sum of an infinite geometric series can be found by $\frac{a_1}{1-r}$.

**52. DEPRECIATION** Chuck purchases a motorcycle for \$11,000, and it depreciates each year that he owns it. The value $v$ of the motorcycle after $t$ years can be estimated by the model $v(t) = 11{,}000(0.76)^t$. (Example 7)

a. Graph the function for $0 \leq t \leq 10$.

b. Use your graph to estimate the value of the motorcycle for $t = 3$, 7, and 10 years.

c. Use your graph to estimate $\lim_{t \to \infty} v(t)$.

d. Explain the relationship between the limit of the function and the value of Chuck's motorcycle.

**For the function below, estimate each limit if it exists.**

**53.** $\lim_{x \to 0^-} f(x)$

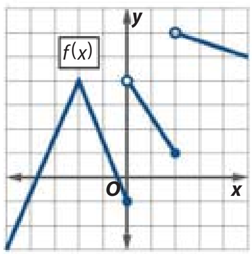
**54.** $\lim_{x \to 0^+} f(x)$

**55.** $\lim_{x \to 0} f(x)$

**56.** $\lim_{x \to 2^-} f(x)$

**57.** $\lim_{x \to 2^+} f(x)$

**58.** $\lim_{x \to 1} f(x)$




**GRAPHING CALCULATOR Determine whether each limit exists. If not, describe what is happening graphically at the limit.**

**59.** $\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1}$

**60.** $\lim_{x \to 2} \frac{x^2 + x}{x^2 - x - 2}$

**61.** $\lim_{x \to 0} 3 \cos \frac{\pi}{x}$

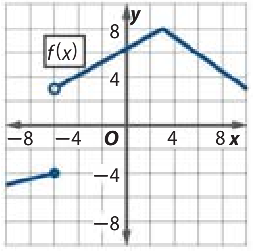
**62.** $\lim_{x \to -5} \frac{|x + 5|}{x + 5}$

**63.** $\lim_{x \to 0} \frac{7}{1 + e^{\frac{1}{x}}}$

**64.** $\lim_{x \to -4} \frac{\sqrt{2 - x} - 3}{x + 4}$

## H.O.T. Problems Use Higher-Order Thinking Skills

**65. ERROR ANALYSIS** Will and Kenyi are finding the limit of the function below as $x$ approaches $-6$. Will says that the limit is $-4$. Kenyi disagrees, arguing that the limit is 3. Is either of them correct? Explain your reasoning.




**66. OPEN ENDED** Give an example of a function $f$ such that $\lim_{x \to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function $g$ such that $g(0)$ exists but $\lim_{x \to 0} g(x)$ does not exist.

**67. CHALLENGE** Suppose $f(x) = \frac{x^2 + 1}{x - 1}$ and $g(x) = \frac{x + 1}{x^2 - 4}$. Estimate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} g(x)$. If $g(a) = 0$ and $f(a) \neq 0$, what can you assume about $\lim_{x \to a} \frac{f(x)}{g(x)}$? Explain your reasoning.

**68. REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

$$\text{If } f(c) = L, \text{ then } \lim_{x \to c} f(x) = L.$$

**69. OPEN ENDED** Sketch the graph of a function such that $\lim_{x \to 0} f(x) = -3$, $f(0) = 2$, $f(2) = 5$, and $\lim_{x \to 2} f(x)$ does not exist.

**70. CHALLENGE** For the function below, estimate each limit if it exists.

$$f(x) = \begin{cases} 2x + 4 & \text{if } x < -1 \\ -1 & \text{if } -1 \leq x \leq 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

a. $\lim_{x \to -1} f(x)$  b. $\lim_{x \to 0^-} f(x)$  c. $\lim_{x \to 2^+} f(x)$

**71. WRITING IN MATH** Explain what method you could use to estimate limits if a function is continuous. Explain how this differs from methods used to estimate functions that are not continuous.

## Spiral Review

**72. FUEL ECONOMY** The table shows various engine sizes available from an auto manufacturer and their respective fuel economies. (Lesson 11-7)

**a.** Make a scatter plot of the data, and identify the relationship.

**b.** Calculate and interpret the correlation coefficient. Determine whether it is significant at the 10% level.

**c.** If the correlation is significant at the 10% level, find the least-squares regression equation and interpret the slope and intercept in context.

**d.** Use the regression equation that you found in part **c** to predict the expected miles per gallon that a car would get for an engine size of 8.0 liters. State whether this prediction is reasonable. Explain.

| Engine Size (liters) | Highway Mileage (MPG) |
|---|---|
| 1.6 | 34 |
| 2.2 | 37 |
| 2.0 | 30 |
| 6.2 | 26 |
| 7.0 | 24 |
| 3.5 | 29 |
| 5.3 | 24 |
| 2.4 | 33 |
| 3.6 | 26 |
| 6.0 | 24 |
| 4.4 | 23 |
| 4.6 | 24 |

**For each statement, write the null and alternative hypotheses and state which hypothesis represents the claim.** (Lesson 11-6)

**73.** A brand of dill pickles claims to contain 4 Calories.

**74.** A student claims that he exercises 85 minutes a day.

**75.** A student claims that she can get ready for school in less than 10 minutes.

**76.** Use Pascal's triangle to expand $\left(3a + \frac{2}{3}b\right)^4$. (Lesson 10-5)

**Write and graph a polar equation and directrix for the conic with the given characteristics.** (Lesson 9-4)

**77.** $e = 1$; vertex at $(0, -2)$

**78.** $e = 3$; vertices at $(0, 3)$ and $(0, 6)$

**Find the angle between each pair of vectors to the nearest tenth of a degree.** (Lesson 8-5)

**79.** $\mathbf{u} = \langle 2, 9, -2 \rangle$, $\mathbf{v} = \langle -4, 7, 6 \rangle$

**80.** $\mathbf{m} = 3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ and $\mathbf{n} = -7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$

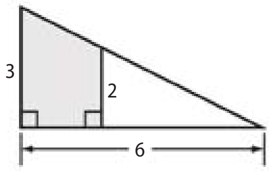
**Use a graphing calculator to graph the conic given by each equation.** (Lesson 7-4)

**81.** $7x^2 - 50xy + 7y^2 = -288$

**82.** $x^2 - 2\sqrt{3}xy + 3y^2 + 16\sqrt{3}x + 16y = 0$

## Skills Review for Standardized Tests

**83. SAT/ACT** What is the area of the shaded region?

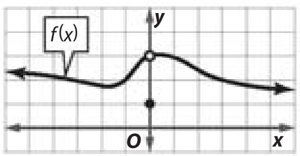



A 5
B 6
C 7
D 8
E 9

**84. REVIEW** Which of the following best describes the end behavior of $f(x) = x^{10} - x^9 + 5x^8$?

F $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -\infty$ as $x \to \infty$
G $f(x) \to \infty$ as $x \to \infty$, $f(x) \to \infty$ as $x \to -\infty$
H $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to \infty$ as $x \to -\infty$
J $f(x) \to -\infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$

**85.** According to the graph of $y = f(x)$, $\lim_{x \to 0} f(x) =$




A 0
B 1
C 3
D The limit does not exist.

**86. REVIEW** Which of the following describes the graph of $g(x) = \frac{1}{x^2}$?

I It has an infinite discontinuity.
II It has a jump discontinuity.
III It has a point discontinuity.

F I only
G II only
H I and II only
J I and III only
K I, II and III

LESSON 12-2

# Evaluating Limits Algebraically

**Then**

- You estimated limits using graphical and numerical methods. (Lesson 12-1)

**Now**

- **1** Evaluate limits of polynomial and rational functions at selected points.
- **2** Evaluate limits of polynomial and rational functions at infinity.

**Why?**

- Suppose the width in millimeters of an animal's pupil is given by $d(x) = \frac{152x^{-0.45} + 85}{4x^{-0.45} + 10}$, where $x$ is the illuminance of the light shining on the pupils measured in lux. You can evaluate limits to find the width of the animal's pupils when the light is at its minimum and maximum intensity.

**NewVocabulary**
direct substitution
indeterminate form

## 1 Compute Limits at a Point

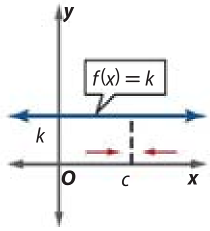
In Lesson 12-1, you learned how to estimate limits by using a graph or by making a table of values. In this lesson, we will explore computational techniques for evaluating limits.

### KeyConcept Limits of Functions

**Limits of Constant Functions**

**Words** The limit of a constant function at any point $c$ is the constant value of the function.

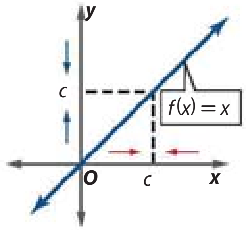
**Symbols** $\lim\limits_{x \to c} k = k$




**Limits of the Identity Function**

**Words** The limit of the identity function at any point $c$ is $c$.

**Symbols** $\lim\limits_{x \to c} x = c$




When combined with the following properties of limits, these constant and identity function limits become very useful.

### KeyConcept Properties of Limits

If $k$ and $c$ are real numbers, $n$ is a positive integer, and $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ both exist, then the following are true.

| | |
|---|---|
| Sum Property | $\lim\limits_{x \to c} [f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$ |
| Difference Property | $\lim\limits_{x \to c} [f(x) - g(x)] = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$ |
| Scalar Multiple Property | $\lim\limits_{x \to c} [kf(x)] = k \lim\limits_{x \to c} f(x)$ |
| Product Property | $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$ |
| Quotient Property | $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, if $\lim\limits_{x \to c} g(x) \neq 0$ |
| Power Property | $\lim\limits_{x \to c} [f(x)^n] = \left[\lim\limits_{x \to c} f(x)\right]^n$ |
| $n$th Root Property | $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$, if $\lim\limits_{x \to c} f(x) > 0$ when $n$ is even. |

**StudyTip**

**Limit Properties** Each of the limit properties given on the previous page also hold for one-sided limits and limits at infinity, so long as each limit exists.

## Example 1 Use Limit Properties

**Use the properties of limits to evaluate each limit.**

**a.** $\lim_{x \to 4} (x^2 - 6x + 3)$

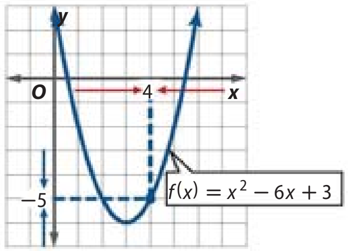
$\lim_{x \to 4} (x^2 - 6x + 3) = \lim_{x \to 4} x^2 - \lim_{x \to 4} 6x + \lim_{x \to 4} 3$ — Sum and Difference Properties

$= \left(\lim_{x \to 4} x\right)^2 - 6 \cdot \lim_{x \to 4} x + \lim_{x \to 4} 3$ — Power and Scalar Multiple Properties

$= 4^2 - 6 \cdot 4 + 3$ — Limits of Constant and Identity Functions

$= -5$ — Simplify.

**CHECK** The graph of $f(x) = x^2 - 6x + 3$ supports this result. ✓




**b.** $\lim_{x \to -2} \frac{4x^3 + 1}{x - 5}$

$\lim_{x \to -2} \frac{4x^3 + 1}{x - 5} = \frac{\lim_{x \to -2} (4x^3 + 1)}{\lim_{x \to -2} (x - 5)}$ — Quotient Property

$= \frac{\lim_{x \to -2} 4x^3 + \lim_{x \to -2} 1}{\lim_{x \to -2} x - \lim_{x \to -2} 5}$ — Sum and Difference Properties

$= \frac{4\left(\lim_{x \to -2} x\right)^3 + \lim_{x \to -2} 1}{\lim_{x \to -2} x - \lim_{x \to -2} 5}$ — Scalar Multiple and Power Properties

$= \frac{4(-2)^3 + 1}{-2 - 5}$ — Limits of Constant and Identity Functions

$= \frac{31}{7}$ — Simplify.

**CHECK** Make a table of values, choosing $x$-values that approach $-2$ from either side. ✓

$x$ approaches $-2$ → ← $x$ approaches $-2$

| x | −2.1 | −2.01 | −2.001 | −2 | −1.999 | −1.99 | −1.9 |
|---|---|---|---|---|---|---|---|
| f(x) | 5.08 | 4.49 | 4.43 | | 4.42 | 4.37 | 3.83 |

**c.** $\lim_{x \to 3} \sqrt{8 - x}$

$\lim_{x \to 3} \sqrt{8 - x} = \sqrt{\lim_{x \to 3} (8 - x)}$ — $n$th Root Property

$= \sqrt{\lim_{x \to 3} 8 - \lim_{x \to 3} x}$ — Difference Property

$= \sqrt{8 - 3}$ — Limits of Constant and Identity Functions

$= \sqrt{5}$ — Simplify.

**GuidedPractice**

**1A.** $\lim_{x \to 2} (-x^3 + 4)$ **1B.** $\lim_{x \to 2} \frac{x - 3}{2x^2 - x - 15}$ **1C.** $\lim_{x \to -1} \sqrt{x + 3}$

Notice that for each of the functions in Example 1, the limit of $f(x)$ as $x$ approaches $c$ is the same value that you would get if you calculated $f(c)$. While this is not true of every function, it is true of polynomial functions and of rational functions as described at the top of the next page.

**StudyTip**

**Well-Behaved Functions** Continuous functions such as polynomial functions are considered *well-behaved*, since limits of these functions at any point can be found by direct substitution. The limits of functions that are not well-behaved over their entire domain can still be found using this method, so long as the function is continuous at the domain value in question.

## KeyConcept Limits of Functions

**Limits of Polynomial Functions**

If $p(x)$ is a polynomial function and $c$ is a real number, then $\lim_{x \to c} p(x) = p(c)$.

**Limits of Rational Functions**

If $r(x) = \frac{p(x)}{q(x)}$ is a rational function and $c$ is a real number, then $\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$ if $q(c) \neq 0$.

More simply stated, limits of polynomial and rational functions can be found by **direct substitution**, so long as the denominator of the rational function evaluated at $c$ is not 0.

## Example 2 Use Direct Substitution

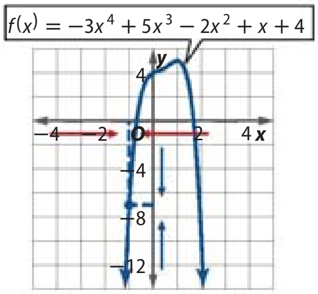
**Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.**

**a.** $\lim_{x \to -1} (-3x^4 + 5x^3 - 2x^2 + x + 4)$

Since this is the limit of a polynomial function, we can apply the method of direct substitution to find the limit.

$$\lim_{x \to -1} (-3x^4 + 5x^3 - 2x^2 + x + 4) = -3(-1)^4 + 5(-1)^3 - 2(-1)^2 + (-1) + 4$$
$$= -3 - 5 - 2 - 1 + 4 \text{ or } -7$$

**CHECK** The graph of $f(x) = -3x^4 + 5x^3 - 2x^2 + x + 4$ supports this result. ✔




**b.** $\lim_{x \to 3} \frac{2x^3 - 6}{x - x^2}$

This is the limit of a rational function, the denominator of which is nonzero when $x = 3$. Therefore, we can apply the method of direct substitution to find the limit.

$$\lim_{x \to 3} \frac{2x^3 - 6}{x - x^2} = \frac{2(3)^3 - 6}{3 - 3^2}$$
$$= \frac{48}{-6} \text{ or } -8$$

**c.** $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

This is the limit of a rational function. Since the denominator of this function is 0 when $x = 1$, the limit cannot be found by direct substitution.

**GuidedPractice**

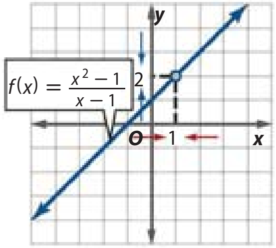
**2A.** $\lim_{x \to 4} (x^3 - 3x^2 - 5x + 7)$ **2B.** $\lim_{x \to -5} \frac{x + 1}{x^2 + 3}$ **2C.** $\lim_{x \to -8} \sqrt{x + 6}$

Suppose you incorrectly applied the Quotient Property of Limits or direct substitution to evaluate $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{\lim_{x \to 1} (x^2 - 1)}{\lim_{x \to 1} (x - 1)} = \frac{1^2 - 1}{1 - 1} \text{ or } \frac{0}{0}$$

This is incorrect because the limit of the denominator is 0.

It is customary to describe the resulting fraction $\frac{0}{0}$ as having an **indeterminate form** because you cannot determine the limit of the function with 0 in the denominator. A limit of this type may exist and have a real number value, or it may not exist, possibly diverging to $\infty$ or $-\infty$. In this case, from the graph of $f(x) = \frac{x^2 - 1}{x - 1}$, it appears that $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$ does exist and has a value of 2.




While an indeterminate form limit results from an incorrect application of limit properties or theorems, an analysis of this form can provide a clue as to what technique *should* be applied to find a limit.

If you evaluate the limit of a rational function and reach the indeterminate form $\frac{0}{0}$, you should try to simplify the expression algebraically by factoring and dividing out a common factor.

## Example 3 Use Factoring

**Evaluate each limit.**

**a.** $\lim\limits_{x \to -4} \frac{x^2 - x - 20}{x + 4}$

By direct substitution, you obtain $\frac{(-4)^2 - (-4) - 20}{-4 + 4}$ or $\frac{0}{0}$. Since this is an indeterminate form, try factoring and dividing out any common factors.

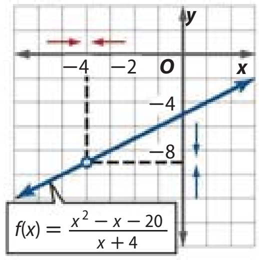
$$\lim_{x \to -4} \frac{x^2 - x - 20}{x + 4} = \lim_{x \to -4} \frac{(x - 5)(x + 4)}{x + 4}$$ Factor the numerator.

$$= \lim_{x \to -4} \frac{(x - 5)\cancel{(x + 4)}}{\cancel{x + 4}}$$ Divide out the common factor.

$$= \lim_{x \to -4} (x - 5)$$ Simplify.

$$= (-4) - 5 \text{ or } -9$$ Apply direct substitution and simplify.

**CHECK** The graph of $f(x) = \frac{x^2 - x - 20}{x + 4}$ supports this result. ✓




**WatchOut!**

**Factoring** If the entire expression in the numerator is divided out, the result is a 1, not a 0.

**b.** $\lim\limits_{x \to 3} \frac{x - 3}{x^3 - 3x^2 - 7x + 21}$

By direct substitution, you obtain $\frac{3 - 3}{3^3 - 3(3)^2 - 7(3) + 21}$ or $\frac{0}{0}$.

$$\lim_{x \to 3} \frac{x - 3}{x^3 - 3x^2 - 7x + 21} = \lim_{x \to 3} \frac{x - 3}{(x^2 - 7)(x - 3)}$$ Factor the denominator.

$$= \lim_{x \to 3} \frac{\cancel{x - 3}}{(x^2 - 7)\cancel{(x - 3)}}$$ Divide out the common factor.

$$= \lim_{x \to 3} \frac{1}{x^2 - 7}$$ Simplify.

$$= \frac{1}{(3)^2 - 7} \text{ or } \frac{1}{2}$$ Apply direct substitution and simplify.

**GuidedPractice**

**3A.** $\lim\limits_{x \to -2} \frac{x^3 - 3x^2 - 4x + 12}{x + 2}$

**3B.** $\lim\limits_{x \to 6} \frac{x^2 - 7x + 6}{3x^2 - 11x - 42}$

While this method of dividing out a common factor is valid, it requires some justification. In Example 3a, dividing out a common factor from $f(x)$ resulted in a new function, $g(x)$, where

$$f(x) = \frac{x^2 - x - 20}{x + 4} \quad \text{and} \quad g(x) = x - 5.$$

These two functions yield the same function values for every $x$ except when $x = -4$. If two functions differ only at a value $c$ in their domain, their limits as $x$ approaches $c$ are the same. This is because the value of a limit at a point is not dependent on the value of the *function* at that point. Therefore, $\lim\limits_{x \to -4} \frac{x^2 - x - 20}{x + 4} = \lim\limits_{x \to -4} (x - 5)$.

Another technique to find limits that have indeterminate form is to rationalize the numerator or denominator of a function and then divide out any common factors.

## Example 4 Use Rationalizing

**Evaluate $\lim\limits_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.**

By direct substitution, you obtain $\frac{\sqrt{9} - 3}{9 - 9}$ or $\frac{0}{0}$. Rationalize the numerator of the fraction, and then divide common factors.

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$$ Multiply the numerator and denominator by $\sqrt{x} + 3$, the conjugate of $\sqrt{x} - 3$.

$$= \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$ Simplify.

$$= \lim_{x \to 9} \frac{\cancel{x - 9}}{\cancel{(x - 9)}(\sqrt{x} + 3)}$$ Divide out the common factor.

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3}$$ Simplify.

$$= \frac{1}{\sqrt{9} + 3}$$ Apply direct substitution.

$$= \frac{1}{6}$$ Simplify.

**CHECK** The graph of $f(x) = \frac{\sqrt{x} - 3}{x - 9}$ in Figure 12.2.1 supports this result. ✔

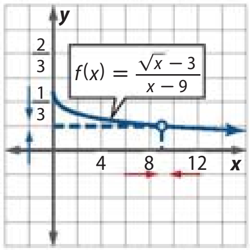



Figure 12.2.1

**GuidedPractice**

**Evaluate each limit.**

**4A.** $\lim\limits_{x \to 25} \frac{x - 25}{\sqrt{x} - 5}$

**4B.** $\lim\limits_{x \to 0} \frac{2 - \sqrt{x + 4}}{x}$

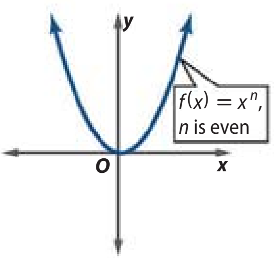
## 2 Compute Limits at Infinity

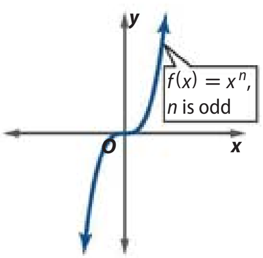
In Lesson 2-1, you learned that all even-degree power functions have the same end behavior, and all odd-degree power functions have the same end behavior. This can be described in terms of limits as shown below.

### KeyConcept Limits of Power Functions at Infinity

For any positive integer $n$,

- $\lim\limits_{x \to \infty} x^n = \infty$.
- $\lim\limits_{x \to -\infty} x^n = \infty$ if $n$ is even.
- $\lim\limits_{x \to -\infty} x^n = -\infty$ if $n$ is odd.




In Lesson 2-2, you also learned that the end behavior of a polynomial function is determined by the end behavior of the power function related to its highest-powered term. This can also be described using limits.

**StudyTip**

**Products with Infinity** Since a limit of $\infty$ means that function values are increasingly large positive numbers, multiplying these numbers by a positive constant does not change this trend. However, multiplying a limit of $\infty$ by a negative constant changes the sign of all the outputs suggested by this notation. Thus, $-1(\infty) = -\infty$.

## KeyConcept Limits of Polynomial Functions at Infinity

Let $p$ be a polynomial function $p(x) = a_n x^n + \ldots + a_1 x + a_0$. Then $\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n$ and $\lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n$.

You can use these properties to evaluate limits of polynomial functions at infinity. Remember, notating that the limit of a function is $\infty$ or $-\infty$ does not indicate that the limit exists but instead describes the behavior of the function as increasing or decreasing without bound, respectively.

## Example 5 Limits of Polynomial Functions at Infinity

**Evaluate each limit.**

**a.** $\lim_{x \to -\infty} (x^3 - 2x^2 + 5x - 1)$

$\lim_{x \to -\infty} (x^3 - 2x^2 + 5x - 1) = \lim_{x \to -\infty} x^3$ — Limits of Polynomial Functions at Infinity

$= -\infty$ — Limits of Power Functions at Infinity

**b.** $\lim_{x \to \infty} (4 + 3x - x^2)$

$\lim_{x \to \infty} (4 + 3x - x^2) = \lim_{x \to \infty} -x^2$ — Limits of Polynomial Functions at Infinity

$= -\lim_{x \to \infty} x^2$ — Scalar Multiple Property

$= -\infty$ — Limits of Power Functions at Infinity

**c.** $\lim_{x \to -\infty} (5x^4 - 3x)$

$\lim_{x \to -\infty} (5x^4 - 3x) = \lim_{x \to -\infty} 5x^4$ — Limits of Polynomial Functions at Infinity

$= 5 \lim_{x \to -\infty} x^4$ — Scalar Multiple Property

$= 5 \cdot \infty$ or $\infty$ — Limits of Power Functions at Infinity

**GuidedPractice** **Evaluate each limit.**

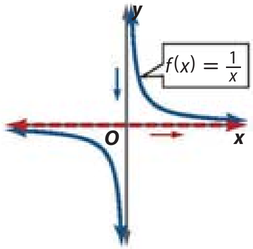
**5A.** $\lim_{x \to \infty} (-x^3 - 4x^2 + 9)$ **5B.** $\lim_{x \to -\infty} (4x^6 + 3x^5 - x)$ **5C.** $\lim_{x \to -\infty} (2x - 6x^2 + 4x^5)$

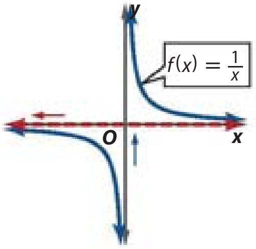
To evaluate limits of rational functions at infinity, we need another limit property.

## KeyConcept Limits of Reciprocal Function at Infinity

**Words** The limit of a reciprocal function at positive or negative infinity is 0.

**Symbols** $\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$




**Corollary** For any positive integer $n$, $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$.

If we divide the numerator and denominator of a rational function by the highest power of $x$ that occurs in the function, we can use this property to find limits of rational functions at infinity.

## Example 6 Limits of Rational Functions at Infinity

**Evaluate each limit.**

**a.** $\lim_{x \to \infty} \frac{4x+5}{8x-3}$

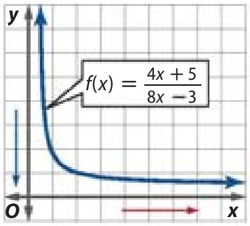
$\lim_{x \to \infty} \frac{4x+5}{8x-3} = \lim_{x \to \infty} \frac{\frac{4x}{x}+\frac{5}{x}}{\frac{8x}{x}-\frac{3}{x}}$ Divide each term by highest-powered term, $x$.

$= \lim_{x \to \infty} \frac{4+\frac{5}{x}}{8-\frac{3}{x}}$ Simplify.

$= \frac{\lim_{x \to \infty} 4 + 5\lim_{x \to \infty} \frac{1}{x}}{\lim_{x \to \infty} 8 - 3\lim_{x \to \infty} \frac{1}{x}}$ Quotient, Sum, Difference and Scalar Multiple Properties

$= \frac{4+5 \cdot 0}{8-3 \cdot 0}$ or $\frac{1}{2}$ Limit of a Constant Function and Limits of Reciprocal Function at Infinity

**CHECK** The graph of $f(x) = \frac{4x+5}{8x-3}$ in supports this result. ✓




**b.** $\lim_{x \to -\infty} \frac{6x^2-x}{3x^3+1}$

$\lim_{x \to -\infty} \frac{6x^2-x}{3x^3+1} = \lim_{x \to -\infty} \frac{\frac{6x^2}{x^3}-\frac{x}{x^3}}{\frac{3x^3}{x^3}+\frac{1}{x^3}}$ Divide each term by highest-powered term, $x^3$.

$= \lim_{x \to -\infty} \frac{\frac{6}{x}-\frac{1}{x^2}}{3+\frac{1}{x^3}}$ Simplify.

$= \frac{6\lim_{x \to -\infty} \frac{1}{x} - \lim_{x \to -\infty} \frac{1}{x^2}}{\lim_{x \to -\infty} 3 + \lim_{x \to -\infty} \frac{1}{x^3}}$ Quotient, Sum, Difference and Scalar Multiple Properties

$= \frac{6 \cdot 0 - 0}{3+0}$ or 0 Limit of a Constant Function and Limits of Reciprocal Function at Infinity

**c.** $\lim_{x \to \infty} \frac{5x^4}{9x^3+2x}$

$\lim_{x \to \infty} \frac{5x^4}{9x^3+2x} = \lim_{x \to \infty} \frac{5}{\frac{9}{x}+\frac{2}{x^3}}$ Divide each term by highest-powered term, $x^4$. Then simplify.

$= \frac{\lim_{x \to \infty} 5}{9\lim_{x \to \infty} \frac{1}{x} + 2\lim_{x \to \infty} \frac{1}{x^3}}$ Quotient, Sum, and Scalar Multiple Properties

$= \frac{5}{9 \cdot 0 + 2 \cdot 0}$ or $\frac{5}{0}$ Limit of a Constant Function and Limits of Reciprocal Function at Infinity

Because the limit of the denominator is 0, we know that we have not correctly applied the Quotient Property of Limits. However, we can argue that as 5 is divided by increasingly smaller values approaching 0, the value of the fraction gets increasingly larger. Therefore, the limit can be described as approaching $\infty$.

### Technology Tip

**Evaluating Limits** Using a calculator is not a foolproof way of evaluating $\lim_{x \to c} f(x)$ or $\lim_{x \to \pm\infty} f(x)$. You may only analyze the values of $f(x)$ for a few values of $x$ near $c$ or for a few large values of $x$. However, the function may do something unexpected as $x$ gets even closer to $c$ or as $x$ gets even larger or smaller. You should use algebraic methods whenever possible to find limits.

### Guided Practice

**6A.** $\lim_{x \to -\infty} \frac{5}{x-10}$

**6B.** $\lim_{x \to \infty} \frac{-3x^2+7}{5x+1}$

**6C.** $\lim_{x \to \infty} \frac{7x^3-3x^2+1}{2x^3+4x}$

In Lesson 10-1, you learned that since a sequence is a function of the natural numbers, the limit of a sequence is the limit of a function as $n \to \infty$. If this limit exists, then its value is the number to which the sequence converges. For example, the sequence $a_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be described as $f(n) = \frac{1}{n}$, where $n$ is a positive integer. Because $\lim_{n\to\infty} \frac{1}{n} = 0$, the sequence converges to 0.

## Example 7 Limits of Sequences

**Write the first five terms of each sequence. Then find the limit of the sequence, if it exists.**

**a.** $a_n = \dfrac{3n + 1}{n + 5}$

The first five terms of this sequence are $\frac{3(1)+1}{1+5}, \frac{3(2)+1}{2+5}, \frac{3(3)+1}{3+5}, \frac{3(4)+1}{4+5}$, and $\frac{3(5)+1}{5+5}$ or approximately 0.667, 1, 1.25, 1.444, and 1.6. To find the limit of the sequence, find $\lim_{n\to\infty} \frac{3n+1}{n+5}$.

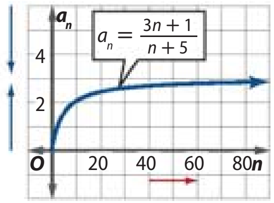
$$\lim_{n\to\infty} \frac{3n+1}{n+5} = \lim_{n\to\infty} \frac{3 + \frac{1}{n}}{1 + \frac{5}{n}}$$ Divide each term by highest-powered term, $n$, and simplify.

$$= \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1 + 5 \lim_{n\to\infty} \frac{1}{n}}$$ Quotient, Sum, and Scalar Multiple Properties

$$= \frac{3+0}{1+5 \cdot 0} \text{ or } 3$$ Limit of a Constant Function and Limits of Reciprocal Function at Infinity

So, the limit of the sequence is 3. That is, the sequence converges to 3.

**CHECK** The graph of $a_n = \frac{3n+1}{n+5}$ supports this result. ✓




**Study Tip**

**Check for Reasonableness** To check the reasonableness of the results in Example 7, find the 100th, 1000th, and 10,000th terms in each sequence. In Example 7a, these terms are 2.867, 2.986, and 2.999, respectively. Since these values appear to be approaching 3, a limit of 3 is reasonable.

**b.** $b_n = \dfrac{5}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

The first five terms of this sequence are approximately 5, 2.813, 2.222, 1.953, and 1.8. Now find the limit of the sequence.

$$\lim_{n\to\infty} \frac{5}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \lim_{n\to\infty} \frac{5}{n^4}\left[\frac{n^2(n^2+2n+1)}{4}\right]$$ Square the binomial.

$$= \lim_{n\to\infty} \frac{5n^4 + 10n^3 + 5n^2}{4n^4}$$ Multiply.

$$= \frac{\lim_{n\to\infty} 5 + 10 \lim_{n\to\infty} \frac{1}{n} + 5 \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 4}$$ Divide each term by highest-powered term. Then use Quotient, Sum, and Scalar Multiple Properties.

$$= \frac{5}{4} \text{ or } 1.25$$ Limit of a Constant Function and Limits of Reciprocal Function at Infinity

So, the limit of $b_n$ is 1.25. That is, the sequence converges to 1.25.

**CHECK** Make a table of values, choosing large $n$-values that grow increasingly larger. ✓

$n$ approaches ∞ →

| $n$ | 10 | 100 | 1000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| $a_n$ | 1.51 | 1.28 | 1.25 | 1.25 | 1.25 |

## Guided Practice

**7A.** $a_n = \dfrac{4}{n^2 + 1}$ **7B.** $b_n = \dfrac{2n^3}{3n + 8}$ **7C.** $c_n = \dfrac{9}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$

## Exercises

= Step-by-Step Solutions begin on page R29.

**Use the properties of limits to evaluate each limit.** (Example 1)

**1.** $\lim\limits_{x \to -3} (5x - 10)$

**2.** $\lim\limits_{x \to 5} \dfrac{x^2 + 4x + 13}{x - 3}$

**3.** $\lim\limits_{x \to -1} (7x^2 - 6x - 3)$

**4.** $\lim\limits_{x \to -2} \dfrac{2x^5 - 4x^3 - 2x - 12}{x^3 + 5x^2}$

**5.** $\lim\limits_{x \to 9} \left(\dfrac{1}{x} + 2x + \sqrt{x}\right)$

**6.** $\lim\limits_{x \to -4} [x^2(x + 1) + 2]$

**7.** $\lim\limits_{x \to 12} \dfrac{x^2 - 10x}{\sqrt{x + 4}}$

**8.** $\lim\limits_{x \to 1} \dfrac{x^3 + 2x - 11}{x + 3}$

**9.** $\lim\limits_{x \to 2} (26 - 3x)$

**10.** $\lim\limits_{x \to -6} \dfrac{x^4 - x^3}{x^2}$

**Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.** (Example 2)

**11.** $\lim\limits_{x \to 16} \dfrac{x^2 + 9}{\sqrt{x} - 4}$

**12.** $\lim\limits_{x \to 2} 4x^3 - 3x^2 + 10$

**13.** $\lim\limits_{x \to 3} \dfrac{x^3 + 9x + 6}{x^2 + 5x + 6}$

**14.** $\lim\limits_{x \to 3} \sqrt{2 - x}$

**15.** $\lim\limits_{x \to -4} \dfrac{5x^5 - 16x^4}{x + 5}$

**16.** $\lim\limits_{x \to 4} \dfrac{x + 4}{x - 4}$

**17.** $\lim\limits_{x \to 5} \dfrac{x^3}{\sqrt{x + 4} - 5}$

**18.** $\lim\limits_{x \to 9} (3x^2 - 10x + 35)$

**19.** $\lim\limits_{x \to 5} \dfrac{2x + 11}{x^2 - x - 20}$

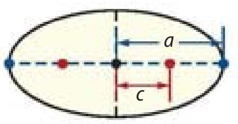
**20.** $\lim\limits_{x \to 1} (-x^2 + 3x + \sqrt{x})$

**21.** **PHYSICS** According to the special theory of relativity developed by Albert Einstein, the mass of an object traveling at speed $v$ is given by $m = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$, where $c$ is the speed of light and $m_0$ is the initial, or *rest mass*, of the object. (Example 2)

**a.** Find $\lim\limits_{v \to 0} m$. Explain the relationship between this limit and $m_0$.

**b.** What happens to the mass of an object if its speed were able to approach the speed of light?

**22.** **GEOMETRY** The area of an ellipse is defined as $A = \pi a\sqrt{a^2 - c^2}$, where $a$ is the distance from the vertices to the center and $c$ is the distance from the foci to the center. (Example 2)




**a.** What is the area of an ellipse for $a = 5$ and $c = 3$?

**b.** What happens to the eccentricity of an ellipse as the foci move closer to the center of the ellipse?

**c.** What is the limit of the area of the ellipse as $c$ approaches 0 in terms of $a$?

**Evaluate each limit.** (Examples 3 and 4)

**23.** $\lim\limits_{x \to 4} \dfrac{2x^2 - 5x - 12}{x - 4}$

**24.** $\lim\limits_{x \to 0} \dfrac{4x}{\sqrt{x + 1} - 1}$

**25.** $\lim\limits_{x \to 1} \dfrac{x^2 + 4x - 5}{x^2 - 1}$

**26.** $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

**27.** $\lim\limits_{x \to -5} \dfrac{4x^2 + 21x + 5}{3x^2 + 17x + 10}$

**28.** $\lim\limits_{x \to 7} \dfrac{5 - \sqrt{18 + x}}{x - 7}$

**29.** $\lim\limits_{x \to -2} \dfrac{x + 2}{\sqrt{6 + x} - 2}$

**30.** $\lim\limits_{x \to \frac{1}{2}} \dfrac{8x^2 + 2x - 3}{12x^2 + 8x - 7}$

**31.** $\lim\limits_{x \to 0} \dfrac{2x}{3 - \sqrt{x + 9}}$

**32.** $\lim\limits_{x \to -3} \dfrac{x^2 - 2x - 15}{x + 3}$

**33.** $\lim\limits_{x \to 6} \dfrac{\sqrt{x + 3} - 3}{x - 6}$

**34.** $\lim\limits_{x \to 0} \dfrac{\sqrt{16 + x} - 4}{x}$

**Evaluate each limit.** (Examples 5 and 6)

**35.** $\lim\limits_{x \to \infty} (5 - 2x^2 + 7x^3)$

**36.** $\lim\limits_{x \to \infty} \dfrac{3x^3 - 10x + 2}{4x^3 + 20x^2}$

**37.** $\lim\limits_{x \to \infty} \dfrac{2x^2 + 7x - 17}{3x^5 + 4x^2 + 2}$

**38.** $\lim\limits_{x \to \infty} (10x + 14 + 6x^2 - x^4)$

**39.** $\lim\limits_{x \to \infty} \dfrac{x^6 + 12x}{3x^6 + 2x^2 + 11x}$

**40.** $\lim\limits_{x \to \infty} \dfrac{14x^3 - 12x}{4x^2 + 13x - 8}$

**41.** $\lim\limits_{x \to \infty} (7x^3 + 4x^4 + x)$

**42.** $\lim\limits_{x \to \infty} \dfrac{6x^5 - 12x^2 + 14x}{2x^5 + 13x^3}$

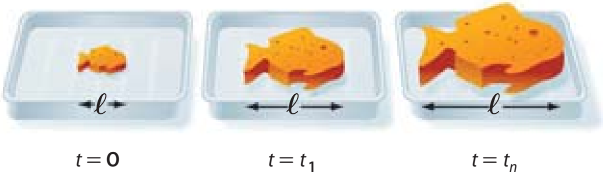
**43.** $\lim\limits_{x \to \infty} (x^3 - 6x^7 + 2x^6)$

**44.** $\lim\limits_{x \to \infty} \dfrac{6x^3 + 2x - 11}{-x^5 + 17x^3 + 4x}$

**45.** $\lim\limits_{x \to \infty} \dfrac{10x^4 - 2}{5x^4 + 3x^3 - 2x}$

**46.** $\lim\limits_{x \to -\infty} (2x^5 - 4x^2 + 10x - 8)$

**47.** **SPONGE** A gel capsule contains a sponge animal. When the capsule is submerged in water, it immediately dissolves, allowing the sponge to absorb water and quickly grow in size. The length $\ell$ in millimete rs of the sponge animal after being submerged in water for $t$ seconds can be defined as $\ell(t) = \dfrac{105t^2}{10 + t^2} + 25$. (Example 6)




**a.** What is the length of the capsule before it is submerged in water?

**b.** What is the limit of this function as $t \to \infty$?

**c.** Explain how the limit of this function relates to the length of the sponge animal.

**48.** **PUPPIES** Suppose the weight $w$ in pounds of a puppy $d$ days after birth can be estimated by $w(d) = \dfrac{50}{2 + 98(0.85)^d}$. (Example 6)

**a.** What is the weight of the puppy at birth?

**b.** How much will the puppy eventually weigh (that is, the weight as $d \to \infty$)?

**Find the limit of each sequence, if it exists.** (Example 7)

**49.** $a_n = \dfrac{n^3 - 2}{n^2}$

**50.** $a_n = \dfrac{8n + 1}{n^2 - 3}$

**51.** $a_n = \dfrac{-4n^2 + 6n - 1}{n^2 + 3n}$

**52.** $a_n = \dfrac{4 - 3n}{2n^3 + 5}$

**53.** $a_n = \dfrac{12n^2 + 2}{6n^2 - 1}$

**54.** $a_n = \dfrac{8n^2 + 5n + 2}{3 + 2n}$

**55.** $a_n = \dfrac{5}{n^2}\left[\dfrac{n(n+1)}{2}\right]$

**56.** $a_n = \dfrac{3}{n^3}\left[\dfrac{n(2n+1)(n+1)}{6}\right]$

**57.** $a_n = \dfrac{1}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right]$

**58.** $a_n = \dfrac{12}{n^2}\left[\dfrac{n(2n+1)(n+1)}{6}\right]$

**59. POPULATION** After being ranked one of the best cities in which to live by a national publication, Northfield Heights experienced a rise in population that could be modeled by $p(t) = \dfrac{36t^3 - 12t + 13}{3t^3 + 90}$, where $p$ is the total rise in population in thousands and $t$ is the number of years after 2006. (Example 7)

| Years Since 2006 | Rise in Population |
|---|---|
| 1 | ? |
| 2 | ? |
| 3 | ? |

**a.** Complete the table for 2007–2009.

**b.** What was the total rise in population by 2011?

**c.** What is the limit of the population growth?

**d.** Explain why a city's population growth may have a limit.

**Find each limit, if it exists, by using direct substitution to evaluate the corresponding one-sided limits.**

**60.** $\lim\limits_{x \to -2} \begin{cases} x - 3 & \text{if } x \le -2 \\ 2x - 1 & \text{if } x > -2 \end{cases}$

**61.** $\lim\limits_{x \to 0} \begin{cases} 4x + 2 & \text{if } x \le 0 \\ 2 - x^2 & \text{if } x > 0 \end{cases}$

**62.** $\lim\limits_{x \to 0} \begin{cases} 5 - x^2 & \text{if } x \le 0 \\ 5 - x & \text{if } x > 0 \end{cases}$

**63.** $\lim\limits_{x \to 2} \begin{cases} (x-2)^2 + 1 & \text{if } x \le 2 \\ x - 6 & \text{if } x > 2 \end{cases}$

**Find each limit, if it exists, using any method.**

**64.** $\lim\limits_{x \to \pi} \dfrac{\sin x}{x}$

**65.** $\lim\limits_{x \to 0} (1 + x + 2^x - \cos x)$

**66.** $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$

**67.** $\lim\limits_{x \to 0} \dfrac{3x - \sin 3x}{x^2 \sin x}$

**68.** $\lim\limits_{x \to 1} \dfrac{\ln x}{\ln (2x - 1)}$

**69.** $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{x - 1}$

**70. BIOLOGY** Suppose the width in millimeters of an animal's pupil is given by $d(x) = \dfrac{152x^{-0.45} + 85}{4x^{-0.45} + 10}$, where $x$ is the illuminance of the light shining on the pupils measured in lux.

**a.** Write a limit to describe the width of the animal's pupils when the light is at its minimum illuminance. Then find the limit, and interpret your results.

**b.** Write a limit to describe the width of the animal's pupils when the light is at its maximum illuminance. Then find the limit, and interpret your results.

**Find $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ for each function.**

**71.** $f(x) = 2x - 1$

**72.** $f(x) = 7 - 9x$

**73.** $f(x) = \sqrt{x}$

**74.** $f(x) = \sqrt{x + 1}$

**75.** $f(x) = x^2$

**76.** $f(x) = x^2 + 8x + 4$

**77. PHYSICS** An object that is in motion possesses an energy of motion called *kinetic energy* because it can do work when it impacts another object. The kinetic energy of an object with mass $m$ is given by $k(t) = \frac{1}{2}m \cdot [v(t)]^2$, where $v(t)$ is the velocity of the object at time $t$ and mass is given in kilograms. Suppose $v(t) = \dfrac{50}{1 + t^2}$ for all $t \ge 0$. What value does the kinetic energy of an object that has a mass of one kilogram approach as time approaches 100?

## H.O.T. Problems Use Higher-Order Thinking Skills

**78. PROOF** Use the properties of limits to show that for any $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ and a real number $c$, $\lim\limits_{x \to c} p(x) = p(c)$.

**79. PROOF** Use mathematical induction to show that if $\lim\limits_{x \to c} f(x) = L$, then $\lim\limits_{x \to c} [f(x)]^n = [\lim\limits_{x \to c} f(x)]^n$ or $L^n$ for any integer $n$.

**80. CHALLENGE** Find

$$\lim_{x \to \infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{b_n x^m + b_{n-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0},$$

where $a_n \ne 0$ and $b_m \ne 0$. (*Hint:* Consider the cases where $m < n$, $n = m$, and $m > n$.)

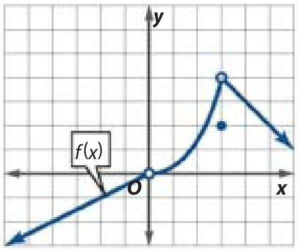
**81. REASONING** If $r(x)$ is a rational function, is it *sometimes*, *always*, or *never* true that $\lim\limits_{x \to c} r(x) = r(c)$? Explain your reasoning.

**82. WRITING IN MATH** Use a spreadsheet or a table to summarize the properties of limits. Give an example of each.

**83. WRITING IN MATH** Consider $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{\infty}{\infty}$. Susan says that this answer means the limit is 1. Why is she incorrect? What further analysis could be used to determine the limit, if it exists?

## Spiral Review

**Use the graph of $y = f(x)$ to find each value.** (Lesson 12-1)




**84.** $\lim_{x \to -2} f(x)$ and $f(-2)$

**85.** $\lim_{x \to 0} f(x)$ and $f(0)$

**86.** $\lim_{x \to 3} f(x)$ and $f(3)$

**87. HEALTH** The table shows the average life expectancy for people born in various years in the United States. (Lesson 11-7)

**a.** Make a scatter plot of the data, and identify the relationship.

**b.** Calculate and interpret the correlation coefficient. Determine whether it is significant at the 5% level.

**c.** If the correlation is significant at the 5% level, find the least-squares regression equation and interpret the slope and intercept in context.

**d.** Use the regression equation that you found in part **c** to predict the average life expectancy for 2080. State whether this prediction is reasonable. Explain.

| Years Since 1900 | Life Expectancy |
|---|---|
| 10 | 50 |
| 20 | 54.1 |
| 30 | 59.7 |
| 40 | 62.9 |
| 50 | 68.2 |
| 60 | 69.7 |
| 70 | 70.8 |
| 80 | 73.7 |
| 90 | 75.4 |
| 100 | 76.9 |

**88. ACOUSTICS** Polar coordinates can be used to model the shape of a concert amphitheater. Suppose the performer is placed at the pole and faces the direction of the polar axis. The seats have been built to occupy the region with $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ and $0.25 \le r \le 3$, where $r$ is measured in hundreds of feet. (Lesson 9-1)

**a.** Sketch this region in the polar plane.

**b.** How many seats are there if each person has 6 square feet of space?

**89.** Write the pair of parametric equations, $x = 2 \sin t$ and $y = 5 \cos t$, in rectangular form. Then sketch the graph. (Lesson 7-5)

## Skills Review for Standardized Tests

**90. SAT/ACT** According to the data in the table, by what percent did the number of applicants to Green College increase from 1995 to 2000?

| Number of Applicants to Green College | |
|---|---|
| **Year** | **Applicants** |
| 1990 | 18,000 |
| 1995 | 20,000 |
| 2000 | 24,000 |
| 2005 | 25,000 |

**A** 15% **B** 20% **C** 25% **D** 27% **E** 29%

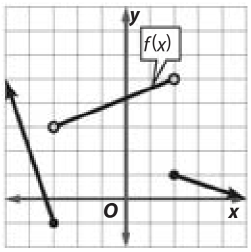
**91. REVIEW** What is $\lim_{h \to 0} \frac{2h^3 - h^2 + 5h}{h}$?

**F** 3 **G** 4 **H** 5 **J** The limit does not exist.

**92.** What value does $g(x) = \frac{x + \pi}{\cos(x + \pi)}$ approach as $x$ approaches 0?

**A** $-\pi$ **B** $-\frac{3}{4}$ **C** $-\frac{1}{2}\pi$ **D** 0

**93. REVIEW** Consider the graph of $y = f(x)$ shown. What is the $\lim_{x \to 2^+} f(x)$?




**F** 0 **G** 1 **H** 5 **J** The limit does not exist.

EXPLORE 12-3

# Graphing Technology Lab
# The Slope of a Curve

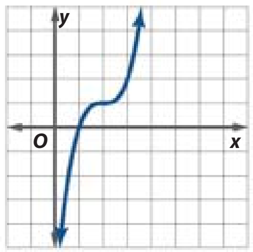
## Objective

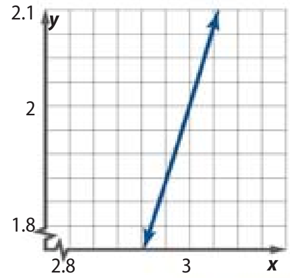
- Use TI-Nspire technology to estimate the slope of a curve

The slope of a line as a constant rate of change is a familiar concept. General curves do not have a constant rate of change because the slope is different at every point on the graph.

However, the graphs of most functions are *locally linear*. That is, if you examine the graph of a function on a very small interval, it will appear linear.

By looking at successive secant lines, it is possible to apply slope to curves.



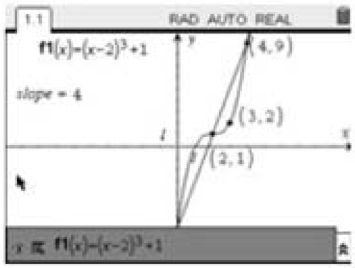




## Activity Secant Lines

**Estimate the slope of the graph of $y = (x - 2)^3 + 1$ at (3, 2).**

**Step 1** Enter $y = (x - 2)^3 + 1$ in f1. Then calculate the slope of the line secant to the graph of $y = (x - 2)^3 + 1$ through $x = 2$ and $x = 4$.

The slope of the secant line is 4.

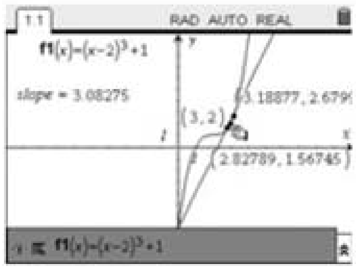



**Step 2** Find the slope of the line secant to the graph of $y = (x - 2)^3 + 1$ through $x = 2.5$ and $x = 3.5$.

The slope of the secant line is 3.25.

**Step 3** Find the slope of the line secant to the graph of $y = (x - 2)^3 + 1$ through $x = 2.8$ and $x = 3.2$.

The slope of the secant line is 3.04.




**Step 4** Find the slope of 3 more secant lines on decreasing intervals around (3, 2).

As the interval around (3, 2) decreases, the slope of the secant line approaches 3. So, the slope of $y = (x - 2)^3 + 1$ at (3, 2) is about 3.

## Exercises

**Estimate the slope of each function at the given point.**

1. $y = (x + 1)^2$; $(-4, 9)$
2. $y = x^3 - 5$; (2, 3)
3. $y = 4x^4 - x^2$; (0.5, 0)
4. $y = \sqrt{x}$; (1, 1)

## Analyze the Results

5. **ANALYZE** Describe the change to a line secant to the graph of a function as the points of intersection approach a given point $(a, b)$.

6. **MAKE A CONJECTURE** Describe how you could determine the exact slope of a curve at a given point.

LESSON 12-3

# Tangent Lines and Velocity

**Then**

- You found average rates of change using secant lines. (Lesson 1-4)

**Now**

- **1** Find instantaneous rates of change by calculating slopes of tangent lines.
- **2** Find average and instantaneous velocity.

**Why?**

- When a skydiver exits a plane, gravity causes the speed of his or her fall to increase. For this reason, the velocity of the sky diver at each instant before terminal velocity is achieved or the parachute is opened varies.

**NewVocabulary**

tangent line
instantaneous rate of change
difference quotient
instantaneous velocity

## 1 Tangent Lines

In Lesson 1-4, you calculated the average rate of change between two points on the graph of a nonlinear function by finding the slope of the secant line through these points. In this lesson, we develop a way to find the slope of such functions at one instant or point on the graph.

The graphs below show successively better approximations of the slope of $y = x^2$ at (1, 1) using secant lines.

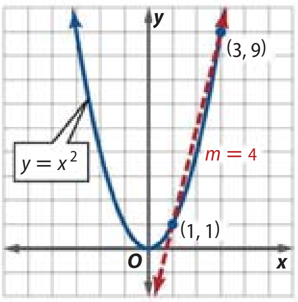



**Figure 12.3.1**

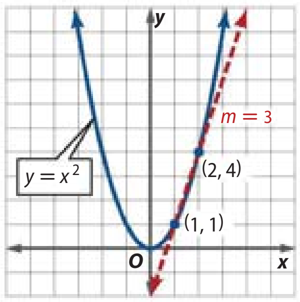



**Figure 12.3.2**

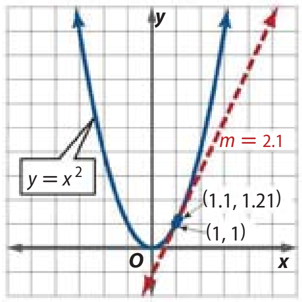



**Figure 12.3.3**

Notice as the rightmost point moves closer and closer to (1, 1), the secant line provides a better linear approximation of the curve near that point. We call the best of these linear approximations the **tangent line** to the graph at (1, 1). The slope of this line represents the rate of change in the slope of the curve at that instant. To define each of these terms more precisely, we use limits.

To define the slope of the tangent line to $y = f(x)$ at the point $(x, f(x))$, find the slope of the secant line through this point and one other point on the curve. Let the $x$-coordinate of the second point be $x + h$ for some small value of $h$. The corresponding $y$-coordinate for this point is then $f(x + h)$, as shown in Figure 12.3.4. The slope of the secant line through these two point is given by

$$m = \frac{f(x + h) - f(x)}{(x + h) - x} \text{ or } \frac{f(x + h) - f(x)}{h}.$$

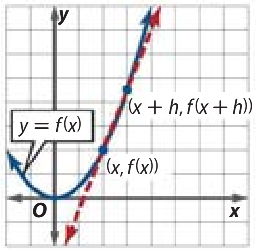



**Figure 12.3.4**

This expression is called the **difference quotient**.

As the second point approaches the first, or as $h \to 0$, the secant line approaches the tangent line at $(x, f(x))$. We define the slope of the tangent line at $x$, which represents the instantaneous rate of change of the function at that point, by finding the limits of the slopes of the secant lines as $h \to 0$.

**KeyConcept** Instantaneous Rate of Change

The instantaneous rate of change of the graph of $f(x)$ at the point $(x, f(x))$ is the slope $m$ of the tangent line at $(x, f(x))$ given by $m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided the limit exists.

You can use this expression to find the slope of the tangent line to a graph for a specified point.

**StudyTip**

**Instantaneous Rate of Change** When calculating the limit of the slopes of the secant lines as $h \to 0$, any term containing a value of $h$ that has not been divided out will become 0.

## Example 1 Slope of a Graph at a Point

**Find the slope of the line tangent to the graph of $y = x^2$ at (1, 1).**

$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ — Instantaneous Rate of Change Formula

$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ — $x = 1$

$= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h}$ — $f(1+h) = (1+h)^2$ and $f(1) = 1^2$

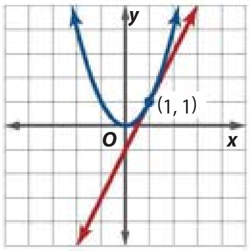
$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$ — Multiply.

$= \lim_{h \to 0} \frac{h(2+h)}{h}$ — Simplify and factor.

$= \lim_{h \to 0} (2 + h)$ — Divide by $h$.

$= 2 + 0$ or 2 — Sum Property of Limits and Limits of Constant and Identity Functions

The slope of the graph at (1, 1) is 2, as shown.




**GuidedPractice**

**Find the slope of the line tangent to the graph of each function at the given point.**

**1A.** $y = x^2$; (3, 9)

**1B.** $y = x^2 + 4$; (−2, 8)

The expression for instantaneous rate of change can also be used to find an equation for the slope of the tangent line to a graph at any point $x$.

## Example 2 Slope of a Graph at Any Point

**Find an equation for the slope of the graph of $y = \frac{4}{x}$ at any point.**

$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ — Instantaneous Rate of Change Formula

$= \lim_{h \to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h}$ — $f(x+h) = \frac{4}{x+h}$ and $f(x) = \frac{4}{x}$

$= \lim_{h \to 0} \frac{\frac{-4h}{x(x+h)}}{h}$ — Add fractions in the numerator and simplify.

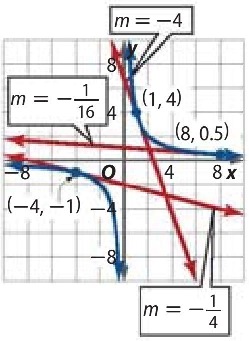
$= \lim_{h \to 0} \frac{-4h}{xh(x+h)}$ — Simplify.

$= \lim_{h \to 0} \frac{-4}{x^2 + xh}$ — Divide by $h$ and multiply.

$= \frac{-4}{x^2 + x(0)}$ — Quotient and Sum Properties of Limits and Limits of Constant and Identity Functions

$= \frac{-4}{x^2}$ — Simplify.

An equation for the slope of the graph at any point is $m = -\frac{4}{x^2}$, as shown.




**GuidedPractice**

**Find an equation for the slope of the graph $m$ of each function at any point.**

**2A.** $y = x^2 - 4x + 2$

**2B.** $y = x^3$

## 2 Instantaneous Velocity

In Lesson 1-4, you calculated the average speed of a dropped object by dividing the distance traveled by the time it took for the object to cover that distance. Velocity is speed with the added dimension of direction. You can calculate average velocity using the same approach that you used when calculating average speed.

### KeyConcept Average Velocity

If position is given as a function of time $f(t)$, for any two points in time $a$ and $b$, the average velocity $v$ is given by

$$v_{avg} = \frac{\text{change in distance}}{\text{change in time}} = \frac{f(b) - f(a)}{b - a}.$$

**Real-WorldLink**

Robert K. Cheruiyot of Kenya completed the 2008 Boston Marathon in less than two hours eight minutes. On average, he completed a mile every four minutes fifty seconds.

**Source:** Boston Athletic Association

### Real-World Example 3 Average Velocity of an Object

**MARATHON The distance in miles that a runner competing in the Boston Marathon has traveled after a certain time $t$ in hours can be found by $f(t) = -1.3t^2 + 12t$. What was the runner's average velocity between the second and third hour of the race?**

First, find the total distance traveled by the runner for $a = 2$ and $b = 3$.

| | | |
|---|---|---|
| $f(t) = -1.3t^2 + 12t$ | Original equation | $f(t) = -1.3t^2 + 12t$ |
| $f(2) = -1.3(2)^2 + 12(2)$ | $a = 2$ and $b = 3$ | $f(3) = -1.3(3)^2 + 12(3)$ |
| $f(2) = 18.8$ | Simplify. | $f(3) = 24.3$ |

Now use the formula for average velocity.

$v_{avg} = \frac{f(b) - f(a)}{b - a}$ — Average Velocity Formula

$= \frac{24.3 - 18.8}{3 - 2}$ — $f(b) = 24.3$, $f(a) = 18.8$, $b = 3$, and $a = 2$

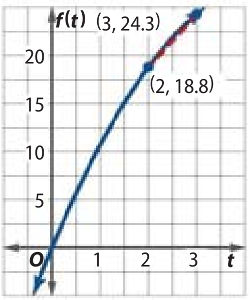
$= 5.5$ — Simplify.

The average velocity of the runner during the third hour was 5.5 miles per hour forward.

**GuidedPractice**

3. **WATER BALLOON** A water balloon is propelled straight up using a launcher. The height of the balloon in feet $t$ seconds after it is launched can be defined by $d(t) = 5 + 65t - 16t^2$. What was the balloon's average velocity between $t = 1$ and 2?

Looking more closely at Example 3, we can see that the velocity was found by calculating the slope of the secant line that connects the two points (2, 18.8) and (3, 24.3), as shown in the graph. The velocity that was calculated is the average velocity traveled by the runner over a period of time and does not represent the **instantaneous velocity**, the velocity or speed the runner achieved at a specific point in time.




To find the actual velocity of the runner at a specific time $t$, we find the instantaneous rate of change of the graph of $f(t)$ at $t$.

### KeyConcept Instantaneous Velocity

If the distance an object travels is given as a function of time $f(t)$, then the instantaneous velocity $v(t)$ at a time $t$ is given by

$$v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h},$$

provided the limit exists.

## Example 4 Instantaneous Velocity at a Point

**A baseball is dropped from the top of a building 2000 feet above the ground. The height of the baseball in feet after $t$ seconds is given by $f(t) = 2000 - 16t^2$. Find the instantaneous velocity $v(t)$ of the baseball at 5 seconds.**

To find the instantaneous velocity, let $t = 5$ and apply the formula for instantaneous velocity.

| | |
|---|---|
| $v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$ | Instantaneous Velocity Formula |
| $v(5) = \lim_{h \to 0} \frac{2000 - 16(5+h)^2 - [2000 - 16(5)^2]}{h}$ | $f(t+h) = 2000 - 16(5+h)^2$ and $f(t) = 2000 - 16(5)^2$ |
| $= \lim_{h \to 0} \frac{-160h - 16h^2}{h}$ | Multiply and simplify. |
| $= \lim_{h \to 0} \frac{h(-160 - 16h)}{h}$ | Factor. |
| $= \lim_{h \to 0} (-160 - 16h)$ | Divide by $h$. |
| $= -160 - 16(0)$ or $-160$ | Difference Property of Limits and Limits of Constant and Identity Functions |

The instantaneous velocity of the baseball at 5 seconds is 160 feet per second. The negative sign indicates that the height of the ball is decreasing.

**WatchOut!**

**Substitution** Remember to distribute the negative sign that precedes $f(t)$ to each term that is substituted.

### GuidedPractice

**4.** A window washer accidentally drops his lunch off his scaffold 1400 feet above the ground. The position of the lunch in relation to the ground is given as $d(t) = 1400 - 16t^2$, where time $t$ is given in seconds and the position of the lunch is given in feet. Find the instantaneous velocity $v(t)$ of the lunch at 7 seconds.

Equations for finding the instantaneous velocity of an object at any time $t$ can also be determined.

## Example 5 Instantaneous Velocity at Any Point

**The distance a particle moves along a path is given by $s(t) = 18t - 3t^3 - 1$, where $t$ is given in seconds and the distance of the particle from its starting point is given in centimeters. Find the equation for the instantaneous velocity $v(t)$ of the particle at any point in time.**

Apply the formula for instantaneous velocity.

| | |
|---|---|
| $v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$ | Instantaneous Velocity Formula |
| $= \lim_{h \to 0} \frac{18(t+h) - 3(t+h)^3 - 1 - [18t - 3t^3 - 1]}{h}$ | $s(t+h) = 18(t+h) - 3(t+h)^3 - 1$ and $s(t) = 18t - 3t^3 - 1$ |
| $= \lim_{h \to 0} \frac{18h - 9t^2h - 9th^2 - 3h^3}{h}$ | Multiply and simplify. |
| $= \lim_{h \to 0} \frac{h(18 - 9t^2 - 9th - 3h^2)}{h}$ | Factor. |
| $= \lim_{h \to 0} (18 - 9t^2 - 9th - 3h^2)$ | Divide by $h$. |
| $= 18 - 9t^2 - 9t(0) - 3(0)^2$ | Difference Property of Limits and Limits of Constant and Identity Functions |
| $= 18 - 9t^2$ | Simplify. |

The instantaneous velocity of the particle at time $t$ is $v(t) = 18 - 9t^2$.

### GuidedPractice

**5.** The distance in feet of a water rocket from the ground after $t$ seconds is given by $s(t) = 90t - 16t^2$. Find the expression for the instantaneous velocity $v(t)$ of the rocket at any time $t$.

## Exercises

= Step-by-Step Solutions begin on page R29.

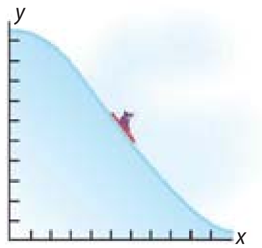
**Find the slope of the lines tangent to the graph of each function at the given points.** (Example 1)

1. $y = x^2 - 5x$; $(1, -4)$ and $(5, 0)$
2. $y = 6 - 3x$; $(-2, 12)$ and $(6, -12)$
3. $y = x^2 + 7$; $(3, 16)$ and $(6, 43)$
4. $y = \frac{3}{x}$; $(1, 3)$ and $(3, 1)$
5. $y = x^3 + 8$; $(-2, 0)$ and $(1, 9)$
6. $y = \frac{1}{x + 2}$; $(2, 0.25)$ and $(-1, 1)$

**Find an equation for the slope of the graph of each function at any point.** (Example 2)

7. $y = 4 - 2x$
8. $y = -x^2 + 4x$
9. $y = x^2 + 3$
10. $y = x^3$
11. $y = 8 - x^2$
12. $y = 2x^2$
13. $y = -2x^3$
14. $y = x^2 + 2x - 3$
15. $y = \frac{1}{\sqrt{x}}$
16. $y = \frac{1}{x^2}$

17. **SLEDDING** A person's vertical position on a sledding hill after traveling a horizontal distance $x$ units away from the top of the hill is given by $y = 0.06x^3 - 1.08x^2 + 51.84$. (Example 2)




a. Find an equation for the hill's slope $m$ at any distance $x$.
b. Find the hill's slope for $x = 2$, 5, and 7.

**The position of an object in miles after $t$ minutes is given by $s(t)$. Find the average velocity of the object in miles per hour for the given interval of time. Remember to convert from minutes to hours.** (Example 3)

18. $s(t) = 0.4t^2 - \frac{1}{20}t^3$ for $3 \le t \le 5$
19. $s(t) = 1.08t - 30$ for $4 \le t \le 8$
20. $s(t) = 0.2t^2$ for $2 \le t \le 4$
21. $s(t) = 0.01t^3 - 0.01t^2$ for $4 \le t \le 7$
22. $s(t) = -0.5(t - 5)^2 + 3$ for $4 \le t \le 4.5$
23. $s(t) = 0.6t + 20$ for $3.8 \le t \le 5.7$

24. **TYPING** The number of words $w$ a person has typed after $t$ minutes is given by $w(t) = 10t^2 - \frac{1}{2}t^3$. (Example 3)
    a. What was the average number of words per minute the person typed between the 2nd and 4th minutes?
    b. What was the average number of words per minute the person typed between the 3rd and 7th minutes?

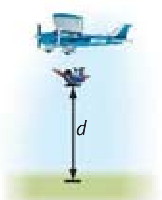
**The distance $d$ an object is above the ground $t$ seconds after it is dropped is given by $d(t)$. Find the instantaneous velocity of the object at the given value for $t$.** (Example 4)

25. $d(t) = 100 - 16t^2$; $t = 3$
26. $d(t) = 38t - 16t^2$; $t = 0.8$
27. $d(t) = -16t^2 - 47t + 300$; $t = 1.5$
28. $d(t) = 500 - 30t - 16t^2$; $t = 4$
29. $d(t) = -16t^2 - 400t + 1700$; $t = 3.5$
30. $d(t) = 150t - 16t^2$; $t = 2.7$
31. $d(t) = 1275 - 16t^2$; $t = 3.8$
32. $d(t) = 853 - 48t - 16t^2$; $t = 1.3$

**Find an equation for the instantaneous velocity $v(t)$ if the path of an object is defined as $s(t)$ for any point in time $t$.** (Example 5)

33. $s(t) = 14t^2 - 7$
34. $s(t) = t - 3t^2$
35. $s(t) = 5t + 8$
36. $s(t) = 18 - t^2 + 4t$
37. $s(t) = t^3 - t^2 + t$
38. $s(t) = 11t^2 - t$
39. $s(t) = \sqrt{t} - 3t^2$
40. $s(t) = 12t^2 - 2t^3$

41. **SKYDIVER** Refer to the beginning of the lesson. The position $d$ of the skydiver in feet relative to the ground can be defined by $d(t) = 15{,}000 - 16t^2$, where $t$ is seconds passed after the skydiver exited the plane. (Example 5)



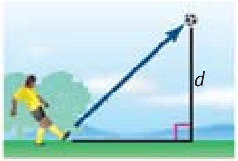

    a. What is the average velocity of the skydiver between the 2nd and 5th seconds of the jump?
    b. What was the instantaneous velocity of the sky diver at 2 and 5 seconds?
    c. Find an equation for the instantaneous velocity $v(t)$ of the skydiver.

42. **DIVING** A cliff diver's distance $d$ in meters above the surface of the water after $t$ seconds is given.

| t | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| d | 43.7 | 42.1 | 40.6 | 33.8 | 25.3 | 14.2 | 0.85 |

    a. Calculate the diver's average velocity for the interval $0.5 \le t \le 1.0$.
    b. Use quadratic regression to find an equation to model $d(t)$. Graph $d(t)$ and the data on the same coordinate plane.
    c. Find an expression for the instantaneous velocity $v(t)$ of the diver and use it to estimate the velocity of the diver at 3 seconds.

**43. SOCCER** A goal keeper can kick a ball at an upward velocity of 75 feet per second. Suppose the height $d$ of the ball in feet $t$ seconds after it is kicked is given by $d(t) = -16t^2 + 75t + 2.5$.




**a.** Find an equation for the instantaneous velocity $v(t)$ of the soccer ball.

**b.** How fast is the ball traveling 0.5 second after it is kicked?

**c.** If the instantaneous velocity of the ball is 0 when the ball reaches its maximum height, at what time will the ball reach its maximum height?

**d.** What is the maximum height of the ball?

**Find an equation for a line that is tangent to the graph of the function and perpendicular to the given line. Then use a graphing calculator to graph the function and both lines on the same coordinate plane.**

**44.** $f(x) = x^2 + 2x;\ y = -\frac{1}{2}x + 3$

**45.** $g(x) = -4x^2;\ y = \frac{1}{4}x + 5$

**46.** $f(x) = -\frac{1}{6}x^2;\ y - x = 2$

**47.** $g(x) = \frac{1}{2}x^2 + 4x;\ y = -\frac{1}{6}x + 9$

**48. PHYSICS** The distance $s$ of a particle moving in a straight line is given by $s(t) = 3t^3 + 8t + 4$, where $t$ is given in seconds and $s$ is measured in meters.

**a.** Find an equation for the instantaneous velocity $v(t)$ of the particle at any given point in time.

**b.** Find the velocity of the particle for $t = 2, 4,$ and 6 seconds.

**Each graph represents an equation for the slope of a function at any point. Match each graph with its original function.**

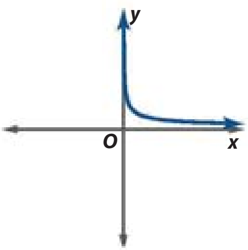
**49.** $f(x) = \frac{a}{x}$

**50.** $g(x) = ax^5$

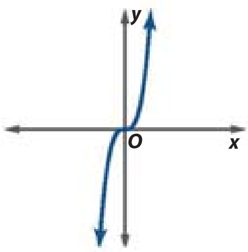
**51.** $h(x) = ax^4$

**52.** $j(x) = a\sqrt{x}$

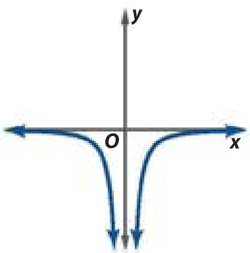
**a.**



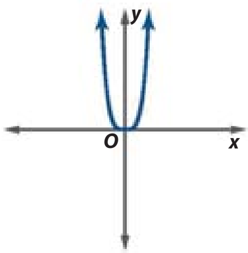

**b.**




**c.**




**d.**




**53. PROJECTILE** When an object is thrown straight down, the total distance $y$ the object falls can be modeled by $y = 16t^2 + v_0t$, where time $t$ is measured in seconds and the initial velocity $v_0$ is measured in feet per second.

**a.** If an object thrown straight down from a height of 816 feet takes 6 seconds to hit the ground, what was the initial velocity of the object?

**b.** What was the average velocity of the object?

**c.** What was the object's velocity when it hit the ground?

**54. MULTIPLE REPRESENTATIONS** In this problem, you will investigate the *Mean Value Theorem*. The theorem states that if a function $f$ is continuous and differentiable on $(a, b)$, then there exists a point $c$ in $(a, b)$ such that the tangent line is parallel to the line passing through $(a, f(a))$ and $(b, f(b))$.

**a. ANALYTICAL** Find the average rate of change for $f(x) = -x^2 + 8x$ on the interval $[1, 6]$, and find an equation for the related secant line through $(1, f(1))$ and $(6, f(6))$.

**b. ANALYTICAL** Find an equation for the slope of $f(x)$ at any point.

**c. ANALYTICAL** Find a point on the interval $(1, 6)$ at which the slope of the tangent line to $f(x)$ is equal to the slope of the secant line found in part **a**. Find an equation for the line tangent to $f(x)$ at this point.

**d. VERBAL** How are the secant line in part **a** and the tangent line in part **b** related? Explain.

**e. GRAPHICAL** Using a graphing calculator, graph $f(x)$, the secant line, and the tangent line on the same screen. Does the graph verify your answer in part **d**? Explain.

## H.O.T. Problems Use Higher-Order Thinking Skills

**55. ERROR ANALYSIS** Chase and Jillian were asked to find an equation for the slope at any point for $f(x) = |x|$. Chase thinks the graph of the slope equation will be continuous because the original function is continuous. Jillian disagrees. Is either of them correct? Explain your reasoning.

**56. CHALLENGE** Find an equation for the slope of $f(x) = 2x^4 + 3x^3 - 2x$ at any point.

**57. REASONING** *True* or *false*: The instantaneous velocity of an object modeled by $s(t) = at + b$ is always $a$.

**58. REASONING** Show that $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ for $f(x) = x^2 + 1$.

**59. WRITING IN MATH** Suppose that $f(t)$ represents the balance in dollars in a bank account $t$ years after the initial deposit. Interpret each of the following.

**a.** $\frac{f(4) - f(0)}{4} \approx 41.2$

**b.** $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \approx 42.9$

## Spiral Review

**Evaluate each limit.** (Lesson 12-2)

**60.** $\lim_{x \to 4} (x^2 + 2x - 2)$

**61.** $\lim_{x \to -1} (-x^4 + x^3 - 2x + 1)$

**62.** $\lim_{x \to 0} (x + \sin x)$

**63. HYDRAULICS** The velocity, in inches per second, of a molecule of liquid flowing through a pipe is given by $v(r) = k(R^2 - r^2)$, where $R$ is the radius of the pipe in inches, $r$ is the distance of the molecule from the center of the pipe in inches, and $k$ is a constant. Suppose for a particular liquid and a particular pipe that $k = 0.65$ and $R = 0.5$. (Lesson 12-1)

**a.** Graph $v(r)$.

**b.** Determine the limiting velocity of molecules closer and closer to the wall of the pipe.

**64. HEIGHT** The mean height of a sample of 100 high school seniors is 68 inches with a standard deviation of 4 inches. Determine the interval of heights such that the probability is 90% that the mean height of the entire population lies within that interval. (Lesson 11-5)

**65. EDUCATION** A college professor plans to grade a test on a curve. The mean score on the test is 65, and the standard deviation is 7. The professor wants the grades distributed as shown in the table. Assume that the grades are normally distributed. (Lesson 11-3)

**a.** What is the lowest score for an A?

**b.** If a D is the lowest passing letter grade, find the lowest passing score.

**c.** What is the interval for the Bs?

| Grade | Percent of Class |
|---|---|
| A | 15 |
| B | 20 |
| C | 30 |
| D | 20 |
| F | 15 |

**Find the specified $n$th term of each geometric sequence.** (Lesson 10-3)

**66.** $a_4 = 50, r = 2, n = 8$

**67.** $a_4 = 1, r = 3, n = 10$

**68.** $a_6$ for $a_n = \frac{1}{5}a_{n-1}, a_1 = -2$

**69.** $a_5$ for $a_n = (-3)a_{n-1}, a_1 = 11$

**Find the indicated arithmetic means for each set of nonconsecutive terms.** (Lesson 10-2)

**70.** 7 means; 62 and −2

**71.** 4 means; 17.2 and 47.7

**72.** 3 means; −5.6 and 8

**73.** 9 means; −45 and 115

## Skills Review for Standardized Tests

**74. SAT/ACT** If the radius of the circle with center $O$ is 4, what is the length of arc $RST$?

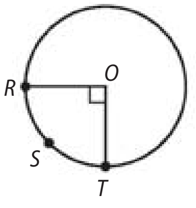
A $2\pi$

B $4\pi$

C $8\pi$

D $12\pi$

E $16\pi$




**75. REVIEW** Which of the following best describes the point at (0, 0) on $f(x) = 2x^5 - 5x^4$?

F absolute maximum

G relative maximum

H relative minimum

J absolute minimum

**76.** When a bowling ball is dropped, the distance $d(t)$ that it falls in $t$ seconds is given by $d(t) = 16t^2$. Its velocity after 2 seconds is given by $\lim_{h \to 0} \frac{d(2+h) - d(2)}{h}$. What is the velocity of the bowling ball after 2 seconds?

A 46 feet per second

B 58 feet per second

C 64 feet per second

D 72 feet per second

**77. REVIEW** The monthly profit $P$ of a manufacturing company depends on the number of units $x$ manufactured and can be described by $P(x) = \frac{1}{3}x^3 - 34x^2 + 1012x, 0 \leq x \leq 50$. How many units should be manufactured monthly in order to maximize profits?

F 15

G 22

H 37

J 46

CHAPTER 12

# Mid-Chapter Quiz

## Lessons 12-1 through 12-3

**Estimate each one-sided or two-sided limit, if it exists.** (Lesson 12-1)

**1.** $\lim_{x\to 0^+} \frac{\sin x}{x}$

**2.** $\lim_{x\to 0} \frac{|x|}{x}$

**3.** $\lim_{x\to 3^-} \frac{2x^2-18}{x-3}$

**4.** $\lim_{x\to 0^-} \frac{\cos x - 1}{x}$

**Estimate each limit, if it exists.** (Lesson 12-1)

**5.** $\lim_{x\to 3} \frac{2x}{x^2+1}$

**6.** $\lim_{x\to 1} \sqrt{x^3+3}$

**7.** $\lim_{x\to -2} e^{2x+3}$

**8.** $\lim_{x\to -4} \frac{\sqrt{x+20}}{x}$

**9.** **COLLECTIBLES** The value of Jorge's baseball card has been increasing every year. The value $v$ of the baseball card after $t$ years can be represented by the model $v(t) = \frac{400t-2}{2t+15}$. (Lesson 12-1)

**a.** Graph the function for $0 \le t \le 10$.

**b.** Use your graph to estimate the value of the baseball card for $t = 2$, 5, and 10 years.

**c.** Use your graph to evaluate $\lim_{t\to\infty} v(t)$.

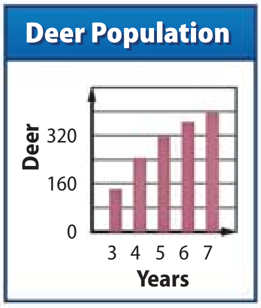
**d.** Explain the relationship between the limit of the function and the value of Jorge's baseball card.

**Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.** (Lesson 12-2)

**10.** $\lim_{x\to 9} \frac{x^2+1}{\sqrt{x}-3}$

**11.** $\lim_{x\to -2} (2x^3 + x^2 - 8)$

**12.** **WILDLIFE** A deer population $P$ in hundreds at a national park after $t$ years can be estimated by the model $P(t) = \frac{10t^3 - 40t + 2}{2t^3 + 14t + 12}$, where $t \ge 3$. The population for 5 years is shown below. What is the maximum number of deer that can live at the national park? (Lesson 12-2)




**Evaluate each limit.** (Lesson 12-2)

**13.** $\lim_{x\to\infty} (15 - x^2 + 8x^3)$

**14.** $\lim_{x\to\infty} \frac{2x^3 - x - 2}{4x^3 + 5x^2}$

**15.** $\lim_{x\to\infty} \frac{2x^2 + 5x - 1}{2x^4 - 14x^2 + 2}$

**16.** $\lim_{x\to\infty} (10x^3 - 4 + x^2 - 7x^4)$

**17.** **MULTIPLE CHOICE** Evaluate $\lim_{x\to 0} \frac{2x^2+5}{10 - e^{\frac{16}{x}}}$. (Lesson 12-1)

**A** does not exist

**B** $\frac{1}{2}$

**C** $\frac{1}{5}$

**D** $\frac{1}{10}$

**Find the slope of the line tangent to the graph of each function at the given points.** (Lesson 12-3)

**18.** $y = x^2 - 3x$; $(2, -2)$ and $(-1, 4)$

**19.** $y = 2 - 5x$; $(-2, 12)$ and $(3, -13)$

**20.** $y = x^3 - 4x^2$; $(1, -3)$ and $(3, -9)$

**21.** **FIREWORKS** Fireworks are launched with an upward velocity of 90 feet per second. Suppose the height $d$ of the fireworks in feet $t$ seconds after they are launched is defined as $d(t) = -16t^2 + 90t + 3.2$. (Lesson 12-3)

**a.** Find an equation for the instantaneous velocity $v(t)$ of the fireworks.

**b.** How fast are the fireworks traveling 0.5 seconds after they are lit?

**c.** What is the maximum height of the fireworks?

**22.** **MULTIPLE CHOICE** Find an equation for the slope of the graph of $y = 7x^2 - 2$ at any point. (Lesson 12-3)

**F** $m = 7x$

**G** $m = 7x - 2$

**H** $m = 14x$

**J** $m = 14x - 2$

**The position of an object in miles after $t$ minutes is given by $s(t)$. Find the average velocity of the object in miles per hour given two values of time $t$. Remember to convert from minutes to hours.** (Lesson 12-3)

**23.** $s(t) = 12 + 0.7t$ for $t = 2$ and 5

**24.** $s(t) = 2.05t - 11$ for $t = 1$ and 7

**25.** $s(t) = 0.9t - 25$ for $3 \le t \le 6$

**26.** $s(t) = 0.5t^2 - 4t$ for $4 \le t \le 8$

**Find an equation for the instantaneous velocity $v(t)$ if the position of an object is defined as $s(t)$ for any point in time $t$.** (Lesson 12-3)

**27.** $s(t) = 4t^2 - 9t$

**28.** $s(t) = 2t - 13t^2$

**29.** $s(t) = 2t - 5t^2$

**30.** $s(t) = 6t^2 - t^3$

# LESSON 12-4 Derivatives

**Then**

- You calculated the slope of tangent lines to find the instantaneous rate of change. (Lesson 12-3)

**Now**

1. Find instantaneous rates of change by calculating derivatives.
2. Use the Product and Quotient Rules to calculate derivatives.

**Why?**

- Zach, who lives on the sixth floor of an apartment building, accidentally drops a ball out of his window. Gabe, standing on the ground outside of Zach's building, retrieves the ball and attempts to throw it back up to Zach. If Gabe can throw the ball at a speed of 65 feet per second, can he get it to Zach's window 70 feet above the ground?

Exactostock/SuperStock

**NewVocabulary**
derivative
differentiation
differential equation
differential operator

## 1 Basic Rules

In Lesson 12-3, you used limits to determine the slope of a line tangent to the graph of a function at any point. This limit is called the derivative of a function. The **derivative** of $f(x)$ is $f'(x)$, which is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. The process of finding a derivative is called **differentiation** and the result is called a **differential equation**.

### Example 1 Derivative of a Function at Any Point

**Find the derivative of $f(x) = 4x^2 - 5x + 8$. Then evaluate the derivative at $x = 1$ and 5.**

| | |
|---|---|
| $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ | Definition of a derivative |
| $= \lim_{h \to 0} \frac{4(x+h)^2 - 5(x+h) + 8 - (4x^2 - 5x + 8)}{h}$ | $f(x+h) = 4(x+h)^2 - 5(x+h) + 8$ and $f(x) = 4x^2 - 5x + 8$ |
| $= \lim_{h \to 0} \frac{8xh + 4h^2 - 5h}{h}$ | Expand and simplify. |
| $= \lim_{h \to 0} \frac{h(8x + 4h - 5)}{h}$ | Factor. |
| $= \lim_{h \to 0} (8x + 4h - 5)$ | Divide by $h$. |
| $= 8x + 4(0) - 5$ or $8x - 5$ | Sum and Difference Properties of Limits and Limits of Constant and Identity Functions |

The derivative of $f(x)$ is $f'(x) = 8x - 5$. Evaluate $f'(x)$ for $x = 1$ and 5.

| | | |
|---|---|---|
| $f'(x) = 8x - 5$ | Original equation | $f'(x) = 8x - 5$ |
| $f'(1) = 8(1) - 5$ | $x = 1$ and 5 | $f'(5) = 8(5) - 5$ |
| $f'(1) = 3$ | Simplify. | $f'(5) = 35$ |

**GuidedPractice**

**Find the derivative of $f(x)$. Then evaluate the derivative for the given values of $x$.**

**1A.** $f(x) = 6x^2 + 7$; $x = 2$ and 5

**1B.** $f(x) = -5x^2 + 2x - 12$; $x = 1$ and 4

The derivative of the function $y = f(x)$ may also be denoted $y'$, $\frac{df}{dx}$, or $\frac{dy}{dx}$. If a function is preceded by a **differential operator** $\frac{d}{dx}$, then you are to take the derivative of the function.

To this point, you have had to evaluate limits as they approach 0 in order to calculate derivatives, slopes of tangent lines, and instantaneous velocity. A very helpful rule for simplifying this process and for reducing calculation errors is the Power Rule. It allows for the evaluation of derivatives without the need to calculate limits.

**ReadingMath**

**Derivatives** The notation for a derivative $f'(x)$ is read *f prime of x* or *the derivative of f with respect to x.*

## KeyConcept Power Rule for Derivatives

**Words** The power of $x$ in the derivative is one less than the power of $x$ in the original function, and the coefficient of the power of $x$ in the derivative is the same as the power of $x$ in the original function.

**Symbols** If $f(x) = x^n$ and $n$ is a real number, then $f'(x) = nx^{n-1}$.

## Example 2 Power Rule for Derivatives

**Find the derivative of each function.**

**a.** $f(x) = x^9$

$f(x) = x^9$ — Original equation

$f'(x) = 9x^{9-1}$ — Power Rule

$= 9x^8$ — Simplify.

**b.** $g(x) = \sqrt[5]{x^7}$

$g(x) = \sqrt[5]{x^7}$ — Original equation

$g(x) = x^{\frac{7}{5}}$ — Rewrite using a rational exponent.

$g'(x) = \frac{7}{5}x^{\frac{7}{5}-1}$ — Power Rule

$= \frac{7}{5}x^{\frac{2}{5}}$ or $\frac{7}{5}\sqrt[5]{x^2}$ — Simplify.

**c.** $h(x) = \frac{1}{x^8}$

$h(x) = \frac{1}{x^8}$ — Original equation

$h(x) = x^{-8}$ — Rewrite using a negative exponent.

$h'(x) = -8x^{-8-1}$ — Power Rule

$= -8x^{-9}$ or $-\frac{8}{x^9}$ — Simplify.

**WatchOut!**

**Negative Derivatives** The derivative of $f(x) = x^{-4}$ is not $f'(x) = -4x^{-3}$. Remember that 1 must be *subtracted* from the exponent and that $-4 - 1 = -4 + (-1)$ or $-5$. Therefore $f'(x) = -4x^{-5}$.

**GuidedPractice**

**2A.** $j(x) = x^4$ **2B.** $k(x) = \sqrt{x^3}$ **2C.** $m(x) = \frac{1}{x^5}$

There are several other rules of derivatives that are useful when finding the derivatives of functions that contain several terms.

## KeyConcept Other Derivative Rules

**Constant** The derivative of a constant function is zero. That is, if $f(x) = c$, then $f'(x) = 0$.

**Constant Multiple of a Power** If $f(x) = cx^n$, where $c$ is a constant and $n$ is a real number, then $f'(x) = cnx^{n-1}$.

**Sum or Difference** If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$.

## Example 3 Derivative Rules

**Find the derivative of each function.**

**a.** $f(x) = 5x^3 + 4$

$f(x) = 5x^3 + 4$ — Original equation

$f'(x) = 5 \cdot 3x^{3-1} + 0$ — Constant, Constant Multiple of a Power, and Sum Rules

$= 15x^2$ — Simplify.

**StudyTip**

**Derivatives** If $f(x) = x$, then $f'(x) = 1$, and if $f(x) = cx$, then $f'(x) = c$.

**b.** $g(x) = x^5(2x^3 + 4)$

$g(x) = x^5(2x^3 + 4)$ — Original equation

$g(x) = 2x^8 + 4x^5$ — Distributive Property

$g'(x) = 2 \cdot 8x^{8-1} + 4 \cdot 5x^{5-1}$ — Constant Multiple of a Power and Sum Rules

$= 16x^7 + 20x^4$ — Simplify.

**c.** $h(x) = \dfrac{5x^3 - 12x + 6\sqrt{x^5}}{x}$

$h(x) = \dfrac{5x^3 - 12x + 6\sqrt{x^5}}{x}$ — Original equation

$h(x) = \dfrac{5x^3}{x} - \dfrac{12x}{x} + \dfrac{6\sqrt{x^5}}{x}$ — Divide each term in the numerator by $x$.

$h(x) = 5x^2 - 12 + 6x^{\frac{3}{2}}$ — $x^{\frac{5}{2}} \cdot x^{-1} = x^{\frac{3}{2}}$

$h'(x) = 5 \cdot 2x^{2-1} + 0 + 6 \cdot \frac{3}{2}x^{\frac{3}{2}-1}$ — Constant, Constant Multiple of a Power, and Sum and Difference Rules

$= 10x + 9x^{\frac{1}{2}}$ or $10x + 9\sqrt{x}$ — Simplify.

**GuidedPractice**

**3A.** $f(x) = 2x^5 - x^3 - 102$ **3B.** $g(x) = 3x^4(x + 2)$ **3C.** $h(x) = \dfrac{4x^4 - 3x^2 + 5x}{x}$

Now that you are familiar with the basic rules of derivatives, problems involving the slopes of tangent lines and instantaneous velocity can be calculated in just a few steps. Example 5 in Lesson 12-3 involved finding an expression for the instantaneous velocity of a particle. Notice the simplification of the problem as a result of the derivative rules.

## Example 4 Instantaneous Velocity

**The distance a particle moves along a path is defined by $s(t) = 18t - 3t^3 - 1$, where $t$ is given in seconds and the distance of the particle is given in centimeters. Find the expression for the instantaneous velocity $v(t)$ of the particle.**

The instantaneous velocity $v(t)$ is equivalent to $s'(t)$.

$s(t) = 18t - 3t^3 - 1$ — Original equation

$s'(t) = 18 \cdot 1t^{1-1} - 3 \cdot 3t^{3-1} - 0$ — Constant, Constant Multiple of a Power, and Difference Rules

$= 18 - 9t^2$ — Simplify.

The instantaneous velocity is $v(t) = 18 - 9t^2$. Notice that this is the same result as found in Example 5 of Lesson 12-3.

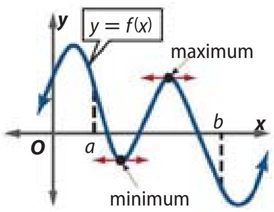
**GuidedPractice**

**4.** A soccer ball is kicked straight up. The height of the ball is defined by $h(t) = 55t - 16t^2$, where time $t$ is given in seconds and the height of the ball is given in feet. Find the expression for the instantaneous velocity $v(t)$ of the ball at any point in time.

In Lesson 1-4, you found relative and absolute extrema of functions graphically and numerically. On a closed interval, these values can be found using the derivative and the following theorem.

## KeyConcept Extreme Value Theorem

If a function $f$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains both a maximum and a minimum on $[a, b]$.




Relative extrema occur only at *critical points* where the slope of the tangent line, the derivative of the function, is 0 or undefined. To locate the maximum and minimum of a polynomial function $f(x)$ on $[a, b]$, evaluate the function at $a$, $b$, and at any values $x$ in $[a, b]$ for which $f'(x) = 0$.

**Real-WorldLink**

Roller coasters have recently achieved speeds that exceed 120 mph and heights over 450 feet.

**Source:** Guinness World Records

## Real-World Example 5 Maximum and Minimum

**ROLLER COASTER The height $h$ in feet of a car travelling along the track of a roller coaster can be modeled by $h(t) = -\frac{1}{3}t^3 + 4t^2 + \frac{11}{3}$ on the interval [1, 12], where time $t$ is given in seconds. Find the maximum and minimum heights of the car.**

Find the derivative of $h(t)$.

$h(t) = -\frac{1}{3}t^3 + 4t^2 + \frac{11}{3}$ — Original equation

$h'(t) = -\frac{1}{3} \cdot 3t^{3-1} + 4 \cdot 2t^{2-1} + 0$ — Constant, Constant Multiple of a Power, and Sum and Difference Rules

$= -t^2 + 8t$ — Simplify.

Solve $h'(t) = 0$ to find where the critical points of $h(x)$ occur.

$-t^2 + 8t = 0$ — $h'(t) = -t^2 + 8t$

$-t(t - 8) = 0$ — Factor.

Critical points for this function occur when $t = 0$ and 8. Note that although $t = 0$ is a critical point of the function $h(t)$, it does not lie on the interval [1, 12]. To find the maximum and minimum of the function on [1, 12] evaluate $h(t)$ for 1, 8, and 12.

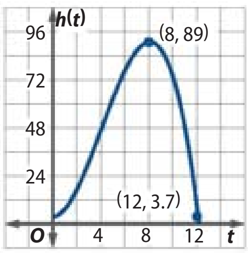
$h(1) = -\frac{1}{3}(1)^3 + 4(1)^2 + \frac{11}{3}$ or 7.33

$h(8) = -\frac{1}{3}(8)^3 + 4(8)^2 + \frac{11}{3}$ or 89 — maximum

$h(12) = -\frac{1}{3}(12)^3 + 4(12)^2 + \frac{11}{3}$ or 3.67 — minimum

The car will achieve a maximum height of 89 feet 8 seconds into the ride and a minimum height of about 3.7 feet 12 seconds into the ride.

**CHECK** The graph of $h(t) = -\frac{1}{3}t^3 + 4t^2 + \frac{11}{3}$ shows that $h(t)$ has a maximum of 89 at $x = 8$ and a minimum of about 3.7 at $x = 12$ on the interval [1, 12]. ✔




**WatchOut!**

**Interpreting Graphs** The graph in Example 5 shows the height of the car over time. It does *not* show the shape of the roller coaster.

### GuidedPractice

5. **BUNGEE JUMPING** A bungee jumper's height $h$ in feet relative to the ground can be modeled by $h(t) = 20t^2 - 160t + 330$ on the interval [0, 6], where time $t$ is given in seconds. Find the maximum and minimum heights of the jumper.

Ethel Davies/Photolibrary

## 2 Product and Quotient Rules

Earlier, you learned that the derivative of the sum of functions is equal to the sum of the individual derivatives. Is the derivative of a product of functions equal to the product of the derivatives? Consider the functions $f(x) = x$ and $g(x) = 3x^3$.

**Derivative of Product**

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[x \cdot 3x^3] = \frac{d}{dx}(3x^4) = 12x^3$$

**Product of Derivatives**

$$\frac{d}{dx}f(x) \cdot \frac{d}{dx}g(x) = \frac{d}{dx}(x) \cdot \frac{d}{dx}(3x^3) = 1 \cdot 9x^2 = 9x^2$$

It is clear that the derivative of the product is not necessarily the product of the derivatives. The following rule can be applied when calculating the derivative of products.

### KeyConcept Product Rule for Derivatives

If $f$ and $g$ are differentiable at $x$, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

You will prove the Product Rule for Derivatives in Exercise 64.

### Example 6 Product Rule

**Find the derivative of each product.**

**a.** $\mathbf{h(x) = (x^3 - 2x + 7)(3x^2 - 5)}$

Let $f(x) = x^3 - 2x + 7$ and $g(x) = 3x^2 - 5$. So, $h(x) = f(x)g(x)$.

$f(x) = x^3 - 2x + 7$ — Original equation

$f'(x) = 3x^2 - 2$ — Power, Constant Multiple of a Power, Constant, and Sum and Difference Rules

$g(x) = 3x^2 - 5$ — Original equation

$g'(x) = 6x$ — Constant Multiple of a Power, Constant, and Difference Rules

Use $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ to find the derivative of $h(x)$.

$h'(x) = f'(x)g(x) + f(x)g'(x)$ — Product Rule

$= (3x^2 - 2)(3x^2 - 5) + (x^3 - 2x + 7)(6x)$ — Substitution

$= 15x^4 - 33x^2 + 42x + 10$ — Distribute and simplify.

> **StudyTip**
> **Product Rule** The Product Rule results in an answer that can still be simplified. Unless there is an easy simplification to make or a reason to do so, you may leave the answer as is.

**b.** $\mathbf{h(x) = (x^3 - 4x^2 + 48x - 64)(6x^2 - x - 2)}$

Let $f(x) = x^3 - 4x^2 + 48x - 64$ and $g(x) = 6x^2 - x - 2$.

$f(x) = x^3 - 4x^2 + 48x - 64$ — Original equation

$f'(x) = 3x^2 - 8x + 48$ — Power, Constant Multiple of a Power, Constant, and Sum and Difference Rules

$g(x) = 6x^2 - x - 2$ — Original equation

$g'(x) = 12x - 1$ — Constant Multiple of a Power, Power, Constant, and Difference Rules

Use $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ to find the derivative of $h(x)$.

$h'(x) = f'(x)g(x) + f(x)g'(x)$ — Product Rule

$= (3x^2 - 8x + 48)(6x^2 - x - 2) + (x^3 - 4x^2 + 48x - 64)(12x - 1)$ — Substitution

$= 30x^4 - 100x^3 + 870x^2 - 848x - 32$ — Distribute and simplify.

### GuidedPractice

**6A.** $h(x) = (x^5 + 13x^2)(7x^3 - 5x^2 + 18)$

**6B.** $h(x) = (x^2 + x^3 + x)(8x^2 + 3)$

The same reasoning used for the derivatives of products can be applied to quotients. The following rule can be applied when calculating the derivative of quotients.

## KeyConcept Quotient Rule for Derivatives

If $f$ and $g$ are differentiable at $x$ and $g(x) \neq 0$, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

You will prove the Quotient Rule for Derivatives in Exercise 67.

## Example 7 Quotient Rule

**Find the derivative of each quotient.**

**a.** $h(x) = \frac{5x^2 - 3}{x^2 - 6}$

Let $f(x) = 5x^2 - 3$ and $g(x) = x^2 - 6$. So, $h(x) = \frac{f(x)}{g(x)}$.

$f(x) = 5x^2 - 3$ Original equation

$f'(x) = 10x$ Constant Multiple of a Power, Constant, and Difference Rules

$g(x) = x^2 - 6$ Original equation

$g'(x) = 2x$ Power, Constant, and Difference Rules

Use $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ to find the derivative of $h(x)$.

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ Quotient Rule

$= \frac{10x(x^2 - 6) - (5x^2 - 3)(2x)}{(x^2 - 6)^2}$ Substitution

$= \frac{10x^3 - 60x - 10x^3 + 6x}{(x^2 - 6)^2}$ Distributive Property

$= \frac{-54x}{(x^2 - 6)^2}$ Simplify.

**StudyTip**

**Quotient Rule** Simplification with the Quotient Rule tends to be more significant and useful. However, it is not necessary to expand the denominator if doing so does not result in further simplification.

**b.** $h(x) = \frac{x^2 + 8}{x^3 - 2}$

Let $f(x) = x^2 + 8$ and $g(x) = x^3 - 2$.

$f(x) = x^2 + 8$ Original equation

$f'(x) = 2x$ Power, Constant, and Sum Rules

$g(x) = x^3 - 2$ Original equation

$g'(x) = 3x^2$ Power, Constant, and Difference Rules

Use $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ to find the derivative of $h(x)$.

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ Quotient Rule

$= \frac{2x(x^3 - 2) - (x^2 + 8)3x^2}{(x^3 - 2)^2}$ Substitution

$= \frac{-x^4 - 24x^2 - 4x}{(x^3 - 2)^2}$ Expand and simplify.

### GuidedPractice

**7A.** $j(x) = \frac{7x - 10}{12x + 5}$ **7B.** $k(x) = \frac{6x}{2x^2 + 4}$

## Exercises

● = Step-by-Step Solutions begin on page R29.

**Evaluate limits to find the derivative of each function. Then evaluate the derivative of each function for the given values of each variable.** (Example 1)

1. $f(x) = 4x^2 - 3$; $x = 2$ and $-1$
2. $g(t) = -t^2 + 2t + 11$; $t = 5$ and $3$
3. $m(j) = 14j - 13$; $j = -7$ and $-4$
4. $v(n) = 5n^2 + 9n - 17$; $n = 7$ and $2$
5. $h(c) = c^3 + 2c^2 - c + 5$; $c = -2$ and $1$
6. $r(b) = 2b^3 - 10b$; $b = -4$ and $-3$

**Find the derivative of each function.** (Examples 2 and 3)

7. $y(f) = -11f$
8. $z(n) = 2n^2 + 7n$
9. $p(v) = 7v + 4$
10. $g(h) = 2h^{\frac{1}{2}} + 6h^{\frac{1}{3}} - 2h^{\frac{3}{2}}$
11. $b(m) = 3m^{\frac{2}{3}} - 2m^{\frac{3}{2}}$
12. $n(t) = \frac{1}{t} + \frac{3}{t^2} + \frac{2}{t^3} + 4$
13. $f(x) = 3x^{\frac{1}{2}} - x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$
14. $q(c) = c^9 - 3c^5 + 5c^2 - 3c$
15. $p(k) = k^{5.2} - 8k^{4.8} + 3k$
16. $f(x) = -5x^3 - 9x^4 + 8x^5$

17. **TEMPERATURE** The temperature in degrees Fahrenheit over a 24-hour period in a certain city can be defined as $f(h) = -0.0036h^3 - 0.01h^2 + 2.04h + 52$, where $h$ is the number of hours since midnight. (Example 4)
    a. Find an equation for the instantaneous rate of change for the temperature.
    b. Find the instantaneous rate of change for $h = 2$, 14, and 20.
    c. Find the maximum temperature for $0 \le h \le 24$.

**Use the derivative to find any critical points of the function. Then find the maximum and minimum points of each graph on the given interval.** (Example 5)

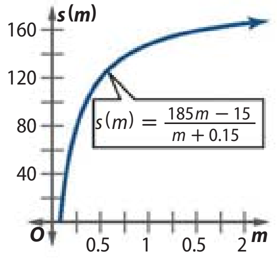
18. $f(x) = 2x^2 + 8x$; $[-5, 0]$
19. $g(m) = m^3 - 4m + 10$; $[-3, 3]$
20. $r(t) = t^4 + 6t^2 - 2$; $[1, 4]$
21. $t(u) = u^3 + 15u^2 + 75u + 115$; $[-6, -3]$
22. $k(p) = p^4 - 8p^2 + 2$; $[0, 3]$
23. $f(x) = -5x^2 - 90x$; $[-11, -8]$
24. $z(k) = k^3 - 3k^2 + 3k$; $[0, 3]$
25. $a(d) = d^4 - 3d^3 + 2$; $[-1, 4]$
26. $c(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 - 6n + 8$; $[-5, 5]$

27. **THROWN OBJECT** Refer to the application at the beginning of the lesson. The height $h$ of the ball in feet after $t$ seconds can be defined as $h(t) = 65t - 16t^2 + 5$ for $0 \le t \le 4$. (Example 5)
    a. Find $h'(t)$.
    b. Find the maximum and minimum points of $h(t)$ on the interval.
    c. Can Gabe throw the ball up to Zach's window?

**Find the derivative of each function.** (Example 6)

28. $f(x) = (4x + 3)(x^2 + 9)$
29. $g(x) = (3x^4 + 2x)(5 - 3x)$
30. $h(x) = (-7x^2 + 4)(2 - x)$
31. $s(t) = \left(t^{\frac{1}{2}} + 2\right)(3t^{11} - 4t)$
32. $g(x) = \left(x^{\frac{3}{2}} + 2x\right)(0.5x^4 - 3x)$
33. $c(t) = (t^3 + 2t - t^7)(t^6 + 3t^4 - 22t)$
34. $p(r) = (r^{2.5} + 8r)(r - 7r^2 + 108)$
35. $q(a) = \left(a^{\frac{9}{8}} + a^{-\frac{1}{4}}\right)\left(a^{\frac{5}{4}} - 13a\right)$
36. $f(x) = (1.4x^5 + 2.7x)(7.3x^9 - 0.8x^5)$
37. $h(x) = \left(\frac{1}{8}x^{\frac{2}{3}} + \frac{2}{5}x^{-\frac{1}{6}}\right)\left(x^{\frac{5}{2}} + x^{\frac{7}{8}}\right)$

38. **BASEBALL** A pitch is struck by a bat with a mass of $m$ kilograms. Suppose the initial speed of the ball after being struck is given by $s(m) = \frac{185m - 15}{m + 0.15}$. (Example 7)




    a. Find an equation for the instantaneous rate of change for the initial speed of the ball.
    b. Use a calculator to graph the equation you found in part **a** on $0 \le m \le 2$. What is happening to the instantaneous rate of change for the initial speed of the ball as the mass of the bat increases?
    c. If the mass of the bat varies inversely with the hitter's control of the swing, is it wise to use a bat that weighs 1.05 kilograms over a bat that weighs 0.80 kilogram? Explain your reasoning.

**Use the Quotient Rule to find the derivative of each function.** (Example 7)

39. $f(m) = \frac{3 - 2m}{3 + 2m}$
40. $g(n) = \frac{3n + 2}{2n + 3}$
41. $r(t) = \frac{t^2 + 2}{3 - t^2}$
42. $m(q) = \frac{q^4 + 2q^2 + 3}{q^3 - 2}$
43. $v(t) = \frac{t^2 - 5t + 3}{t^3 - 4t}$
44. $c(m) = \frac{m^4 + 1}{-m^3 + 2m}$
45. $f(x) = \frac{x^3 + 2x}{-x^2 + 3}$
46. $q(r) = \frac{1.5r^3 + 5 - r^2}{r^3}$
47. $t(w) = \frac{w + w^4}{w^2}$
48. $m(x) = \frac{x^5 + 3x}{-x^4 - 2x^3 - 2x - 3}$

**49.** **ECONOMICS** Nayla and Deirdre are selling sweatshirts to raise money for the junior class. Their weekly revenue is given by $r(x) = 0.125x^3 - 11.25x^2 + 250x$, where $x$ is the cost of one sweatshirt.

**a.** Find $r'(x)$.

**b.** Find the solutions of $r'(x) = 0$.

**c.** What do the solutions you found in part **b** represent in terms of the given situation?

**Find the equation of the line tangent to $f(x)$ at the given point. Verify your answer graphically.**

**50.** $f(x) = 3x^2 + 2x - 7; (1, -2)$

**51.** $f(x) = -5x^2 - 10x + 25; (-2, 25)$

**52.** $f(x) = -0.2x^2 + 1.5x - 0.75; (5, 1.75)$

**53.** $f(x) = 4x^2 - 12x - 35; (-1.2, -14.84)$

**54.** $f(x) = 0.8x^2 + 0.64x - 12; (10, 74.4)$

**55.** **DERIVATIVES** Let $f'(x)$ be the derivative of a function $f(x)$. If it exists, we can calculate the derivative of $f'(x)$, which is called the second derivative, and is denoted $f''(x)$ or $f^{(2)}(x)$. We can continue and find the derivative of $f''(x)$, which is called the third derivative and is denoted $f'''(x)$ or $f^{(3)}(x)$. These are examples of higher-order derivatives. Find the indicated derivative of each function.

**a.** second derivative of $f(x) = 4x^5 - 2x^3 + 6$

**b.** third derivative of $g(x) = -2x^7 + 4x^4 - 7x^3 + 10x$

**c.** fourth derivative of $h(x) = 3x^{-3} + 2x^{-2} + 4x^2$

**Sketch a graph for a function that has the given characteristics.**

**56.** The derivative is 0 at $x = -1$ and $x = 1$.

**57.** The derivative is $-2$ at $x = -1$, $x = 0$, and $x = 2$.

**58.** The derivative is 0 at $x = -1$, $x = 2$, and $x = 4$.

**59.** The derivative is undefined at $x = 4$.

**60.** **STUDYING** Sayra kept track of the amount of time $t$ in minutes that she studied the night before an exam and the percentage $p$ that she earned on the exam.

| t | 30 | 60 | 90 | 120 | 180 | 210 | 240 |
|---|---|---|---|---|---|---|---|
| p | 39 | 68 | 86 | 96 | 90 | 76 | 56 |

**a.** Find a quadratic equation $p(t)$ that can be used to model the data. Round the coefficients to the nearest ten thousandth. Graph the data and $p(t)$ on the same screen.

**b.** Use $p'(t)$ to find the maximum test score that Sayra can earn and the amount of time that she would need to study to achieve this score.

**c.** Explain why more time spent studying does not necessarily result in a higher exam score.

**61.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate how derivatives relate to some geometric properties.

**a.** **ANALYTICAL** Find the derivatives of the formulas for the area $A$ of a circle and the volume $V$ of a sphere in terms of $r$.

**b.** **VERBAL** Explain the relationship between each formula and its derivative.

**c.** **GEOMETRIC** Draw a square and its apothem $a$. Draw a cube and the apothem $a$ for three faces that are joined at a shared vertex.

**d.** **ANALYTICAL** Write formulas for the area $A$ of the square and the volume $V$ of the cube in terms of the apothem $a$. Find the derivative of each formula in terms of $a$.

**e.** **VERBAL** Explain the relationship between each formula and its derivative.

## H.O.T. Problems Use Higher-Order Thinking Skills

**62.** **ERROR ANALYSIS** Alicia and Ryan are finding $[f'(x)]^2$, where $f(x) = 6x^2 + 4x$. Ryan thinks that the answer is $144x^2 + 96x + 16$, but Alicia thinks that the answer is $144x^3 + 144x^2 + 32x$. Is either of them correct? Explain your reasoning.

**63** **CHALLENGE** Find $f'(y)$ if $f(y) = 10x^2y^3 + 5xz^2 - 6xy^2 + 8x^5 - 11x^8yz^7$.

**64.** **PROOF** Prove the Product Rule for Derivatives by showing that

$$f'(x)g(x) + f(x)g'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

(*Hint:* Work with the right side. Add and subtract $f(x)g(x + h)$ in the numerator.)

**65.** **REASONING** Determine whether the following statement is *true* or *false*. Explain your reasoning. If $f(x) = x^{5n+3}$, then $f'(x) = (5n + 3)x^{5n+2}$.

**66.** **PREWRITE** Use a plot pyramid to map out the process of finding the derivative of $f(x) = 4x^2 - 2x + 5$ evaluated at $x = 1$.

**67.** **PROOF** Prove the Quotient Rule for Derivatives by showing that

$$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}.$$

(*Hint:* Work with the right side. Add and subtract $f(x)g(x)$ in the numerator.)

**68.** **WRITING IN MATH** Can two different functions have the same derivative? Explain why this is or is not possible, and provide examples to support your answer.

## Spiral Review

**Find the slope of the lines tangent to the graph of each function at the given points.** (Lesson 12-3)

**69.** $y = x^2 - 3x$; (0, 0) and (3, 0)

**70.** $y = 4 - 2x$; (−2, 8) and (6, −8)

**71.** $y = x^2 + 9$; (3, 18) and (6, 45)

**Evaluate each limit.** (Lesson 12-2)

**72.** $\lim_{x \to -4} \frac{x^2 - 16}{x + 4}$

**73.** $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + x - 2}$

**74.** $\lim_{x \to 2} \frac{3x + 9}{x^2 - 5x - 24}$

**75.** **EXERCISE** A gym teacher asked his students to track how many days they exercised each week. Use the frequency distribution shown to construct and graph a probability distribution for the random variable $X$, rounding each probability to the nearest hundredth. (Lesson 11-2)

| Days, $X$ | Frequency |
|---|---|
| 0 | 3 |
| 1 | 6 |
| 2 | 7 |
| 3 | 8 |
| 4 | 4 |
| 5 | 2 |

**76.** **SPORTS** The number of hours per week members of the North High School basketball team spent practicing, either as a team or individually, are listed below. (Lesson 11-1)

15, 18, 16, 20, 22, 18, 19, 20, 24, 18, 16, 18

**a.** Construct a histogram and use it to describe the shape of the distribution.

**b.** Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

**Use the fifth partial sum of the trigonometric series for cosine or sine to approximate each value to three decimal places.** (Lesson 10-6)

**77.** $\cos \frac{2\pi}{11}$

**78.** $\sin \frac{3\pi}{14}$

**79.** $\sin \frac{\pi}{13}$

**Write an explicit formula and a recursive formula for finding the *n*th term of each geometric sequence.** (Lesson 10-3)

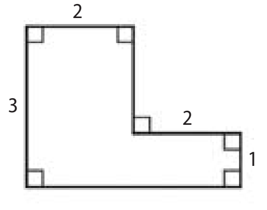
**80.** 1.25, −1.5, 1.8, …

**81.** 1.4, −3.5, 8.75, …

**82.** $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$

## Skills Review for Standardized Tests

**83.** **SAT/ACT** The figure shows the dimensions, in feet, of a stone slab. How many slabs are required to construct a rectangular patio 24 feet long and 12 feet wide?




A 18
B 20
C 24
D 36
E 40

**84.** **REVIEW** What is the slope of the line tangent to the graph of $y = 2x^2$ at the point (1, 2)?

F 1
G 2
H 4
J 8

**85.** The Better Book Company finds that the cost in dollars to print $x$ copies of a book is given by $C(x) = 1000 + 10x - 0.001x^2$. The derivative $C'(x)$ is called the *marginal cost function*. The marginal cost is the approximate cost of printing one more book after $x$ copies have been printed. What is the marginal cost when 1000 books have been printed?

A $7
B $8
C $9
D $10

**86.** **REVIEW** Find the derivative of $f(x) = 5\sqrt[3]{x^8}$.

F $f'(x) = \frac{40}{3}x^{\frac{5}{3}}$

G $f'(x) = \frac{40}{3}x^{\frac{8}{3}}$

H $f'(x) = 225x^{\frac{5}{3}}$

J $f'(x) = 225x^{\frac{8}{3}}$

LESSON 12-5

# Area Under a Curve and Integration

**Then**

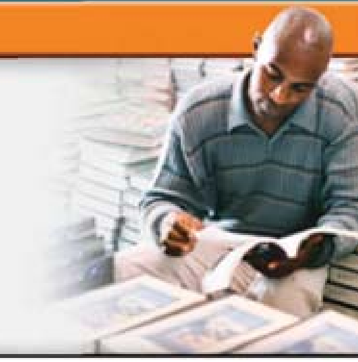
- You computed limits algebraically by using the properties of limits. (Lesson 12-2)

**Now**

1. Approximate the area under a curve using rectangles.
2. Approximate the area under a curve using definite integrals and integration.

**Why?**

- *Marginal cost* is the approximate cost that a company incurs to produce an additional unit of a product. The marginal cost equation is the derivative of the *actual cost* equation. The marginal cost function for a particular publisher is $f(x) = 10 - 0.002x$, where $x$ is the number of books manufactured and $f(x)$ is in dollars.



**NewVocabulary**

- regular partition
- definite integral
- lower limit
- upper limit
- right Riemann sum
- integration

## 1 Area Under a Curve

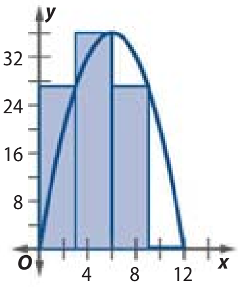
In geometry, you learned how to calculate the area of a basic figure, such as a triangle, a rectangle, or a regular polygon. You also learned how to calculate the area of a composite figure, that is, a region comprising basic shapes. However, many regions are not a collection of the basic shapes. As a result, you need a general approach for calculating the area of any two-dimensional figure.

We can approximate the area of an irregular shape by using a basic figure with a known formula for area, the rectangle. For example, consider the graph of $f(x) = -x^2 + 12x$ on the interval [0, 12]. We can *approximate* the area between the curve and the $x$-axis using rectangles of equal width.

### Example 1 Area Under a Curve Using Rectangles

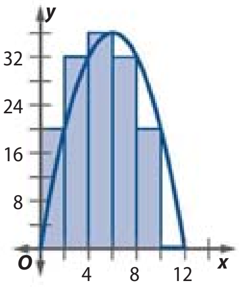
**Approximate the area between the curve $f(x) = -x^2 + 12x$ and the $x$-axis on the interval [0, 12] using 4, 6, and 12 rectangles. Use the right endpoint of each rectangle to determine the height.**

Using the figures from below for reference, notice that the rectangles were drawn with a height equal to $f(x)$ at each right endpoint. For example, the heights of the rectangles in the first figure are $f(3)$, $f(6)$, $f(9)$, and $f(12)$. We can use these heights and the length of the base of each rectangle to approximate the area under the curve.



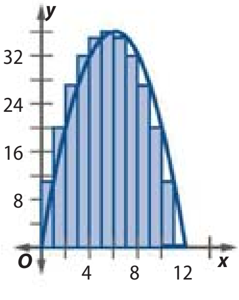

Area using 4 rectangles

$R_1 = 3 \cdot f(3)$ or 81
$R_2 = 3 \cdot f(6)$ or 108
$R_3 = 3 \cdot f(9)$ or 81
$R_4 = 3 \cdot f(12)$ or 0
total area = 270




Area using 6 rectangles

$R_1 = 2 \cdot f(2)$ or 40
$R_2 = 2 \cdot f(4)$ or 64
$R_3 = 2 \cdot f(6)$ or 72
$R_4 = 2 \cdot f(8)$ or 64
$R_5 = 2 \cdot f(10)$ or 40
$R_6 = 2 \cdot f(12)$ or 0
total area = 280




Area using 12 rectangles

$R_1 = 1 \cdot f(1)$ or 11
$R_2 = 1 \cdot f(2)$ or 20
$R_3 = 1 \cdot f(3)$ or 27
$R_4 = 1 \cdot f(4)$ or 32
$R_5 = 1 \cdot f(5)$ or 35
$R_6 = 1 \cdot f(6)$ or 36
$R_7 = 1 \cdot f(7)$ or 35
$R_8 = 1 \cdot f(8)$ or 32
$R_9 = 1 \cdot f(9)$ or 27
$R_{10} = 1 \cdot f(10)$ or 20
$R_{11} = 1 \cdot f(11)$ or 11
$R_{12} = 1 \cdot f(12)$ or 0
total area = 286

The approximation for the area under the curve using 4, 6, and 12 rectangles is 270 square units, 280 square units, and 286 square units, respectively.

**TechnologyTip**

**Tables** To help generate multiple heights of rectangles with your graphing calculator, enter the function using the [Y=] menu. Then use the TABLE function by pressing [2nd] [TABLE]. This will generate a list of heights for different values of *x*. You can also change the interval for the *x* values in your table by pressing [2nd] [TBLSET] and adjusting the TBLSET options.

### GuidedPractice

**1.** Approximate the area between the curve $f(x) = -x^2 + 24x$ and the *x*-axis on the interval [0, 24] using 6, 8, and 12 rectangles. Use the right endpoint of each rectangle to determine the height.

Notice that the thinner the rectangles are, the better they fit the region, and the better their total area approximates the area of the region. Also, the rectangles were drawn so that the right endpoint of each one evaluated at $f(x)$ represents the height. The left endpoints may also be used to determine the height of each rectangle and can produce a different result for the approximated area.

Using right or left endpoints may result in adding or excluding area that does or does not lie between the curve and the *x*-axis. In some cases, better approximations may be obtained by calculating the area using both left and right endpoints and then averaging the results.

## Example 2 Area Under a Curve Using Left and Right Endpoints

**Approximate the area between the curve $f(x) = x^2$ and the *x*-axis on the interval [0, 4] by first using the right endpoints and then by using the left endpoints of the rectangles. Use rectangles with a width of 1.**

Using right endpoints for the height of each rectangle produces four rectangles with a width of 1 unit (Figure 12.5.1). Using left endpoints for the height of each rectangle produces four rectangles with a width of 1 unit (Figure 12.5.2).

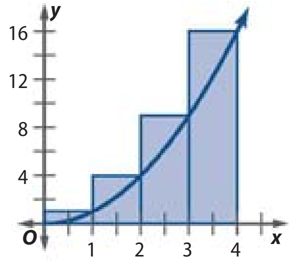



**Figure 12.5.1**

Area using right endpoints

$R_1 = 1 \cdot f(1)$ or 1

$R_2 = 1 \cdot f(2)$ or 4

$R_3 = 1 \cdot f(3)$ or 9

$R_4 = 1 \cdot f(4)$ or 16

total area = 30

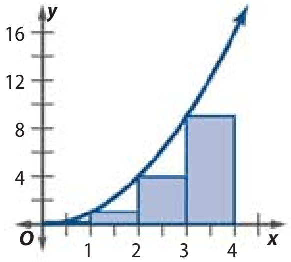



**Figure 12.5.2**

Area using left endpoints

$R_1 = 1 \cdot f(0)$ or 0

$R_2 = 1 \cdot f(1)$ or 1

$R_3 = 1 \cdot f(2)$ or 4

$R_4 = 1 \cdot f(3)$ or 9

total area = 14

The area using the right and left endpoints is 30 and 14 square units, respectively. We now have lower and upper estimates for the area of the region, $14 < \text{area} < 30$. Averaging the two areas would give a better approximation of 22 square units.

### GuidedPractice

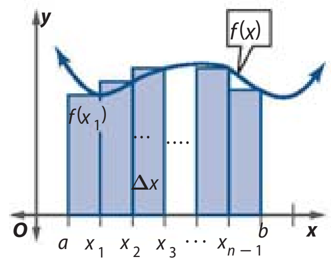
**2.** Approximate the area between the curve $f(x) = \frac{12}{x}$ and the *x*-axis on the interval [1, 5] by first using the right endpoints and then by using the left endpoints. Use rectangles with a width of 1 unit. Then find the average of the two approximations.

Any point within the width of the rectangles may be used as the heights when approximating the area between the graph of a curve and the *x*-axis. The most commonly used are the left endpoints, the right endpoints, and the midpoints.

## 2 Integration

As we saw in Example 1, as the rectangles get thinner, their total area approaches the exact area of the region under the curve. We can conclude that the area of the region under a curve is the limit of the total area of the rectangles as the widths of the rectangles approach 0.

In the figure, the interval from $a$ to $b$ has been subdivided into $n$ equal subintervals. This is called a **regular partition**. The length of the entire interval from $a$ to $b$ is $b - a$, so the width of each of the $n$ rectangles is $\frac{b-a}{n}$ and is denoted $\Delta x$. The height of each rectangle at the right endpoint corresponds with the value of the function at that point. Thus, the height of the first rectangle is $f(x_1)$, the height of the second is $f(x_2)$, and so on, with the height of the last rectangle being $f(x_n)$.




**ReadingMath**

**Sigma Notation** $\sum_{i=1}^{n} f(x_i)\,\Delta x$ is read *the summation of the product of f of x sub i from 1 to n and the change in x.*

The area of each rectangle can now be calculated by taking the product of $\Delta x$ and the corresponding height. The area of the first rectangle is $f(x_1)\Delta x$, the area of the second rectangle is $f(x_2)\Delta x$, and so on. The total area $A$ of the $n$ rectangles is given by the sum of the areas and can be written in sigma notation.

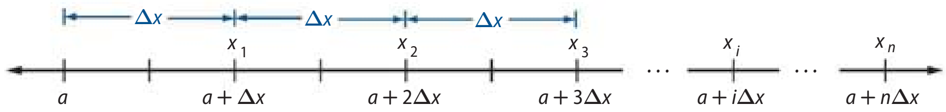
$$A = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \quad \text{Add the areas.}$$

$$A = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{Factor out } \Delta x.$$

$$A = \Delta x \sum_{i=1}^{n} f(x_i) \quad \text{Write the sum of the heights in sigma notation.}$$

$$A = \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{Commutative Property of Multiplication}$$

To assist in future calculations, we can derive a formula to find any $x_i$. The width $\Delta x$ of each rectangle is the distance between successive $x_i$-values. Consider the $x$-axis.




We can see that $x_i = a + i\Delta x$. This formula will be useful when finding the area under the graph of any function.

To make the width of the rectangles approach 0, we let the number of rectangles approach infinity. This limit is called a **definite integral** and is given special notation.

## KeyConcept Definite Integral

The area of a region under the graph of a function is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $a$ and $b$ are the **lower limit** and **upper limit** respectively, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$. This method is referred to as the **right Riemann sum**.

The Riemann sum is named for the German mathematician Bernhard Riemann (1826–1866). He is credited with formulating the expression for approximating the area under a curve using limits. The expression may be altered to use left endpoints or midpoints.

The process of evaluating an integral is called **integration**. The following summation formulas will assist in evaluating definite integrals.

$$\sum_{i=1}^{n} c = cn,\ c \text{ is a constant} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}$$

**WatchOut!**

**Summations** The sum of a constant $c$ is $cn$, not 0 or $\infty$.

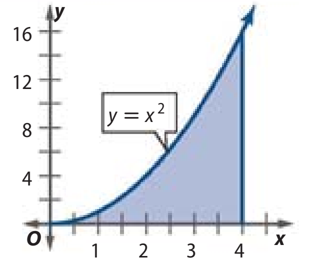
For instance, $\sum_{i=1}^{n} 5 = 5n$.

There are two summation properties that are needed to evaluate some integrals.

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n} ci = c\sum_{i=1}^{n} i,\ c \text{ is a constant}$$

## Example 3 Area Under a Curve Using Integration

**Use limits to find the area of the region between the graph of $y = x^2$ and the $x$-axis on the interval [0, 4], or $\int_0^4 x^2\,dx$.**




First, find $\Delta x$ and $x_i$.

| | |
|---|---|
| $\Delta x = \frac{b-a}{n}$ | Formula for $\Delta x$ |
| $= \frac{4-0}{n}$ or $\frac{4}{n}$ | $b = 4$ and $a = 0$ |
| $x_i = a + i\Delta x$ | Formula for $x_i$ |
| $= 0 + i\frac{4}{n}$ or $\frac{4i}{n}$ | $a = 0$ and $\Delta x = \frac{4}{n}$ |

Calculate the definite integral that gives the area.

| | |
|---|---|
| $\int_0^4 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$ | Definition of a definite integral |
| $= \lim_{n\to\infty}\sum_{i=1}^{n}(x_i)^2\,\Delta x$ | $f(x_i) = x_i^2$ |
| $= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4i}{n}\right)^2\left(\frac{4}{n}\right)$ | $x_i = \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ |
| $= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(\frac{4i}{n}\right)^2$ | Factor. |
| $= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\frac{16i^2}{n^2}$ | Expand. |
| $= \lim_{n\to\infty}\frac{4}{n}\left(\frac{16}{n^2}\sum_{i=1}^{n} i^2\right)$ | Factor. |
| $= \lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$ | $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ |
| $= \lim_{n\to\infty}\frac{4}{n}\left[\frac{16n(2n^2+3n+1)}{6n^2}\right]$ | Multiply and expand. |
| $= \lim_{n\to\infty}\frac{64n(2n^2+3n+1)}{6n^3}$ | Multiply. |
| $= \lim_{n\to\infty}\frac{64(2n^2+3n+1)}{6n^2}$ | Divide by $n$. |
| $= \lim_{n\to\infty}\frac{64}{6}\left[\frac{(2n^2+3n+1)}{n^2}\right]$ | Factor. |
| $= \lim_{n\to\infty}\frac{64}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$ | Divide each term by $n^2$. |
| $= \left(\lim_{n\to\infty}\frac{64}{6}\right)\left[\lim_{n\to\infty} 2 + \left(\lim_{n\to\infty} 3\right)\left(\lim_{n\to\infty}\frac{1}{n}\right) + \lim_{n\to\infty}\frac{1}{n^2}\right]$ | Limit theorems |
| $= \frac{64}{6}[2 + 3(0) + 0]$ or $\frac{64}{3}$ | The area is $\frac{64}{3}$ or $21\frac{1}{3}$ square units. |

**StudyTip**
**Limits** Factor each summation until the expression remaining involves only a constant or $i$. Then apply the necessary summation formula.

### GuidedPractice

**Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.**

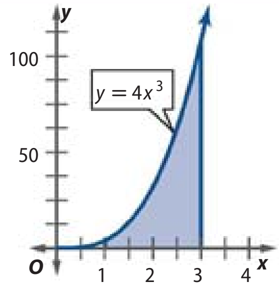
**3A.** $\int_0^1 3x^2\,dx$

**3B.** $\int_0^3 x\,dx$

The areas of regions that do not have a lower limit at the origin can also be found using limits.

## Example 4 Area Under a Curve Using Integration

**Use limits to find the area of the region between the graph of $y = 4x^3$ and the $x$-axis on the interval [1, 3], or $\int_1^3 4x^3\,dx$.**




First, find $\Delta x$ and $x_i$.

$\Delta x = \frac{b-a}{n}$ — Formula for $\Delta x$

$= \frac{3-1}{n}$ or $\frac{2}{n}$ — $b = 3$ and $a = 1$

$x_i = a + i\Delta x$ — Formula for $x_i$

$= 1 + i\frac{2}{n}$ or $1 + \frac{2i}{n}$ — $a = 1$ and $\Delta x = \frac{2}{n}$

Calculate the definite integral that gives the area.

$\int_1^3 4x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ — Definition of a definite integral

$= \lim_{n\to\infty}\sum_{i=1}^{n} 4(x_i)^3\Delta x$ — $f(x_i) = 4(x_i)^3$

$= \lim_{n\to\infty}\sum_{i=1}^{n} 4\left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$ — $x_i = 1 + \frac{2i}{n}$ and $\Delta x = \frac{2}{n}$

$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^3$ — Factor

$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left[1 + 3\left(\frac{2i}{n}\right) + 3\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right)^3\right]$ — Expand.

$= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)$ — Simplify.

$= \lim_{n\to\infty}\frac{8}{n}\left(\sum_{i=1}^{n} 1 + \sum_{i=1}^{n}\frac{6i}{n} + \sum_{i=1}^{n}\frac{12i^2}{n^2} + \sum_{i=1}^{n}\frac{8i^3}{n^3}\right)$ — Apply summations.

$= \lim_{n\to\infty}\frac{8}{n}\left(\sum_{i=1}^{n} 1 + \frac{6}{n}\sum_{i=1}^{n} i + \frac{12}{n^2}\sum_{i=1}^{n} i^2 + \frac{8}{n^3}\sum_{i=1}^{n} i^3\right)$ — Factor constants.

$= \lim_{n\to\infty}\frac{8}{n}\left[n + \frac{6}{n}\cdot\frac{n(n+1)}{2} + \frac{12}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^3}\cdot\frac{n^2(n+1)^2}{4}\right]$ — Summation formulas

$= \lim_{n\to\infty}\left[\frac{8n}{n} + \frac{48n(n+1)}{2n^2} + \frac{96n(2n^2+3n+1)}{6n^3} + \frac{64n^2(n^2+2n+1)}{4n^4}\right]$ — Distribute $\frac{8}{n}$.

$= \lim_{n\to\infty}\left[8 + \frac{24(n+1)}{n} + \frac{16(2n^2+3n+1)}{n^2} + \frac{16(n^2+2n+1)}{n^2}\right]$ — Simplify.

$= \lim_{n\to\infty}\left[8 + 24\left(1 + \frac{1}{n}\right) + 16\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 16\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\right]$ — Factor and perform division.

$= \lim_{n\to\infty} 8 + 24\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) + 16\lim_{n\to\infty}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 16\lim_{n\to\infty}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$ — Limit Theorems

$= 8 + 24(1 + 0) + 16(2 + 0 + 0) + 16(1 + 0 + 0)$ or 80 — Simplify.

The area of the region is 80 square units.

**WatchOut!**

**Limits** When finding the area under a curve by using limits, evaluate the summation expressions for the given values of *i* before distributing the width $\Delta x$ or any other constants.

### GuidedPractice

**Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.**

**4A.** $\int_1^3 x^2\,dx$

**4B.** $\int_2^4 x^3\,dx$

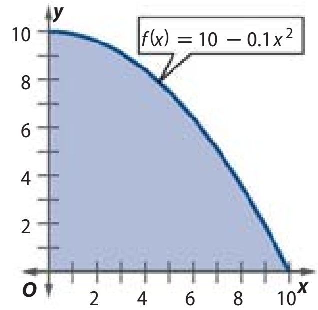
## Real-World Career

**Landscape Architect**
Employment opportunities as a landscape architect are projected to increase by 16% by 2016. Landscape architects design golf courses, college campuses, public parks, and residential areas. A professional license and a bachelor's degree in landscape architecture are generally required.

Definite integrals can be used to find the areas of other irregular shapes.

### Real-World Example 5 Area Under a Curve

**LANDSCAPING** **Foster charges \$2.40 per square foot of mulch to deliver and install it. He is hired to create two identical flower beds in the back corners of a residential lot. If the area of each flower bed can be found by $\int_0^{10}(10 - 0.1x^2)dx$, how much will Foster charge for the flower beds if $x$ is given in terms of feet?**




First, find $\Delta x$ and $x_i$.

$\Delta x = \frac{b-a}{n}$ — Formula for $\Delta x$

$= \frac{10-0}{n}$ or $\frac{10}{n}$ — $b = 10$ and $a = 0$

$x_i = a + i\Delta x$ — Formula for $x_i$

$= 0 + i\frac{10}{n}$ or $\frac{10i}{n}$ — $a = 0$ and $\Delta x = \frac{10}{n}$

Calculate the definite integral that gives the area.

$\int_0^{10}(10 - 0.1x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ — Definition of a definite integral

$= \lim_{n\to\infty}\sum_{i=1}^{n}(10 - 0.1x_i^2)\Delta x$ — $f(x_i) = 10 - 0.1x_i^2$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[10 - 0.1\left(\frac{10i}{n}\right)^2\right]\cdot\frac{10}{n}$ — $x_i = \frac{10i}{n}$ and $\Delta x = \frac{10}{n}$

$= \lim_{n\to\infty}\frac{10}{n}\sum_{i=1}^{n}\left(10 - \frac{10i^2}{n^2}\right)$ — Expand and simplify.

$= \lim_{n\to\infty}\frac{10}{n}\left(\sum_{i=1}^{n}10 - \sum_{i=1}^{n}\frac{10i^2}{n^2}\right)$ — Apply summations.

$= \lim_{n\to\infty}\frac{10}{n}\left(\sum_{i=1}^{n}10 - \frac{10}{n^2}\sum_{i=1}^{n}i^2\right)$ — Factor.

$= \lim_{n\to\infty}\frac{10}{n}\left(10n - \frac{10}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right)$ — Summation formulas

$= \lim_{n\to\infty}\left(\frac{100n}{n} - \frac{100n(2n^2+3n+1)}{3n^2}\right)$ — Distribute $\frac{10}{n}$.

$= \lim_{n\to\infty}\left(100 - \frac{50(2n^2+3n+1)}{3n^2}\right)$ — Divide by $n$.

$= \lim_{n\to\infty}\left[100 - \frac{50}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right]$ — Factor and perform division.

$= \lim_{n\to\infty}100 - \frac{50}{3}\lim_{n\to\infty}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$ — Limit Theorems

$= 100 - \frac{50}{3}(2 + 0 + 0)$ or $66\frac{2}{3}$ — Simplify.

The area of one flower bed is about 66.67 square feet. For Foster to install both flower beds, he would have to charge \$2.40 • $\left(66\frac{2}{3}\cdot 2\right)$ or \$320.

### Guided Practice

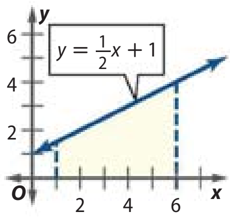
**5.** **PAINTING** Mrs. Sung's art class is painting a large mural depicting a winter ski scene. The students want to paint a ski hill at both the beginning and the end of the mural, but they only have enough paint to cover 30 square feet. If the area of each ski hill can be found by $\int_0^5(5 - 0.2x^2)dx$, will the students have enough paint for both hills? Explain.

## Exercises

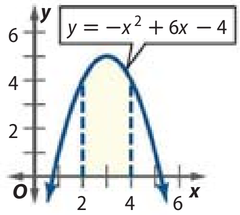
● = Step-by-Step Solutions begin on page R29.

**Approximate the area of the shaded region for each function using the indicated number of rectangles. Use the specified endpoints to determine the heights of the rectangles.** (Example 1)

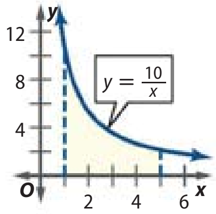
**1.** 5 rectangles
right endpoints



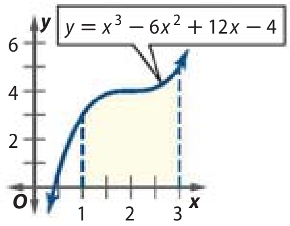

**2.** 4 rectangles
left endpoints



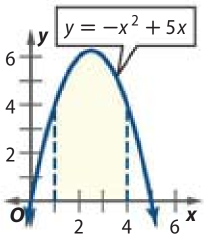

**3.** 8 rectangles
right endpoints



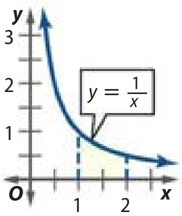

**4.** 8 rectangles
left endpoints




**5.** 4 rectangles
left endpoints



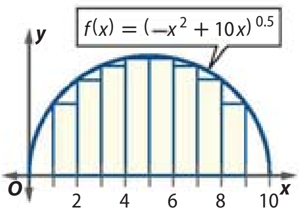

**6.** 5 rectangles
right endpoints




**7** **FLOORING** Len is installing a hardwood floor and has to cover a semicircular section defined by $f(x) = (-x^2 + 10x)^{0.5}$. (Example 1)

**a.** Approximate the area of the semicircular region using left endpoints and rectangles one unit in width.

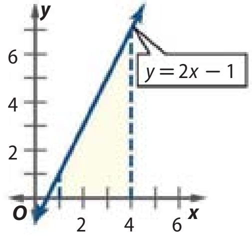
**b.** Len decides that using both left and right endpoints may give a better estimate because it would eliminate any area outside the semicircular region. Approximate the area of the semicircular region as shown in the figure.



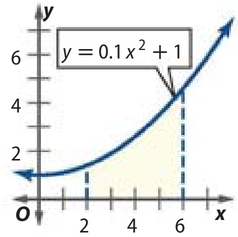

**c.** Find the area of the region by using the formula for the area of a semicircle. Which approximation is closer to the actual area of the region? Explain why this estimate gives a better approximation.

**Approximate the area of the shaded region for each function by first using the right endpoints and then by using the left endpoints. Then find the average for both approximations. Use the specified width for the rectangles.** (Example 2)

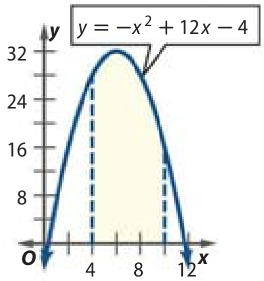
**8.** width = 0.5



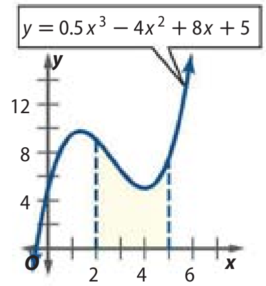

**9.** width = 1.0



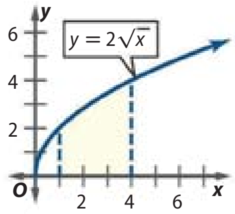

**10.** width = 0.75



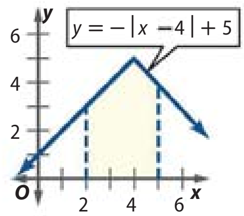

**11.** width = 0.5




**12.** width = 0.75




**13.** width = 0.5




**Use limits to find the area between the graph of each function and the *x*-axis given by the definite integral.** (Examples 3 and 4)

**14.** $\int_1^4 4x^2\,dx$

**15.** $\int_2^6 (2x + 5)\,dx$

**16.** $\int_0^2 6x\,dx$

**17.** $\int_1^3 (2x^2 + 3)\,dx$

**18.** $\int_2^5 (x^2 + 4x - 2)\,dx$

**19.** $\int_1^2 8x^3\,dx$

**20.** $\int_0^4 (4x - x^2)\,dx$

**21.** $\int_3^4 (-x^2 + 6x)\,dx$

**22.** $\int_0^3 (x^3 + x)\,dx$

**23.** $\int_2^4 (-3x + 15)\,dx$

**24.** $\int_1^5 (x^2 - x + 1)\,dx$

**25.** $\int_1^3 12x\,dx$

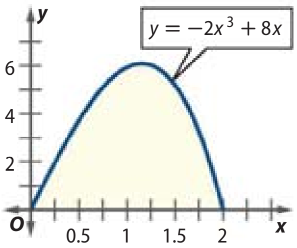
**26.** $\int_0^3 (8 - 0.6x^2)\,dx$

**27.** $\int_3^8 0.5x^2\,dx$

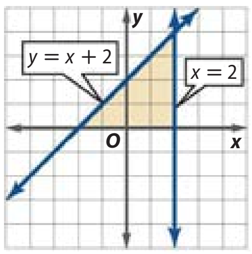
**28.** **PUBLISHING** Refer to the beginning of the lesson. The publisher would like to increase daily production from 1000 books to 1500 books. Find the cost of the increase if it is defined as $\int_{1000}^{1500} (10 - 0.002x)\, dx$. (Example 5)

**29.** **ARCHWAY** The homecoming committee decided the entrance into the dance would be a balloon archway. In addition, the committee wanted to hang streamers that would extend from the top of the arch down to the floor, completely covering the entrance. Find the area beneath the balloon archway if it can be defined as $\int_{1}^{13} (-0.2x^2 + 2.8x - 1.8)\, dx$, where $x$ is given in feet. (Example 5)

**30.** **LOGO** Part of a company's logo is in the shape of the region shown. If this part of the logo is to be sewn on a flag, how much material is required if $x$ is given in feet? (Example 5)




**31.** **NEGATIVE LIMITS** Definite integrals can be calculated for both positive and negative limits.




**a.** Find the height and length of the base of the triangle. Then calculate the area of the triangle using its height and base.

**b.** Calculate the area of the triangle by evaluating $\int_{-2}^{2} (x + 2)\, dx$.

**Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.**

**32.** $\int_{-1}^{1} x^2\, dx$

**33.** $\int_{-1}^{0} (x^3 + 2)\, dx$

**34.** $\int_{-4}^{-2} (-x^2 - 6x)\, dx$

**35.** $\int_{-3}^{-2} -5x\, dx$

**36.** $\int_{-2}^{0} (2x + 6)\, dx$

**37.** $\int_{-1}^{0} (x^3 - 2x)\, dx$

**Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.**

**38.** $\int_{-3}^{-1} (-2x^2 - 7x)\, dx$

**39.** $\int_{-2}^{0} (-x^3)\, dx$

**40.** $\int_{-4}^{3} 2\, dx$

**41.** $\int_{-2}^{-1} \left(-\frac{1}{2}x + 3\right) dx$

**42.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the process of finding the area between two curves.

**a.** **GRAPHICAL** Graph $f(x) = -x^2 + 4$ and $g(x) = x^2$ on the same coordinate plane and shade the areas represented by $\int_{0}^{1} (-x^2 + 4)\, dx$ and $\int_{0}^{1} x^2\, dx$.

**b.** **ANALYTICAL** Evaluate $\int_{0}^{1} (-x^2 + 4)\, dx$ and $\int_{0}^{1} x^2\, dx$.

**c.** **VERBAL** Explain why the area between the two curves is equal to $\int_{0}^{1} (-x^2 + 4)\, dx - \int_{0}^{1} x^2\, dx$. Then calculate this value using the values found in part **b**.

**d.** **ANALYTICAL** Find $f(x) - g(x)$. Then evaluate $\int_{0}^{1} [f(x) - g(x)]\, dx$.

**e.** **VERBAL** Make a conjecture about the process of finding the area between two curves.

## H.O.T. Problems Use Higher-Order Thinking Skills

**43.** **ERROR ANALYSIS** Pete says that when you use the right endpoints of rectangles to estimate the area between a curve and the $x$-axis, the total area of the rectangles will always be greater than the actual area. Lara says that the area of the rectangles is always greater when you use the left endpoints. Is either of them correct? Explain.

**44.** **CHALLENGE** Evaluate $\int_{0}^{1} (5x^4 + 3x^2 - 2x + 1)\, dx$.

**45.** **REASONING** Suppose every vertical cross section of a tunnel can be modeled by $f(x)$ on $[a, b]$. Explain how the volume of the tunnel can be calculated by using $\ell \cdot \int_{a}^{b} f(x)\, dx$, where $\ell$ is the length of the tunnel.

**46.** **PREWRITE** Write an outline that could be used to describe the steps involved in estimating the area between the $x$-axis and the curve of a function on a given interval.

**47.** **CHALLENGE** Evaluate $\int_{0}^{t} (x^2 + 2)\, dx$.

**48.** **WRITING IN MATH** Explain the effectiveness of using triangles and circles to approximate the area between a curve and the $x$-axis. Which shape do you think gives the best approximation?

## Spiral Review

**Find the derivative of each function.** (Lesson 12-4)

**49.** $j(x) = (2x^3 + 11x)(2x^8 - 12x^2)$

**50.** $f(k) = (k^{15} + k^2 + 2k)(k - 7k^2)$

**51.** $s(t) = (\sqrt{t} - 7)(3t^8 - 5t)$

**Find the slope of the line tangent to the graph of each function at $x = 1$.** (Lesson 12-3)

**52.** $y = x^3$

**53.** $y = x^3 - 7x^2 + 4x + 9$

**54.** $y = (x + 1)(x - 2)$

**Evaluate each limit.** (Lesson 12-2)

**55.** $\lim_{x \to 0} \frac{x^2 + 3x}{x}$

**56.** $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$

**57.** $\lim_{x \to 3} \frac{x^2 - 9}{x^3 - 27}$

**58. MOVIES** Veronica believes that the price of a ticket for a movie is still under \$7.00. She randomly visits 14 movie theatres and records the prices of a ticket. Find the $p$-value and determine whether there is enough evidence to support the claim at $\alpha = 0.10$. (Lesson 11-6)

| Ticket Prices (\$) | | | | | | |
|---|---|---|---|---|---|---|
| 5.25 | 7.27 | 5.46 | 7.63 | 7.75 | 5.42 | 6.00 |
| 6.63 | 7.38 | 6.97 | 7.85 | 7.03 | 6.53 | 6.87 |

**59. VIDEO GAMES** A random sample of 85 consumers of video games showed that the average price of a video game was \$36.50. Assume that the standard deviation from previous studies was \$11.30. Find the maximum error of estimate given a 99% confidence level. Then create a confidence interval for the mean price of a video game. (Lesson 11-5)

**60. SHOPPING** In a recent year, 33% of Americans said that they were planning to go shopping on Black Friday, the day after Thanksgiving. What is the probability that in a random sample of 45 people, fewer than 14 people plan to go shopping on Black Friday? (Lesson 11-4)

**Classify each random variable $X$ as *discrete* or *continuous*. Explain your reasoning.** (Lesson 11-2)

**61.** $X$ represents the number of mobile phone calls made by a randomly chosen student during a given day.

**62.** $X$ represents the time it takes a randomly selected student to run a mile.

## Skills Review for Standardized Tests

**63. SAT/ACT** If the statement below is true, then which of the following must also be true?
If at least 1 bear is sleepy,
then some ponies are happy.

A If all bears are sleepy, then all ponies are happy.

B If all ponies are happy, then all bears are sleepy.

C If no bear is sleepy, then no pony is happy.

D If no pony is happy, then no bear is sleepy.

E If some ponies are happy, then at least 1 bear is sleepy.

**64. REVIEW** What is $\lim_{x \to 3} \frac{x^2 + 3x - 10}{x^2 + 5x + 6}$?

F $\frac{1}{15}$

H $\frac{3}{15}$

G $\frac{2}{15}$

J $\frac{4}{15}$

**65.** Find the area of the region between the graph of $y = -x^2 + 3x$ and $x$-axis on the interval [0, 3] or $\int_0^3 (-x^2 + 3x)\, dx$.

A $3\frac{3}{4}$ units$^2$

C $21\frac{1}{4}$ units$^2$

B $4\frac{1}{2}$ units$^2$

D $22\frac{1}{2}$ units$^2$

**66. REVIEW** Find the derivative of $n(a) = \frac{4}{a} - \frac{5}{a^2} + \frac{3}{a^4} + 4a$.

F $n'(a) = 8a - 5a^2 + 3a^4$

H $n'(a) = 4a^2 - 5a^3 + 3a^4 + 4$

G $n'(a) = -\frac{4}{a^2} + \frac{5}{a^3} - \frac{3}{a^5} + 4$

J $n'(a) = -\frac{4}{a^2} + \frac{10}{a^3} - \frac{12}{a^5} + 4$

LESSON 12-6

# The Fundamental Theorem of Calculus

**Then**

- You used limits to approximate the area under a curve. (Lesson 12-5)

**Now**

- 1 Find antiderivatives.
- 2 Use the Fundamental Theorem of Calculus.

**Why?**

- During the initial ascent of a hot-air balloon ride, Kristina realized that she had her brother's cell phone in her pocket. Before the balloon rose too high, Kristina dropped the phone over the edge to her brother waiting on the ground. Knowing that the velocity of the phone could be described as $v(t) = -32t$, where $t$ is given in seconds and velocity is given in feet per second, Kristina was able to determine how high she was from the ground when she dropped the phone over the edge.

**NewVocabulary**

antiderivative
indefinite integral
Fundamental Theorem of Calculus

## 1 Antiderivatives and Indefinite Integrals

In Lessons 12-3 and 12-4, you learned that if the position of an object is defined as $f(x) = x^2 + 2x$, then an expression for the velocity of the object is the derivative of $f(x)$ or $f'(x) = 2x + 2$. However, if you are given an expression for velocity, but need to know the formula from which it was obtained, we need a way to work backward or undo the differentiation.

In other words, if given $f(x)$, we want to find an equation $F(x)$ such that $F'(x) = f(x)$. The function $F(x)$ is an **antiderivative** of $f(x)$.

### Example 1 Find Antiderivatives

**Find an antiderivative for each function.**

**a.** $f(x) = 3x^2$

We need to find a function that has a derivative of $3x^2$. Recall that the derivative has an exponent that is one less than the exponent in the original function. Therefore, $F(x)$ will be raised to a power of three. Also, the coefficient of a derivative is determined in part by the exponent of the original function. $F(x) = x^3$ fits this description. The derivative of $x^3 = 3x^{3-1}$ or $3x^2$.

However, $x^3$ is not the only function that works. The function $G(x) = x^3 + 10$ is another because its derivative is $G'(x) = 3x^{3-1} + 0$ or $3x^2$. Another answer could be $H(x) = x^3 - 37$.

**b.** $f(x) = -\frac{8}{x^9}$

Rewrite $f(x)$ with a negative exponent, $-8x^{-9}$. Again, the derivative has an exponent that is one less than the exponent in the original function, so $F(x)$ will be raised to the negative eighth power. We can try $F(x) = x^{-8}$. The derivative of $x^{-8}$ is $-8x^{-8-1}$ or $-8x^{-9}$.

$G(x) = x^{-8} + 3$ and $H(x) = x^{-8} - 12$ are other antiderivatives.

**GuidedPractice**

**Find two different antiderivatives for each function.**

**1A.** $2x$ 　　 **1B.** $-3x^{-4}$

In Example 1, notice that adding or subtracting constants to the original antiderivative produced other antiderivatives. In fact, because the derivative of any constant is 0, adding or subtracting a constant $C$ to an antiderivative will not affect its derivative. Therefore, there are an infinite number of antiderivatives for a given function. Antiderivatives that include a constant $C$ are said to be in *general form*.

Marty Melville/Getty Images News/Getty Images

As with derivatives, there are rules for finding antiderivatives.

### KeyConcept Antiderivative Rules

**Power Rule** If $f(x) = x^n$, where $n$ is a rational number other than $-1$, $F(x) = \frac{x^{n+1}}{n+1} + C$.

**Constant Multiple of a Power** If $f(x) = kx^n$, where $n$ is a rational number other than $-1$ and $k$ is a constant, then $F(x) = \frac{kx^{n+1}}{n+1} + C$.

**Sum and Difference** If the antiderivatives of $f(x)$ and $g(x)$ are $F(x)$ and $G(x)$ respectively, then an antiderivative of $f(x) \pm g(x)$ is $F(x) \pm G(x)$.

**StudyTip**

**Antiderivatives** An antiderivative of a constant $k$ is $kx$. For instance, if $f(x) = 3$, then $F(x) = 3x$.

### Example 2 Antiderivative Rules

**Find all antiderivatives for each function.**

**a.** $f(x) = 4x^7$

$f(x) = 4x^7$ — Original equation

$F(x) = \frac{4x^{7+1}}{7+1} + C$ — Constant Multiple of a Power

$= \frac{1}{2}x^8 + C$ — Simplify.

**b.** $f(x) = \frac{2}{x^4}$

$f(x) = \frac{2}{x^4}$ — Original equation

$= 2x^{-4}$ — Rewrite with a negative exponent.

$F(x) = \frac{2x^{-4+1}}{-4+1} + C$ — Constant Multiple of a Power

$= -\frac{2}{3}x^{-3} + C$ or $-\frac{2}{3x^3} + C$ — Simplify.

**c.** $f(x) = x^2 - 8x + 5$

$f(x) = x^2 - 8x + 5$ — Original equation

$= x^2 - 8x^1 + 5x^0$ — Rewrite the function so each term has a power of $x$.

$F(x) = \frac{x^{2+1}}{2+1} - \frac{8x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} + C$ — Antiderivative Rule

$= \frac{1}{3}x^3 - 4x^2 + 5x + C$ — Simplify.

**GuidedPractice**

**2A.** $f(x) = 6x^4$ **2B.** $f(x) = \frac{10}{x^3}$ **2C.** $f(x) = 8x^7 + 6x + 2$

The general form of an antiderivative is given a special name and notation.

### KeyConcept Indefinite Integral

The **indefinite integral** of $f(x)$ is defined by $\int f(x)\,dx = F(x) + C$, where $F(x)$ is an antiderivative of $f(x)$ and $C$ is any constant.

**Real-WorldLink**

In egg drop competitions, participants attempt to protect their eggs from a two-story drop. Scoring may be based on the weight of the protective device, the number of parts the device includes, whether the device hits a target, and of course, whether the egg breaks.

**Source:** *Salem-Winston Journal*

## Real-World Example 3 Indefinite Integral

**EGG DROP** **Students of Ms. Nair's technology class are participating in an egg drop competition. Each team of students must build a protective device that will keep an egg from cracking after a 30-foot drop. The instantaneous velocity of an egg can be defined as $v(t) = -32t$, where $t$ is given in seconds and velocity is measured in feet per second.**

**a. Find the position function $s(t)$ of a dropped egg.**

To find the function for the position of the egg, find the antiderivative of $v(t)$.

| | |
|---|---|
| $s(t) = \int v(t)\, dt$ | Relationship between position and velocity |
| $= \int -32t\, dt$ | $v(t) = -32t$ |
| $= -\dfrac{32t^{1+1}}{1+1} + C$ | Constant Multiple of a Power |
| $= -16t^2 + C$ | Simplify. |

Find $C$ by substituting 30 feet for the initial height and 0 for the initial time.

| | |
|---|---|
| $s(t) = -16t^2 + C$ | Antiderivative of $v(t)$ |
| $30 = -16(0)^2 + C$ | $s(t) = 30$ and $t = 0$ |
| $30 = C$ | Simplify. |

The position function for the egg is $s(t) = -16t^2 + 30$.

**b. Find how long it will take for the egg to hit the ground.**

Solve for $t$ when $s(t) = 0$.

| | |
|---|---|
| $s(t) = -16t^2 + 30$ | Position function for the egg |
| $0 = -16t^2 + 30$ | $s(t) = 0$ |
| $-30 = -16t^2$ | Subtract 30 from each side. |
| $1.875 = t^2$ | Divide each side by $-16$. |
| $1.369 = t$ | Take the positive square root of each side. |

The egg will hit the ground in about 1.37 seconds.

### GuidedPractice

3. **FALLING OBJECT** A maintenance worker, securely standing on a catwalk in a gymnasium, is fixing a light positioned 120 feet above the floor when his wallet accidentally falls out of his pocket. The instantaneous velocity of his wallet can be defined as $v(t) = -32t$, where $t$ is given in seconds and velocity is measured in feet per second.
   - **A.** Find the position function $s(t)$ of the dropped wallet.
   - **B.** Find how long it will take for the wallet to hit the floor.

## 2 Fundamental Theorem of Calculus

Notice that the notation used for indefinite integrals looks very similar to the notation used in Lesson 12-5 for definite integrals. The only difference appears to be the lack of upper and lower limits in indefinite integrals. In fact, finding an antiderivative of a function is a shortcut to calculating a definite integral by evaluating a Riemann sum. This connection between definite integrals and antiderivatives is so important that it is called the **Fundamental Theorem of Calculus**.

### KeyConcept Fundamental Theorem of Calculus

If $f$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The difference $F(b) - F(a)$ is commonly denoted by $F(x)\Big|_a^b$.

**Math HistoryLink**

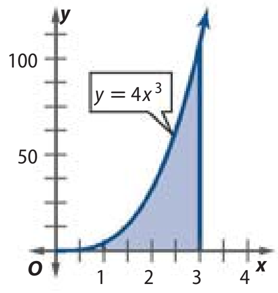
**Maria Gaetana Agnesi**
**(1718–1799)**
An Italian linguist, mathematician, and philosopher, Maria Agnesi authored *Analytical Institutions*, the first book to discuss both differential and integral calculus. She is also known for an equation she developed for a curve called the "Witch of Agnesi."

A by-product of the Fundamental Theorem of Calculus is that it makes a connection between integrals and derivatives. Integration is the process of calculating antiderivatives, while differentiation is the process of calculating derivatives. Thus, integration and differentiation are inverse processes. We can use the Fundamental Theorem of Calculus to evaluate definite integrals without having to use limits.

## Example 4 Area Under a Curve

**Use the Fundamental Theorem of Calculus to find the area of the region between the graph of each function and the *x*-axis on the given interval.**

**a.** $y = 4x^3$ **on the interval [1, 3], or** $\int_1^3 4x^3\,dx$.




First, find the antiderivative.

$$\int 4x^3\,dx = \frac{4x^{3+1}}{3+1} + C \quad \text{Constant Multiple of a Power}$$

$$= x^4 + C \quad \text{Simplify.}$$

Now evaluate the antiderivative at the upper and lower limits, and find the difference.

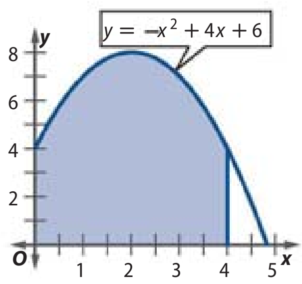
$$\int_1^3 4x^3\,dx = x^4 + C\Big|_1^3 \quad \text{Fundamental Theorem of Calculus}$$

$$= (3^4 + C) - (1^4 + C) \quad b = 3 \text{ and } a = 1$$

$$= 81 - 1 \text{ or } 80 \quad \text{Simplify.}$$

The area under the graph on the interval [1, 3] is 80 square units.

**b.** $y = -x^2 + 4x + 6$ **on the interval [0, 4], or** $\int_0^4 (-x^2 + 4x + 6)\,dx$




First, find the antiderivative.

$$\int (-x^2 + 4x + 6)\,dx$$

$$= -\frac{x^{2+1}}{2+1} + \frac{4x^{1+1}}{1+1} + \frac{6x^{0+1}}{0+1} + C \quad \text{Antiderivative Rule}$$

$$= -\frac{x^3}{3} + 2x^2 + 6x + C \quad \text{Simplify.}$$

Now evaluate the antiderivative at the upper and lower limits, and find the difference.

$$\int_0^4 (-x^2 + 4x + 6)\,dx = -\frac{x^3}{3} + 2x^2 + 6x + C\Big|_0^4 \quad \text{Fundamental Theorem of Calculus}$$

$$= \left(-\frac{(4)^3}{3} + 2(4)^2 + 6(4) + C\right) - \left(-\frac{(0)^3}{3} + 2(0)^2 + 6(0) + C\right) \quad b = 4 \text{ and } a = 0$$

$$= 34.67 - 0 \text{ or } 34.67 \quad \text{Simplify.}$$

The area under the graph on the interval [0, 4] is 34.67 square units.

### GuidedPractice

**Evaluate each definite integral.**

**4A.** $\int_2^5 3x^2\,dx$

**4B.** $\int_1^2 (16x^3 - 6x^2)\,dx$

Notice that when the antiderivative is evaluated at the upper and lower limits and the difference is found, we do not have an exact value for *C*. However, regardless of the value of the constant, because it is present in each antiderivative, the difference between the constants is 0. Therefore, when evaluating definite integrals using the Fundamental Theorem of Calculus, you can disregard the constant term when writing the antiderivative.

Before evaluating an integral, determine whether it is indefinite or definite.

**WatchOut!**

**Integrals** Although the constant $C$ can be excluded when evaluating definite integrals, it must be included when evaluating indefinite integrals because it is part of the antiderivative.

## Example 5 Indefinite and Definite Integrals

**Evaluate each integral.**

**a.** $\int (9x - x^3)\, dx$

This is an indefinite integral. Use the antiderivative rules to evaluate.

$$\int (9x - x^3)\, dx = \frac{9x^{1+1}}{1+1} - \frac{x^{3+1}}{3+1} + C$$ Constant Multiple of a Power

$$= \frac{9}{2}x^2 - \frac{x^4}{4} + C$$ Simplify.

**b.** $\int_2^3 (9x - x^3)\, dx$

This is a definite integral. Evaluate the integral using the given upper and lower limits.

$$\int_2^3 (9x - x^3)\, dx = \left(\frac{9}{2}x^2 - \frac{x^4}{4}\right)\Big|_2^3$$ Fundamental Theorem of Calculus

$$= \left(\frac{9}{2}(3)^2 - \frac{3^4}{4}\right) - \left[\frac{9}{2}(2)^2 - \frac{2^4}{4}\right]$$ $b = 3$ and $a = 2$

$$= 20.25 - 14 \text{ or } 6.25$$ Simplify.

The area under the graph on the interval [2, 3] is 6.25 square units.

**GuidedPractice**

**5A.** $\int (6x^2 + 8x - 3)\, dx$

**5B.** $\int_1^3 (-x^4 + 8x^3 - 24x^2 + 30x - 4)\, dx$

Notice that indefinite integrals give the antiderivative of a function while definite integrals not only give the antiderivative but also require it to be evaluated for given upper and lower limits. Thus, an indefinite integral gives a function, the antiderivative, for finding the area under a curve for *any* set of limits. The integral becomes definite when a set of limits is provided and the antiderivative can be evaluated.

## Example 6 Definite Integrals

**The work, in joules, required to stretch a certain spring a distance of 0.5 meter beyond its natural length is given by $\int_0^{0.5} 360x\, dx$. How much work is required?**

Evaluate the definite integral for the given upper and lower limits.

$$\int_0^{0.5} 360x\, dx = 180x^2\Big|_0^{0.5}$$ Constant Multiple of a Power and the Fundamental Theorem of Calculus

$$= 180(0.5)^2 - 180(0)^2$$ Let $b = 0.5$ and $a = 0$ and subtract.

$$= 45 - 0 \text{ or } 45$$ Simplify.

The work required is 45 joules.

**GuidedPractice**

**Find the work required to stretch a spring if it is defined by the following integrals.**

**6A.** $\int_0^{0.7} 476x\, dx$

**6B.** $\int_0^{1.4} 512x\, dx$

## Exercises

= Step-by-Step Solutions begin on page R29.

**Find all antiderivatives for each function.** (Examples 1 and 2)

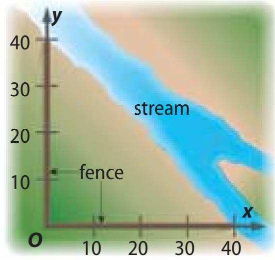
1. $f(x) = x^5$
2. $h(b) = -5b - 3$
3. $f(z) = \sqrt[3]{z}$
4. $n(t) = \frac{1}{4}t^4 - \frac{2}{3}t^2 + \frac{3}{4}$
5. $q(r) = \frac{3}{4}r^{\frac{2}{5}} + \frac{5}{8}r^{\frac{1}{3}} + r^{\frac{1}{2}}$
6. $w(u) = \frac{2}{3}u^5 + \frac{1}{6}u^3 - \frac{2}{5}u$
7. $g(a) = 8a^3 + 5a^2 - 9a + 3$
8. $u(d) = \frac{12}{d^5} + \frac{5}{d^3} - 6d^2 + 3.5$
9. $m(t) = 16t^3 - 12t^2 + 20t - 11$
10. $p(h) = 72h^8 + 24h^5 - 12h^2 + 14$

11. **CELL PHONE** Refer to the beginning of the lesson. Suppose the phone took exactly two seconds to drop from the balloon to the ground. (Example 3)
    a. Evaluate $s(t) = \int -32t\, dt$.
    b. Solve for $C$ in the position function $s(t)$ by substituting two seconds for $t$ and 0 for $s(t)$.
    c. How far from the ground is the phone after 1.5 seconds of falling?

**Evaluate each integral.** (Examples 4 and 5)

12. $\int (6m + 12m^3)\, dm$
13. $\int (20n^3 - 9n^2 - 18n + 4)\, dn$
14. $\int_1^4 2x^3\, dx$
15. $\int_2^5 (a^2 - a + 6)\, da$
16. $\int_1^2 (4g + 6g^2)\, dg$
17. $\int_2^{10} \left(\frac{2}{5}p^{\frac{1}{8}} + \frac{5}{4}p^{\frac{2}{7}} + \frac{1}{4}\right) dp$
18. $\int_1^3 \left(\frac{1}{2}h^2 + \frac{2}{3}h^3 - \frac{1}{5}h^4\right) dh$
19. $\int_0^2 (-v^4 + 2v^3 + 2v^2 + 6)\, dv$
20. $\int (3.4t^4 - 1.2t^3 + 2.3t - 5.7)\, dt$
21. $\int (14.2w^{6.1} - 20.1w^{5.7} + 13.2w^{2.3} + 3)\, dw$

22. **SURVEYOR** A plot of land has two perpendicular fences and a stream for borders as shown.




Suppose the edge of the stream that borders the plot can be modeled by $f(x) = -0.00005x^3 + 0.004x^2 - 1.04x + 40$, where the fences are the $x$ and $y$-axes, and $x$ is given in miles. Evaluate $\int_0^{40} f(x)\, dx$ to find the area of the land. (Example 6)

23. **INSECTS** The velocity of a flea's jump can be defined as $v(t) = -32t + 34$, where $t$ is given in seconds and velocity is given in feet per second. (Example 6)
    a. Find the position function $s(t)$ for the flea's jump. Assume that for $t = 0$, $s(t) = 0$.
    b. After a flea jumps, how long does it take before the flea lands on the ground?

24. **NATIONAL MONUMENT** A magician wants to make the Gateway Arch in St. Louis disappear. To attempt the illusion, he needs to cover the arch with a large canopy. Before constructing the canopy, the magician wants an approximation of the area under the arch. One equation that can be used to model the shape of the arch is $y = -\frac{x^2}{157.5} + 4x$, where $x$ is given in feet. Find the area under the arch. (Example 6)

25. **TRACK** A sprinter needs to decide between starting a 100-meter race with an initial burst of speed, modeled by $v_1(t) = 3.25t - 0.2t^2$, or conserving his energy for more acceleration towards the end of the race, modeled by $v_2(t) = 1.2t + 0.03t^2$, where velocity is measured in meters per second after $t$ seconds.
    a. Use a graphing calculator to graph both velocity functions on the same screen for $0 \le t \le 12$.
    b. Find the position function $s(t)$ for $v_1(t)$ and $v_2(t)$.
    c. How long does it take for the sprinter to finish a 100-meter race using each strategy?

**Evaluate each integral.**

26. $\int_{-3}^{1} 3\, dx$
27. $\int_{-1}^{2} (-x^2 + 10)\, dx$
28. $\int_{-6}^{-3} (-x^2 - 9x - 10)\, dx$
29. $\int_{-3}^{-1} (x^3 + 8x^2 + 21x + 20)\, dx$
30. $\int_{-2}^{-1} \left(\frac{x^5}{2} + \frac{5x^4}{4}\right) dx$
31. $\int_{-1}^{1} (x^4 - 2x^3 - 4x + 8)\, dx$

**32.** **CATAPULT** Pumpkins are launched by catapults at The Pumpkin Chunking World Championships in Delaware. A pumpkin launched by a catapult has a velocity of $v(t) = -32t + 120$ feet per second after $t$ seconds. After 3 seconds, the pumpkin has a height of 228 feet.

**a.** Find the maximum height of the pumpkin.

**b.** Find the pumpkin's velocity when it hits the ground.

**Evaluate each integral.**

**33.** $\int_x^2 (3t^2 + 8t)\, dt$

**34.** $\int_5^x (10t^4 - 12t^2 + 5)\, dt$

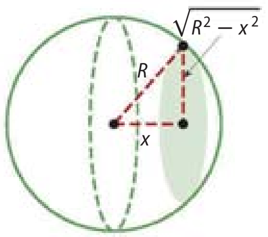
**35.** $\int_3^{2x} (4t^3 + 10t + 2)\, dt$

**36.** $\int_{-x}^6 (-9t^2 + 4t)\, dt$

**37.** $\int_x^{x^2} (16t^3 - 15t^2 + 7)\, dt$

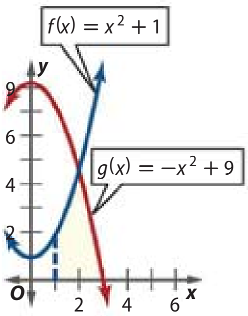
**38.** $\int_{2x}^{x+3} (3t^2 + 6t + 1)\, dt$

**39.** **SPHERE** The volume of a sphere with radius $R$ can be found by slicing the sphere vertically into circular cross sections and then integrating the areas.




The radius of each cross section is $\sqrt{R^2 - x^2}$. Thus, the area of a cross-section is equal to $\pi\left(\sqrt{R^2 - x^2}\right)^2$. Evaluate $\int_{-R}^{R} (\pi R^2 - \pi x^2)dx$ to find the volume of a sphere.

**40.** **AREA** Calculate the area bound by $f(x)$, $g(x)$, and the $x$-axis on the interval $1 \le x \le 3$.




**The integral** $\int_0^{n+0.5} x^k\, dx$ **gives a reasonable estimate of the sum of the series** $\sum_{i=1}^{n} i^k$. **Use the integral to estimate each sum and then find the actual sum.**

**41.** $\sum_{i=1}^{20} i^3$

**42.** $\sum_{i=1}^{100} i^2$

**43.** $\sum_{i=1}^{25} i^4$

**44.** $\sum_{i=1}^{30} i^5$

**45.** **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between the *total area* and the *signed area* of a region bound by a curve and the $x$-axis.

**a.** **GEOMETRIC** Graph $f(x) = x^3 - 6x^2 + 8x$, and shade the region bound by $f(x)$ and the $x$-axis for $0 \le x \le 4$.

**b.** **ANALYTICAL** Evaluate $\int_0^2 (x^3 - 6x^2 + 8x)\, dx$ and $\int_2^4 (x^3 - 6x^2 + 8x)\, dx$.

**c.** **VERBAL** Make a conjecture for the area found above and below the $x$-axis.

**d.** **ANALYTICAL** Calculate the *signed area* by evaluating $\int_0^4 (x^3 - 6x^2 + 8x)\, dx$. Then calculate the *total area* by evaluating

$$\left|\int_0^2 (x^3 - 6x^2 + 8x)\, dx\right| + \left|\int_2^4 (x^3 - 6x^2 + 8x)\, dx\right|.$$

**e.** **VERBAL** Make a conjecture for the difference between the *signed area* and the *total area*.

## H.O.T. Problems Use Higher-Order Thinking Skills

**46.** **CHALLENGE** Evaluate $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$, where $r$ is a constant. (*Hint*: Find the area between the graph of $y = \sqrt{r^2 - x^2}$ and the $x$-axis.)

**REASONING** **Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.**

**47.** $\int_a^b f(x)\, dx = \int_b^a f(x)\, dx$

**48.** $\int_a^b f(x)\, dx = \int_{-b}^{-a} f(x)\, dx$

**49.** $\int_a^b f(x)\, dx = \int_{|a|}^{|b|} f(x)\, dx$

**50.** **PROOF** Prove that for constants $n$ and $m$

$$\int_a^b (n + m)\, dx = \int_a^b n\, dx + \int_a^b m\, dx.$$

**51.** **REASONING** Describe the values of $f(x)$, $\sum_{i=1}^{n} f(x_i)\Delta x$, and $\int_a^b f(x)\, dx$, where the graph of $f(x)$ lies below the $x$-axis for $a \le x \le b$.

**52.** **WRITING IN MATH** Explain why the constant term $C$ in an antiderivative can be disregarded when evaluating a definite integral.

**53.** **WRITING IN MATH** Write an outline that could be used to describe the steps involved in finding the area of the region between the graph of $y = 6x^2$ and the $x$-axis on the interval [0, 2].

## Spiral Review

**Use limits to approximate the area between the graph of each function and the $x$-axis given by the definite integral.** (Lesson 12-5)

**54.** $\int_{-2}^{2} 14x^2\,dx$

**55.** $\int_{0}^{6} (x + 2)\,dx$

**Use the Quotient Rule to find the derivative of each function.** (Lesson 12-4)

**56.** $j(k) = \dfrac{k^8 - 7k}{2k^4 + 11k^3}$

**57.** $g(n) = \dfrac{2n^3 + 4n}{n^2 + 1}$

**58. FASHION** The average prices for three designer handbags on an online auction site are shown. (Lesson 11-4)

| Handbag Style | Average Price ($) |
|---|---|
| A | 135 |
| B | 145 |
| C | 152 |

**a.** If a random sample of 35 A-style handbags is selected, find the probability that the mean price is more than $138 if the standard deviation of the population is $9.

**b.** If a random sample of 40 C-style handbags is selected, find the probability that the mean price is between $150 and $155 if the standard deviation of the population is $12.

**59. BASEBALL** The average age of a major league baseball player is normally distributed with a mean of 28 and a standard deviation of 4 years. (Lesson 11-3)

**a.** About what percent of major league baseball players are younger than 24?

**b.** If a team has 35 players, about how many are between the ages of 24 and 32?

**60.** Find two pairs of polar coordinates for the point with the given rectangular coordinates (3, 8), if $-2\pi \leq \theta \leq 2\pi$. (Lesson 9-3)

## Skills Review for Standardized Tests

**61. SAT/ACT** In the figure, $C$ and $E$ are midpoints of the edges of the cube. A triangle is to be drawn with $R$ and $S$ as two of the vertices. Which of the following points should be the third vertex of the triangle if it is to have the largest possible perimeter?

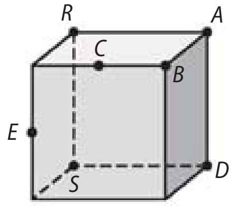
**A** $A$

**B** $B$

**C** $C$

**D** $D$

**E** $E$




**62.** The work in joules required to pump all of the water out of a 10-meter by 5-meter by 2-meter swimming pool is given by $\int_{0}^{2} 490{,}000x\,dx$. If you evaluate this integral, what is the required work?

**F** 980,000 J

**G** 985,000 J

**H** 990,000 J

**J** 995,000 J

**63. FREE RESPONSE** A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 3t + 10$ where $s$ is measured in meters and $t$ is measured in seconds.

**a.** Find the displacement of the particle during the first 4 seconds. That is, how far has the particle moved from its original starting position after 4 seconds?

**b.** Find the average velocity of the particle during the first 4 seconds.

**c.** Write an equation for the instantaneous velocity of the particle at any time $t$.

**d.** Find the instantaneous velocity of the particle when $t = 1$ and $t = 4$.

**e.** At what value for $t$ does $s(t)$ attain a minimum value?

**f.** What does the value of $t$ you found in part **d** represent in terms of the motion of the particle?

# CHAPTER 12 Study Guide and Review

## Chapter Summary

### KeyConcepts

**Estimating Limits Graphically** (Lesson 12-1)

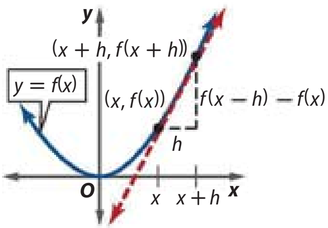
- The limit of a function $f(x)$ as $x$ approaches $c$ exists if and only if both one-sided limits exist and are equal.
- A limit of $f(x)$ as $x$ approaches $c$ does not exist if $f(x)$ approaches a different value from the left of $c$ than from the right of $c$, increases or decreases without bound from the left and/or right of $c$, or *oscillates* back and forth between two values.

**Evaluating Limits Algebraically** (Lesson 12-2)

- Limits of polynomial and rational functions can often be found by using direct substitution.
- If you evaluate a limit and reach the indeterminate form $\frac{0}{0}$, simplify the expression algebraically by factoring and dividing out a common factor or by rationalizing the numerator or denominator and then dividing out any common factors.

**Tangent Lines and Velocity** (Lesson 12-3)

- The instantaneous rate of change of the graph of $f(x)$ at the point $(x, f(x))$ is the slope $m$ of the tangent line given by $m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.




**Derivatives** (Lesson 12-4)

- The derivative of $f(x) = x^n$ is $f'(x)$ given by $f'(x) = nx^{n-1}$, where $n$ is a real number.

**Area Under a Curve and Integration** (Lesson 12-5)

- The area of a region under the graph of a function is $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, where $a$ and $b$ are the lower and upper limits, respectively, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$.

**The Fundamental Theorem of Calculus** (Lesson 12-6)

- The antiderivative of $f(x) = x^n$ is $F(x)$ given by $F(x) = \frac{x^{n+1}}{n+1} + C$, for some constant $C$.
- If $F(x)$ is the antiderivative of the continuous function $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

### KeyVocabulary

antiderivative (p. 784)
definite integral (p. 777)
derivative (p. 766)
difference quotient (p. 758)
differential equation (p. 766)
differential operator (p. 766)
differentiation (p. 766)
direct substitution (p. 748)
indefinite integral (p. 785)
indeterminate form (p. 749)
instantaneous rate of change (p. 758)
instantaneous velocity (p. 760)
integration (p. 777)
lower limit (p. 777)
one-sided limit (p. 738)
regular partition (p. 777)
right Riemann sum (p. 777)
tangent line (p. 758)
two-sided limit (p. 738)
upper limit (p. 777)

### VocabularyCheck

Choose the correct term to complete each sentence.

1. The slope of a nonlinear graph at a specific point is the ________ and can be represented by the slope of the tangent line to the graph at that point.
2. The process of evaluating an integral is called ________.
3. Limits of polynomial and rational functions can be found by ________, so long as the denominator of the rational function evaluated at $c$ is not 0.
4. The function $F(x)$ is the ________ of $f(x)$.
5. Since it is not possible to determine the limit of a function with 0 in the denominator, it is customary to describe the resulting fraction $\frac{0}{0}$ as having a(n) ________.
6. To find the limits of rational functions at infinity, divide the numerator and denominator by the ________ power of $x$ that occurs in the function.
7. The process of finding a derivative is called ________.
8. If a function is preceded by a(n) ________ $\frac{d}{dx}$, then you are to take the derivative of the function.
9. The velocity or speed achieved at any specific point in time is the ________.
10. The indefinite integral of $f(x)$ is defined by $\int f(x)\,dx =$ ________.

# Lesson-by-Lesson Review

## 12-1 Estimating Limits Graphically (pp. 736–745)

Estimate each limit using a graph. Support your conjecture using a table of values.

**11.** $\lim_{x \to 3} (2x - 7)$

**12.** $\lim_{x \to 1} (0.5x^4 + 3x^2 - 5)$

Estimate each one-sided or two-sided limit, if it exists.

**13.** $\lim_{x \to 2^+} \frac{x^2 + x - 6}{x - 2}$

**14.** $\lim_{x \to 4} \frac{x^2 + x + 20}{x - 4}$

Estimate each limit, if it exists.

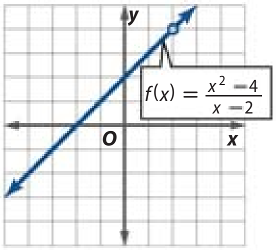
**15.** $\lim_{x \to 4} \frac{9}{x^2 - 8x + 16}$

**16.** $\lim_{x \to 2} \frac{x^2 - 7x - 10}{x - 2}$

### Example 1

Estimate $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ using a graph. Support your conjecture using a table of values.

**Analyze Graphically** The graph of $f(x) = \frac{x^2 - 4}{x - 2}$ suggests that as $x$ gets closer to 2, the corresponding function value approaches 4. Therefore, we can estimate that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ is 4.




**Support Numerically** Make a table of values, choosing $x$-values that approach 2 from either side. ✔

→ $x$ approaches 2 → ← $x$ approaches 2 ←

| $x$ | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 3.9 | 3.99 | 3.999 | | 4.001 | 4.01 | 4.1 |

## 12-2 Evaluating Limits Algebraically (pp. 746–756)

Use the properties of limits to evaluate each limit.

**17.** $\lim_{x \to 5} \frac{x^2 + 2x + 10}{x}$

**18.** $\lim_{x \to -1} (5x^2 - 2x + 12)$

Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.

**19.** $\lim_{x \to 25} \frac{x^2 + 1}{\sqrt{x} - 5}$

**20.** $\lim_{x \to 2} (-3x^3 - 2x^2 + 15)$

Evaluate each limit.

**21.** $\lim_{x \to -2} \frac{x + 2}{x^2 - 2x - 8}$

**22.** $\lim_{x \to \infty} \left(2 - 4x^3 + x^2\right)$

### Example 2

Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.

**a.** $\lim_{x \to 2} (2x^3 - x^2 + 4x + 1)$

This is the limit of a polynomial function. Therefore, direct substitution can be used to find the limit.

$$\lim_{x \to 2} (2x^3 - x^2 + 4x + 1) = 2(2)^3 - 2^2 + 4(2) + 1$$
$$= 16 - 4 + 8 + 1 \text{ or } 21$$

**b.** $\lim_{x \to -4} \frac{2x - 7}{2 - x^2}$

This is the limit of a rational function, the denominator of which is nonzero when $x = -4$. Therefore, direct substitution can be used to find the limit.

$$\lim_{x \to -4} \frac{2x - 7}{2 - x^2} = \frac{2(-4) - 7}{2 - (-4)^2} = \frac{-8 - 7}{2 - 16} \text{ or } \frac{15}{14}$$

# Study Guide and Review *Continued*

## 12-3 Tangent Lines and Velocity (pp. 758–764)

Find the slope of the lines tangent to the graph of each function at the given points.

**23.** $y = 6 - x$; $(-1, 7)$ and $(3, 3)$

**24.** $y = x^2 + 2$; $(0, 2)$ and $(-1, 3)$

The distance $d$ an object is above the ground $t$ seconds after it is dropped is given by $d(t)$. Find the instantaneous velocity of the object at the given value for $t$.

**25.** $y = -x^2 + 3x$

**26.** $y = x^3 + 4x$

Find the instantaneous velocity if the position of an object in feet is defined as $h(t)$ for given values of time $t$ given in seconds.

**27.** $h(t) = 15t + 16t^2$; $t = 0.5$

**28.** $h(t) = -16t^2 - 35t + 400$; $t = 3.5$

Find an equation for the instantaneous velocity $v(t)$ if the path of an object is defined as $h(t)$ for any point in time $t$.

**29.** $h(t) = 12t^2 - 5$ **30.** $h(t) = 8 - 2t^2 + 3t$

### Example 3

Find the slope of the line tangent to the graph of $y = x^2$ at (2, 4).

$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ Instantaneous Rate of Change Formula

$= \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$ $x = 2$

$= \lim_{h \to 0} \frac{(2 + h)^2 - 2^2}{h}$ $f(2 + h) = (2 + h)^2$ and $f(2) = 2^2$

$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$ Multiply.

$= \lim_{h \to 0} \frac{h(4 + h)}{h}$ Simplify and factor.

$= \lim_{h \to 0} (4 + h)$ Divide by $h$.

$= 4 + 0$ or 4 Sum Property of Limits and Limits of Constant and Identity Functions

Therefore, the slope of the line tangent to the graph of $y = x^2$ at (2, 4) is 4.

## 12-4 Derivatives (pp. 766–774)

Evaluate limits to find the derivative of each function. Then evaluate the derivative of each function for the given values of each variable.

**31.** $g(t) = -t^2 + 5t + 11$; $t = -4$ and 1

**32.** $m(j) = 10j - 3$; $j = 5$ and $-3$

Find the derivative of each function.

**33.** $p(v) = -9v + 14$ **34.** $z(n) = 4n^2 + 9n$

**35.** $t(x) = -3\sqrt[5]{x^6}$ **36.** $g(h) = 4h^{\frac{3}{4}} - 8h^{\frac{1}{2}} + 5$

Use the quotient rule to find the derivative of each function.

**37.** $f(m) = \frac{5 - 3m}{5 + 2m}$ **38.** $\frac{m(q) = 2q^4 - q^2 + 9}{q2 - 12}$

### Example 4

Find the derivative of $h(x) = \frac{x^2 - 5}{x^3 + 2}$.

Let $f(x) = x^2 - 5$, and $g(x) = x^3 + 2$. So, $h(x) = \frac{f(x)}{g(x)}$. Find the derivatives of $f(x)$ and $g(x)$

$f(x) = x^2 - 5$ Original equation

$f'(x) = 2x$ Power and Constant Rules

$g(x) = x^3 + 2$ Original equation

$g'(x) = 3x^2$ Power and Constant Rules

Use $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ to find the derivative of $h(x)$.

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ Quotient Rule

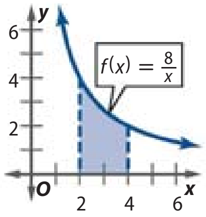
$= \frac{2x(x^3 + 2) - (x^2 - 5)\, 3x^2}{(x^3 + 2)^2}$ Substitution

$= \frac{-x^4 + 15x^2 + 4x}{(x^3 + 2)^2}$ Simplify.

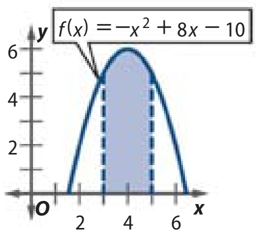
## 12-5 Area Under a Curve and Integration (pp. 775–783)

Approximate the area of the shaded region for each function using right endpoints and 5 rectangles.

**39.**




**40.**




Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.

**41.** $\int_1^2 2x^2\,dx$

**42.** $\int_0^3 (2x^3 - 1)\,dx$

**43.** $\int_0^2 (x^2 + x)\,dx$

**44.** $\int_1^4 (3x^2 - x)\,dx$

### Example 5

Use limits to find the area of the region between the graph of $y = 2x^2$ and the $x$-axis on the interval [0, 2], or $\int_0^2 2x^2\,dx$.

First, find $\Delta x$ and $x_i$.

$\Delta x = \frac{b-a}{n}$ — Formula for $\Delta x$

$\Delta x = \frac{2-0}{n}$ or $\frac{2}{n}$ — $b = 2$ and $a = 0$

$x_i = 0 + i\frac{2}{n}$ or $\frac{2i}{n}$ — $a = 0$ and $\Delta x = \frac{2}{n}$

$\int_0^2 2x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$ — $x_i = \frac{2i}{n}$ and $\Delta x = \frac{2}{n}$

$= \lim_{n\to\infty} \frac{4}{n}\left(\sum_{i=1}^{n} \frac{4i^2}{n^2}\right)$ — Simplify.

$= \lim_{n\to\infty} \frac{4}{n}\left(\frac{4}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right)$ — $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} \left(\frac{8(2n^2+3n+1)}{3n^2}\right)$ — Multiply and divide by $n$.

$= \lim_{n\to\infty} \left[\frac{8}{3} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right]$ — Factor and divide each term by $n^2$.

$= \frac{16}{3}$ or $5\frac{1}{3}$ — Limit theorems and simplify.

## 12-6 The Fundamental Theorem of Calculus (pp. 784–791)

Find all antiderivatives for each function.

**45.** $g(n) = 5n - 2$

**46.** $r(q) = -3q^2 + 9q - 2$

**47.** $m(t) = 6t^3 - 12t^2 + 2t - 11$

**48.** $p(h) = 7h^6 + 4h^5 - 12h^3 - 4$

Evaluate each integral.

**49.** $\int 8x^2\,dx$

**50.** $\int (2x^2 - 4)\,dx$

**51.** $\int_3^5 (2x^2 - 4 + 5x^3 + 3x^4)\,dx$

**52.** $\int_1^4 (-x^2 + 4x - 2x^3 + 5x^5)\,dx$

### Example 6

Find all antiderivatives for each function.

**a.** $f(x) = \frac{4}{x^5}$

$f(x) = 4x^{-5}$ — Rewrite with a negative exponent.

$F(x) = \frac{4x^{-5+1}}{-5+1} + C$ — Constant Multiple of a Power

$= -1x^{-4} + C$ or $-\frac{1}{x^4} + C$ — Simplify.

**b.** $f(x) = x^2 - 7$

$f(x) = x^2 - 7$ — Original equation

$= x^2 - 7x^0$ — Rewrite the function so each term has a power of $x$.

$F(x) = \frac{x^{2+1}}{2+1} - \frac{7x^{0+1}}{0+1} + C$ — Antiderivative Rule

$= \frac{1}{3}x^3 - 7x + C$ — Simplify.

# Study Guide and Review *Continued*

## Applications and Problem Solving

**53. STAMPS** Suppose the value $v$ of a stamp in dollars after $t$ years can be represented as $v(t) = \frac{450}{5 + 25(0.4)^t}$. (Lesson 12-1)

**a.** Complete the following table.

| Years | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Value | | | | |

**b.** Graph the function for $0 \leq t \leq 10$.

**c.** Use the graph to estimate $\lim_{t \to \infty} v(t)$, if it exists.

**d.** Explain the relationship between the limit of the function and the value of the stamp.

**54. ANIMALS** A wildlife conservation's population $P$ in hundreds after $t$ years can be estimated by $P(t) = \frac{120}{1 + 24e^{-0.25t}}$. (Lesson 12-1)

**a.** Use a graphing calculator to graph the function for $0 \leq t \leq 50$.

**b.** Estimate $\lim_{t \to \infty} \frac{120}{1 + 24e^{-0.25t}}$, if it exists.

**c.** Interpret your results from part **b**.

**55. COLLECTORS** The value of Eldin's coin collection has been increasing every year over the last five years. Following this trend, the value $v$ of the coins after $t$ years can be modeled by $v(t) = \frac{800t - 21}{4t + 19}$. (Lesson 12-2)

**a.** Find $\lim_{t \to \infty} v(t)$.

**b.** What does the limit of the function imply about the value of Eldin's coin collection? Do you agree?

**c.** After 10 years, a coin dealer offers Eldin \$300 for his collection. Should Eldin sell his collection? Explain your answer.

**56. ROCKETS** A rocket is launched into the sky with an upward velocity of 150 feet per second. Suppose the height $d$ of the rocket in feet $t$ seconds after it is launched is modeled by $d(t) = -16t^2 + 150t + 8.2$. (Lesson 12-3)

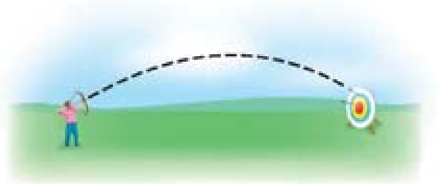
**a.** Find an equation for the instantaneous velocity $v(t)$ of the rocket.

**b.** How fast is the rocket traveling 1.5 seconds after it is launched?

**c.** For what value of $t$ will the rocket reach its maximum height?

**d.** What is the maximum height that the rocket will reach?

**57. ARCHERY** An archer shoots an arrow with a velocity of 35 feet per second towards the target. Suppose the height $s$ of the arrow in feet $t$ seconds after it is shot is defined as $s(t) = -16t^2 + 35t + 1.5$. (Lesson 12-3)



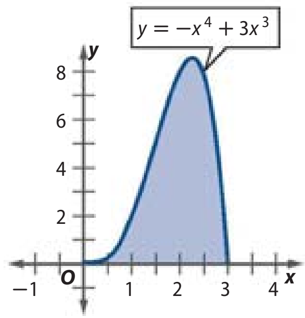
**a.** Find an equation for the instantaneous velocity $v(t)$ of the arrow.

**b.** How fast is the arrow traveling 0.5 second after it is shot?

**c.** For what value of $t$ will the arrow reach its maximum height?

**d.** What is the maximum height of the arrow?

**58. DESIGN** The owner of a ski resort is designing a new logo to put on his employees' uniforms. The design is in the shape of the region shown in the figure. If this part of the design is to be sewn on to the uniforms, how much material is required if $x$ is given in inches? (Lesson 12-6)




**59. FROGS** A frog can jump with a velocity modeled by $v(t) = -32t + 26$, where $t$ is given in seconds and velocity is given in feet per second. (Lesson 12-6)

**a.** Find the position function $s(t)$ for the frog's jump. Assume that for $t = 0$, $s(t) = 0$.

**b.** How long will the frog stay in the air when it jumps?

**60. BIRDS** A cardinal in a tree 20 feet above the ground drops some food. The instantaneous velocity of his food can be defined as $v(t) = -32t$, where $t$ is given in seconds and velocity is measured in feet per second. (Lesson 12-6)

**a.** Find the position function $s(t)$ of the dropped food.

**b.** Find how long it will take for the food to hit the ground.

# CHAPTER 12 Practice Test

Estimate each one-sided or two-sided limit, if it exists.

**1.** $\lim_{x \to 0^+} \sqrt{x+4} - 8$

**2.** $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

Estimate each limit, if it exists.

**3.** $\lim_{x \to 7} \dfrac{6}{x - 7}$

**4.** $\lim_{x \to \infty} x^3 + 5x^2 - 2x + 21$

**5.** **ELECTRONICS** The average cost $C$ in dollars of $x$ number of personal digital assistants can be modeled by $C(x) = \dfrac{100x + 7105}{x}$.

**a.** Determine the limit of the function as $x$ approaches infinity.

**b.** Interpret the results from part **a**.

Use direct substitution, if possible, to evaluate each limit. If not possible, explain why not.

**6.** $\lim_{x \to 5} \dfrac{x^2}{\sqrt{x-4} - 2}$

**7.** $\lim_{x \to 9} (2x^3 - 12x + 3)$

**8.** **SCHOOL** The number of students $S$ absent with the flu after $t$ days at a school can be modeled by $S(t) = \dfrac{2000t^2 + 4}{1 + 50t^2}$.

**a.** How many students are initially sick?

**b.** How many students will eventually be sick?

Evaluate each limit.

**9.** $\lim_{x \to \infty} (x^2 - 7x + 2)$

**10.** $\lim_{x \to \infty} (2x^3 - 8x^2 - 5)$

**11.** $\lim_{x \to \infty} \dfrac{2x^3 - x - 1}{-x^4 + 7x^3 + 4}$

**12.** $\lim_{x \to \infty} \dfrac{\sqrt{25 + x} - 4}{x}$

**13.** **MULTIPLE CHOICE** Evaluate $\lim_{x \to 0} \dfrac{\frac{1}{x+3} - \frac{1}{3}}{x}$.

**A** $-\frac{1}{9}$

**B** 0

**C** $\frac{1}{9}$

**D** does not exist

Find the slope of the lines tangent to the graph of each function at the given points.

**14.** $y = x^2 + 2x - 8$; $(-5, 7)$ and $(-2, -8)$

**15.** $y = \dfrac{4}{x^3} + 2$; $(-1, -2)$ and $\left(2, \frac{5}{2}\right)$

**16.** $y = (2x + 1)^2$; $(-3, 25)$ and $(0, 1)$

Find an equation for the instantaneous velocity $v(t)$ if the path of an object is defined as $h(t)$ for any point in time $t$.

**17.** $h(t) = 9t + 3t^2$

**18.** $h(t) = 10t^2 - 7t^3$

**19.** $h(t) = 3t^3 - 2 + 4t$

Find the derivative of each function.

**20.** $f(x) = -3x - 7$

**21.** $b(c) = 4c^{\frac{1}{2}} - 8c^{\frac{2}{3}} + 5c^{\frac{4}{5}}$

**22.** $w(y) = 3y^{\frac{4}{3}} + 6y^{\frac{1}{2}}$

**23.** $g(x) = (x^2 - 4)(2x - 5)$

**24.** $h(t) = \dfrac{t^3 + 4t^2 + t}{t^2}$

**25.** **FOOTBALL** The marginal cost $c$ of producing $x$ footballs is represented by $c(x) = 15 - 0.005x$.

**a.** Determine the function that represents the actual cost function.

**b.** Determine the cost of increasing the daily production from 1500 footballs to 2000 footballs.

Use limits to find the area between the graph of each function and the $x$-axis given by the definite integral.

**26.** $\int_1^4 (x^2 - 3x + 4)\, dx$

**27.** $\int_3^8 10x^4\, dx$

**28.** $\int_2^5 (7 - 2x + 4x^2)\, dx$

Find all antiderivatives for each function.

**29.** $d(a) = 4a^3 + 9a^2 - 2a + 8$

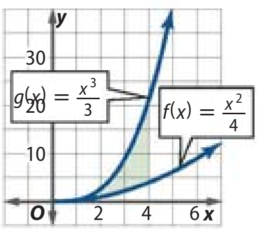
**30.** $w(z) = \frac{3}{4}z^4 + \frac{1}{6}z^2 - \frac{2}{5}$

Evaluate each integral.

**31.** $\int (5x^3 - 6x^2 + 4x - 3)\, dx$

**32.** $\int_1^4 (x^2 + 4x - 2)\, dx$

**33.** **AREA** Calculate the area bound by $f(x)$ and $g(x)$ on the interval $2 \leq x \leq 4$.




**F** $17\frac{5}{12}$

**G** $17\frac{1}{3}$

**H** $15\frac{1}{3}$

**J** 16

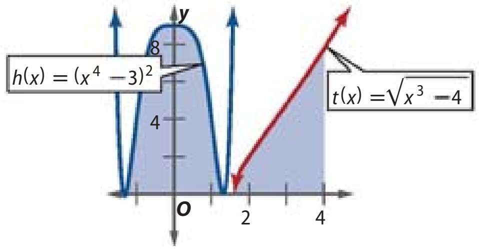
CHAPTER 12

# Connect to AP Calculus
# The Chain Rule

## Objective

- Differentiate composite functions using the Chain Rule.

Learning to differentiate polynomials by using the Power Rule led to the evaluation of definite integrals. This allowed for the calculation of the area found between a curve and the $x$-axis. However, our integration work was limited to basic polynomial functions. So, how can we calculate the areas found between the $x$-axis and the curves defined by composite functions such as $h(x) = (x^4 - 3)^2$ or $t(x) = \sqrt{x^3 - 4}$?




Just as in Lesson 12-4, you must learn how to differentiate these functions before you can integrate. Begin with $h(x)$. You can use the Product Rule to develop the derivative.

$$h(x) = (x^4 - 3)^2 \quad \text{Original equation}$$
$$= (x^4 - 3)(x^4 - 3) \quad \text{Rewrite without the power.}$$
$$h'(x) = 4x^3(x^4 - 3) + (x^4 - 3)4x^3 \quad \text{Product Rule}$$
$$= 2(x^4 - 3)4x^3 \quad \text{Simplify.}$$

Although the derivative of $h(x)$ can be simplified further, leave it as shown in order to derive a rule for composite functions.

### Activity 1 Derivative of a Composite Function

**Find the derivative of $k(x) = (x^4 - 3)^3$.**

**Step 1** Rewrite $k(x)$ to include the factor $(x^4 - 3)^2$.

$$k(x) = (x^4 - 3)(x^4 - 3)^2$$

**Step 2** Let $m(x) = (x^4 - 3)$ and $n(x) = (x^4 - 3)^2$. Calculate the derivative of each function.

$m'(x) = 4x^3$ Power Rule $\qquad n'(x) = 2(x^4 - 3)4x^3$ Derived above

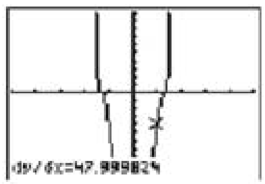
**Step 3** Use the Product Rule to find $k'(x)$.

$$k'(x) = m'(x)n(x) + m(x)n'(x)$$
$$= 4x^3(x^4 - 3)^2 + (x^4 - 3)2(x^4 - 3)4x^3 \quad \text{Substitution}$$
$$= (x^4 - 3)^2 4x^3 + 2(x^4 - 3)^2 4x^3 \quad \text{Simplify.}$$
$$= 3(x^4 - 3)^2 4x^3 \quad \text{Add.}$$

Therefore, $k'(x) = 3(x^4 - 3)^2 4x^3$.

**CHECK** You can use a graphing calculator to evaluate the derivative of a function at a point. Graph $k(x)$, and from the CALC menu, select 6:dy/dx. After the screen returns to the graphing window, press 1 and then ENTER. The value of $k'(1) = 48$. Now verify the answer found in Step 3 by substituting $x = 1$ into $k'(x)$.

$k'(1) = 3(1^4 - 3)^2 4(1)^3$ or 48 ✔




[−5, 5] scl: 1 by [−20, 20] scl: 2

### Analyze the Results

1. Make a conjecture as to why $h'(x)$ and $k'(x)$ have $4x^3$ as a factor.
2. Make a conjecture as to why $h'(x)$ has 2 as a factor and $k'(x)$ has 3 as a factor.
3. Without rewriting $p(x)$ as a product, find the derivative of $p(x) = (x^4 - 3)^4$.

Notice that for the derivatives of $h(x)$ and $k(x)$, each expression was multiplied by its exponent, 1 was subtracted from its exponent, and then each expression was multiplied by the derivative of the expression inside the parentheses.

Multiply by exponent. Subtract 1 from exponent.

$h(x) = (x^4 - 3)^2$

Multiply by derivative.

$h'(x) = \mathbf{2}(x^4 - 3)^{2 - \mathbf{1}}\mathbf{4x^3}$

$= 2(x^4 - 3)4x^3$

Multiply by exponent. Subtract 1 from exponent.

$k(x) = (x^4 - 3)^3$

Multiply by derivative.

$k'(x) = \mathbf{3}(x^4 - 3)^{3 - \mathbf{1}}\mathbf{4x^3}$

$= 3(x^4 - 3)^2 4x^3$

Recall from Lesson 1-6 that we can decompose a function. For example, $h(x) = (x^4 - 3)^2$ can be written as $h(x) = f[g(x)]$, where $f(x) = x^2$ and $g(x) = (x^4 - 3)$. This can be used to develop the Chain Rule.

## KeyConcept Function

**Words** The derivative of a composite function $f[g(x)]$ is the derivative of the outer function $f$ evaluated at the inner function $g$ multiplied by the derivative of the inner function $g$.

**Symbols** If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then

$$\frac{d}{dx} f[g(x)] = f'[g(x)]g'(x).$$

## Activity 2 Derivative of a Composite Function

**Find the derivative of $t(x) = \sqrt{x^3 - 4}$.**

**Step 1** Decompose $t(x)$ into two functions $f$ and $g$.

$f(x) = \sqrt{x}$ $\qquad$ $g(x) = x^3 - 4$

**Step 2** Find $f'(x)$ and $g'(x)$.

$f(x) = \sqrt{x}$ or $x^{\frac{1}{2}}$ — Original equation — $g(x) = x^3 - 4$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ — Power Rule — $g'(x) = 3x^2$

**Step 3** Substitute into the Chain Rule.

$t'(x) = f'[g(x)]\,g'(x)$ — Chain Rule

$= \frac{1}{2}(x^3 - 4)^{-\frac{1}{2}}\,3x^2$ — $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $g(x) = x^3 - 4$, and $g'(x) = 3x^2$

$= \dfrac{3x^2}{2\sqrt{x^3 - 4}}$ — Simplify.

$= \dfrac{3x^2\sqrt{x^3 - 4}}{2(x^3 - 4)}$ — Simplify

### Analyze the Results

4. Can $t'(x)$ be found using the same method that was used earlier to find $h'(x)$? Explain.

**WatchOut!**

Substitution To help with the Chain Rule, $g(x)$ can be represented by $u$. For example, if $y = f[g(x)]$, then $y = f(u)$. The derivative of y becomes $y' = f'(u) \cdot u'$.

## Model and Apply

**Find the derivative of each function.**

5. $s(t) = (t^2 - 1)^4$
6. $b(x) = (1 - 5x)^6$
7. $c(r) = (3r - 2r^2)^3$
8. $h(x) = (x^3 + x - 1)^3$
9. $f(x) = \sqrt{100 + 8x}$
10. $g(m) = \sqrt{m^2 + 4}$

# Student Handbook

This **Student Handbook** can help you answer these questions.

## What if I Need to Recall Characteristics of a Specific Function?



## What if I Need to Check a Homework Answer?



## What if I Forget a Vocabulary Word?



## What if I Need to Find Something Quickly?



## What if I Forget a Formula?



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