CHAPTER 10

# Sequences and Series

## Then

In **Chapters 1–4**, you modeled data using various types of functions.

## Now

In **Chapter 10,** you will:

- Relate sequences and functions.
- Represent and calculate sums of series with sigma notation.
- Use arithmetic and geometric sequences and series.
- Prove statements by using mathematical induction.
- Expand powers by using the Binomial Theorem.

## Why?

**MARCHING BAND** Sequences and series can be used to predict patterns. For example, arithmetic sequences can be used to determine the number of band members in a specified row of a pyramid formation.

**PREREAD** Use the text on this page to predict the organization of Chapter 10.

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# Get Ready for the Chapter

**Diagnose Readiness** You have two options for checking Prerequisite Skills.

**1 Textbook Option** Take the Quick Check below.

## QuickCheck

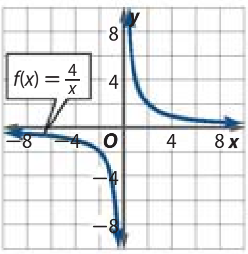
**Expand each binomial.** (Prerequisite Skill)

**1.** $(x + 3)^3$

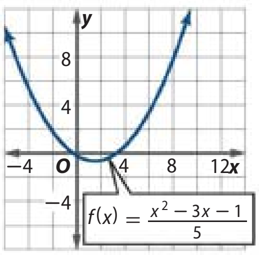
**2.** $(2x - 1)^4$

**Use the graph of each function to describe its end behavior. Support the conjecture numerically.** (Lesson 1-3)

**3.**




**4.**




**Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.** (Lesson 3-1)

**5.** $f(x) = 3^{-x}$

**6.** $r(x) = 5^{-x}$

**7.** $h(x) = 0.1^{x+2}$

**8.** $k(x) = -2^x$

**Evaluate each expression.** (Lesson 3-2)

**9.** $\log_2 16$

**10.** $\log_{10} 10$

**11.** $\log_6 \frac{1}{216}$

**12.** **MUSIC** The table shows the type and number of CDs that Adam and Lindsay bought. Write and solve a system of equations to determine the price of each type of CD. (Lesson 6-1)

| Buyer | New CD | Used CD | Price ($) |
|---|---|---|---|
| Adam | 2 | 5 | 49 |
| Lindsay | 3 | 4 | 56 |

**2 Online Option** Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

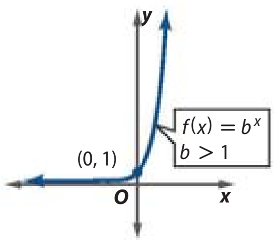
## NewVocabulary

| English | | Español |
|---|---|---|
| sequence | p. 590 | sucesión |
| term | p. 590 | término |
| finite sequence | p. 590 | sucesión finita |
| infinite sequence | p. 590 | sucesión infinita |
| recursive sequence | p. 591 | sucesión recursiva |
| explicit sequence | p. 591 | sucesión explícita |
| Fibonacci sequence | p. 591 | sucesión de Fibonacci |
| converge | p. 592 | converge |
| diverge | p. 592 | diverge |
| series | p. 593 | serie |
| finite series | p. 593 | serie finita |
| *n*th partial sum | p. 593 | suma parcial enésima |
| infinite series | p. 593 | serie infinita |
| sigma notation | p. 594 | notación de suma |
| arithmetic sequence | p. 599 | sucesión aritmética |
| common difference | p. 599 | diferencia común |
| arithmetic series | p. 602 | serie aritmética |
| common ratio | p. 608 | razón común |
| geometric means | p. 611 | medios geométricos |
| geometric series | p. 611 | serie geométrica |
| binomial coefficients | p. 628 | coeficientes binomiales |
| power series | p. 636 | serie de potencias |

## ReviewVocabulary

exponential function p. 158 funciones exponenciales a function in which the base is a constant and the exponent is a variable

**Exponential Growth Function**



LESSON 10-1

# Sequences, Series, and Sigma Notation

**Then**

- You used functions to generate ordered pairs and used graphs to analyze end behavior. (Lesson 1-1 and 1-3)

**Now**

1. Investigate several different types of sequences.
2. Use sigma notation to represent and calculate sums of series.

**Why?**

- Khari developed a Web site where students at her high school can post their own social networking Web pages. A student at the high school is given a free page if he or she refers the Web site to five friends. The site starts with one page created by Khari, who in turn, refers five friends that each create a page. Those five friends refer five more people each, all of whom develop pages, and so on.

**NewVocabulary**

sequence
term
finite sequence
infinite sequence
recursive sequence
explicit sequence
Fibonacci sequence
converge
diverge
series
finite series
*n*th partial sum
infinite series
sigma notation

## 1 Sequences

In mathematics, a **sequence** is an ordered list of numbers. Each number in the sequence is known as a **term**. A **finite sequence**, such as 1, 3, 5, 7, 9, 11, contains a finite number of terms. An **infinite sequence**, such as 1, 3, 5, 7, …, contains an infinite number of terms.

Each term of a sequence is a function of its position. Therefore, an infinite sequence is a function whose domain is the set of natural numbers and can be written as $f(1) = a_1, f(2) = a_2, f(3) = a_3, \ldots,$ $f(n) = a_n, \ldots,$ where $a_n$ denotes the $n$th term. If the domain of the function is only the first $n$ natural numbers, the sequence is finite.

Infinitely many sequences exist with the same first few terms. To sufficiently define a *unique* sequence, a formula for the $n$th term or other information *must* be given. When defined *explicitly*, an **explicit formula** gives the $n$th term $a_n$ as a function of $n$.

### Example 1 Find Terms of Sequences

**a. Find the next four terms of the sequence 2, 7, 12, 17, … .**

The $n$th term of this sequence is not given. One possible pattern is that each term is 5 greater than the previous term. Therefore, a sample answer for the next four terms is 22, 27, 32, and 37.

**b. Find the next four terms of the sequence 2, 5, 10, 17, … .**

The $n$th term of this sequence is not given. If we subtract each term from the term that follows, we start to see a possible pattern.

$a_2 - a_1 = 5 - 2$ or 3 $\qquad a_3 - a_2 = 10 - 5$ or 5 $\qquad a_4 - a_3 = 17 - 10$ or 7

It appears that each term is generated by adding the next successive odd number. However, looking at the pattern, it may also be determined that each term is 1 more than each perfect square, or $a_n = n^2 + 1$. Using either pattern, a sample answer for the next four terms is 26, 37, 50, and 65.

**c. Find the first four terms of the sequence given by $a_n = 2n(-1)^n$.**

Use the explicit formula given to find $a_n$ for $n = 1, 2, 3,$ and 4.

$a_1 = 2 \cdot 1 \cdot (-1)^1$ or $-2$ $\quad n = 1 \qquad a_2 = 2 \cdot 2 \cdot (-1)^2$ or 4 $\quad n = 2$

$a_3 = 2 \cdot 3 \cdot (-1)^3$ or $-6$ $\quad n = 3 \qquad a_4 = 2 \cdot 4 \cdot (-1)^4$ or 8 $\quad n = 4$

The first four terms in the sequence are −2, 4, −6, and 8.

**GuidedPractice**

**Find the next four terms of each sequence.**

**1A.** 32, 16, 8, 4, …

**1B.** 1, 2, 4, 7, 11, 16, 22, …

**1C.** Find the first four terms of the sequenc e given by $a_n = n^3 - 10$.

Sequences can also be defined *recursively*. Recursively defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms. The formula defining the $n$th term of the sequence is called a **recursive formula** or a *recurrence relation*.

**StudyTip**

**Notation** The term denoted $a_n$ represents the $n$th term of a sequence. The term denoted $a_{n-1}$ represents the term immediately before $a_n$. The term $a_{n-2}$ represents the term two terms before $a_n$.

## Example 2 Recursively Defined Sequences

**Find the fifth term of the recursively defined sequence $a_1 = 1$, $a_n = a_{n-1} + 2n - 1$, where $n \geq 2$.**

Since the sequence is defined recursively, all the terms before the fifth term must be found first. Use the given first term, $a_1 = 2$, and the recursive formula for $a_n$.

$$\begin{aligned} a_2 &= a_{2-1} + 2(2) - 1 && n = 2 \\ &= a_1 + 3 && \text{Simplify.} \\ &= 2 + 3 \text{ or } 5 && a_1 = 2 \end{aligned}$$

$$\begin{aligned} a_3 &= a_{3-1} + 2(3) - 1 && n = 3 \\ &= a_2 + 5 \text{ or } 10 && a_2 = 5 \end{aligned}$$

$$\begin{aligned} a_4 &= a_{4-1} + 2(4) - 1 && n = 4 \\ &= a_3 + 7 \text{ or } 17 && a_3 = 10 \end{aligned}$$

$$\begin{aligned} a_5 &= a_{5-1} + 2(5) - 1 && n = 5 \\ &= a_4 + 9 \text{ or } 26 && a_4 = 17 \end{aligned}$$

**GuidedPractice**

**Find the sixth term of each sequence.**

**2A.** $a_1 = 3$, $a_n = (-2)a_{n-1}$, $n \geq 2$

**2B.** $a_1 = 8$, $a_n = 2a_{n-1} - 7$, $n \geq 2$

The **Fibonacci sequence** describes many patterns found in nature. This sequence is often defined recursively.

Nacivet/Taxi/Getty Images

**Real-WorldLink**

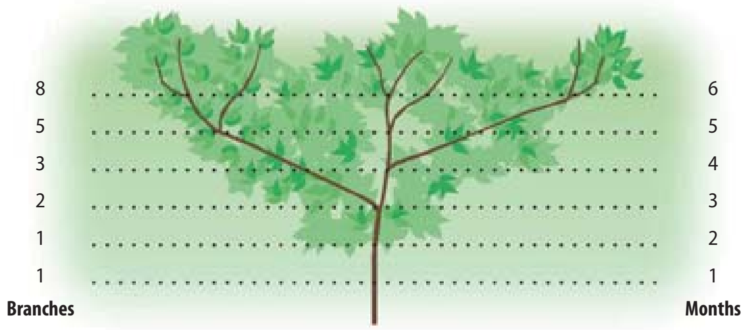
Along with being found in flower petals, sea shells, and the bones in a human hand, Fibonacci sequences can also be found in pieces of art, music, poetry, and architecture.

**Source:** *Universal Principles of Design*

## Real-World Example 3 Fibonacci Sequence

**NATURE Suppose that when a plant first starts to grow, the stem has to grow for two months before it is strong enough to support branches. At the end of the second month, it sprouts a new branch and will continue to sprout one new branch each month. The new branches also each grow for two months and then start to sprout one new branch each month. If this pattern continues, how many branches will the plant have after 10 months?**

During the first two months, there will only be one branch, the stem. At the end of the second month, the stem will produce a new branch, making the total for the third month two branches. The new branch will grow and develop two months before producing a new branch of its own, but the original branch will now produce a new branch each month.



**WatchOut!**

**Notation** The first term of a sequence is occasionally denoted as $a_0$. When this occurs, the domain of the function describing the sequence is the set of whole numbers.

The following table shows the pattern.

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Branches | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |

Each term is the sum of the previous two terms. This pattern can be written as the recursive formula $a_0 = 1, a_1 = 1, a_n = a_{n-2} + a_{n-1}$, where $n \geq 2$.

**GuidedPractice**

**3. NATURE** How many branches will a plant like the one described in Example 3 have after 15 months if no branches are removed?

In Lesson 1-3, you examined the end behavior of the graphs of functions. You learned that as the domains of some functions approach $\infty$, the ranges approach a unique number called a limit. As a function, an infinite sequence may also have a limit. If a sequence has a limit such that the terms approach a unique number, then it is said to **converge**. If not, the sequence is said to **diverge**.

**TechnologyTip**

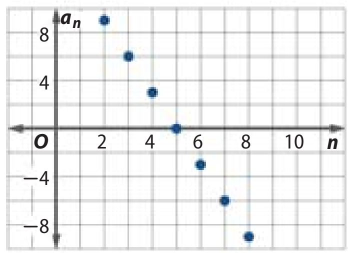
**Convergent or Divergent Sequences** If an explicit formula for a sequence is known, you can enter the formula in the Y= menu of a graphing calculator and graph the related function. Analyzing the end behavior of the graph can help you to determine whether the sequence is convergent or divergent.

## Example 4 Convergent and Divergent Sequences

**Determine whether each sequence is *convergent* or *divergent*.**

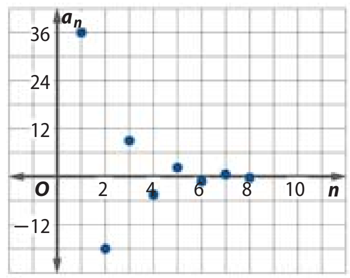
**a.** $a_n = -3n + 12$

The first eight terms of this sequence are 12, 9, 6, 3, 0, −3, −6, and −9. From the graph at the right, you can see that $a_n$ does not approach a finite number. Therefore, this sequence is divergent.




**b.** $a_1 = 36, a_n = -\frac{1}{2}a_{n-1}, a \geq 2$

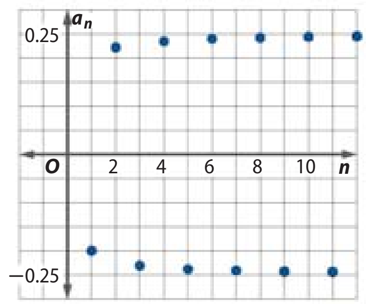
The first eight terms of this sequence are 36, −18, 9, −4.5, 2.25, −1.125, 0.5625, −0.28125, and 0.140625. From the graph at the right, you can see that $a_n$ approaches 0 as $n$ increases. This sequence has a limit and is therefore convergent.




**c.** $a_n = \dfrac{(-1)^n \cdot n}{4n + 1}$

The first twelve terms of this sequence are given or approximated below.

| | |
|---|---|
| $a_1 = -0.2$ | $a_2 \approx 0.222$ |
| $a_3 \approx -0.231$ | $a_4 \approx 0.235$ |
| $a_5 \approx -0.238$ | $a_6 = 0.24$ |
| $a_7 \approx -0.241$ | $a_8 \approx 0.242$ |
| $a_9 \approx -0.243$ | $a_{10} \approx 0.244$ |
| $a_{11} \approx -0.244$ | $a_{12} \approx 0.245$ |




It appears that when $n$ is odd, $a_n$ approaches $-\frac{1}{4}$, and when $n$ is even, $a_n$ approaches $\frac{1}{4}$. Since $a_n$ does not approach one particular value, the sequence has no limit. Therefore, the sequence is divergent.

**GuidedPractice**

**4A.** $a_n = \dfrac{64}{2n}$ **4B.** $a_1 = 9, a_n = a_{n-1} + 4$ **4C.** $a_n = 3(-1)^n$

## 2 Series

A **series** is the indicated sum of all of the terms of a sequence. Like sequences, series can be finite or infinite. A **finite series** is the indicated sum of all the terms of a finite sequence, and an **infinite series** is the indicated sum of all the terms of an infinite sequence.

| | Sequence | Series |
|---|---|---|
| **Finite** | 1, 3, 5, 7, 9 | $1 + 3 + 5 + 7 + 9$ |
| **Infinite** | 1, 3, 5, 7, 9, … | $1 + 3 + 5 + 7 + 9 + \ldots$ |

The sum of the first $n$ terms of a series is called the **$n$th partial sum** and is denoted $S_n$. The $n$th partial sum of any series can be found by calculating each term up to the $n$th term and then finding the sum of those terms.

### Example 5 The $n$th Partial Sum

**a. Find the fourth partial sum of $a_n = (-2)^n + 3$.**

Find the first four terms.

$a_1 = (-2)^1 + 3$ or 1 $\quad n = 1$

$a_2 = (-2)^2 + 3$ or 7 $\quad n = 2$

$a_3 = (-2)^3 + 3$ or $-5$ $\quad n = 3$

$a_4 = (-2)^4 + 3$ or 19 $\quad n = 4$

The fourth partial sum is $S_4 = 1 + 7 + (-5) + 19$ or 22.

**b. Find $S_3$ of $a_n = \frac{4}{10^n}$.**

Find the first three terms.

$a_1 = \frac{4}{10^1}$ or 0.4 $\quad n = 1$

$a_2 = \frac{4}{10^2}$ or 0.04 $\quad n = 2$

$a_3 = \frac{4}{10^3}$ or 0.004 $\quad n = 3$

The third partial sum is $S_3 = 0.4 + 0.04 + 0.004$ or 0.444.

**GuidedPractice**

**5A.** Find the sixth partial sum of $a_1 = 8$, $a_n = 0.5(a_{n-1})$, $n \geq 2$.

**5B.** Find the seventh partial sum of $a_n = 3\left(\frac{1}{10}\right)^n$.

**StudyTip**

**Converging Infinite Sequences** While it is necessary for an infinite sequence to converge to 0 in order for the corresponding infinite series to have a sum, it is not sufficient. Some infinite sequences converge to 0 and the corresponding infinite series still do not have sums.

Since an infinite series does not have a finite number of terms, you might assume that an infinite series has no sum $S$. This is true for the series below.

| Infinite Sequence | Infinite Series | Sequence of First Four Partial Sums |
|---|---|---|
| 1, 4, 7, 10, … | $1 + 4 + 7 + 10 + \ldots$ | 1, 5, 12, 22, … |

However, some infinite series do have sums. For an infinite series to have a fixed sum $S$, the infinite sequence associated with this series must converge to 0. Notice the sequence of partial sums in the infinite series below appears to approach a sum of $0.\overline{1}$ or $\frac{1}{9}$.

| Infinite Sequence | Infinite Series | Sequence of First Three Partial Sums |
|---|---|---|
| 0.1, 0.01, 0.001, … | $0.1 + 0.01 + 0.001 + \ldots$ | 0.1, 0.11, 0.111, … |

We will take a closer look at sums of infinite sequences in Lesson 10-3.

Series are often more conveniently notated using the uppercase Greek letter sigma $\Sigma$. A series written using this letter is said to be expressed using *summation notation* or **sigma notation**.

## KeyConcept Sigma Notation

For any sequence $a_1, a_2, a_3, a_4, \ldots$, the sum of the first $k$ terms is denoted

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k,$$

where $n$ is the index of summation, $k$ is the upper bound of summation, and 1 is the lower bound of summation.

**ReadingMath**

**Sigma Notation** $\sum_{n=1}^{k} a_n$ is read *the summation from $n = 1$ to $k$ of a sub $n$.*

In this notation, the lower bound indicates where to begin summing the terms of the sequence and the upper bound indicates where to end the sum. If the upper bound is given as $\infty$, the sigma notation represents an infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

## Example 6 Sums in Sigma Notation

**Find each sum.**

**a.** $\sum_{n=1}^{5} (4n - 3)$

$$\sum_{n=1}^{5} (4n - 3) = [4(1) - 3] + [4(2) - 3] + [4(3) - 3] + [4(4) - 3] + [4(5) - 3]$$

$$= 1 + 5 + 9 + 13 + 17 \text{ or } 45$$

**b.** $\sum_{n=3}^{7} \frac{6n - 3}{2}$

$$\sum_{n=3}^{7} \frac{6n - 3}{2} = \frac{6(3) - 3}{2} + \frac{6(4) - 3}{2} + \frac{6(5) - 3}{2} + \frac{6(6) - 3}{2} + \frac{6(7) - 3}{2}$$

$$= 7.5 + 10.5 + 13.5 + 16.5 + 19.5 \text{ or } 67.5$$

**c.** $\sum_{n=1}^{\infty} \frac{7}{10^n}$

$$\sum_{n=1}^{\infty} \frac{7}{10^n} = \frac{7}{10^1} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \ldots$$

$$= 0.7 + 0.07 + 0.007 + 0.0007 + 0.00007 + \ldots$$

$$= 0.77777\ldots \text{ or } \frac{7}{9}$$

**WatchOut!**

**Variations in Sigma Notation** The index of summation does not have to be the letter $n$. It can be represented by any variable. For example, the summation in Example 6a could also be written as $\sum_{i=1}^{5} (4i - 3)$.

### GuidedPractice

**6A.** $\sum_{n=1}^{5} \frac{n^2 - 1}{2}$

**6B.** $\sum_{n=7}^{13} (n^3 - n^2)$

**6C.** $\sum_{n=1}^{\infty} \frac{6}{10^n}$

Note that while the lower bound of a summation is often 1, a sum can start with any term $p$ in a sequence as long as $p < k$. In Example 6b, the summation started with the 3rd term of the sequence and ended with the 7th term.

## Exercises

= Step-by-Step Solutions begin on page R29.

**Find the next four terms of each sequence.** (Example 1)

1. 1, 8, 15, 22, …
2. 3, −6, 12, −24, …
3. 81, 27, 9, 3, …
4. 1, 3, 7, 13, …
5. −2, −15, −28, −41…
6. 1, 4, 10, 19, …

**Find the first four terms of each sequence.** (Example 1)

7. $a_n = n^2 - 1$
8. $a_n = -2^n + 7$
9. $a_n = \frac{n + 7}{9 - n}$
10. $a_n = (-1)^{n+1} + n$

11. **AUTOMOBILE LEASES** Lease agreements often contain clauses that limit the number of miles driven per year by charging a per-mile fee over that limit. For the car shown below, the lease requires that the number of miles driven each year must be no more than 15,000. (Example 2)


a. Write the sequence describing the maximum number of allowed miles on the car at the end of every 12 months of the lease if the car has 1350 miles at the beginning of the lease.

b. Write the first 4 terms of the sequence that gives the cumulative cost of the lease for a given month.

c. Write an explicit formula to represent the sequence in part **b**.

d. Determine the total amount of money paid by the end of the lease.

**Find the specified term of each sequence.** (Example 2)

12. 4th term, $a_1 = 5$, $a_n = -3a_{n-1} + 10$, $n \geq 2$
13. 7th term, $a_1 = 14$, $a_n = 0.5a_{n-1} + 3$, $n \geq 2$
14. 4th term, $a_1 = 0$, $a_n = 3^{a_{n-1}}$, $n \geq 2$
15. 3rd term, $a_1 = 3$, $a_n = (a_{n-1})^2 - 5a_{n-1} + 4$, $n \geq 2$

16. **WEB SITE** Khari, the student from the beginning of the lesson, had great success expanding her Web site. Each student who received a referral developed a Web page and referred five more students to Khari's site. (Example 3)

a. List the first five terms of a sequence modeling the number of new Web pages created through Khari's site.

b. Suppose the school has 1576 students. After how many rounds of referrals did the entire student body have a Web page?

17. **BEES** Female honeybees come from fertilized eggs (male and female parent), while male honeybees come from unfertilized eggs (one female parent). (Example 3)

a. Draw a family tree showing the 3 previous generations of a male honeybee (parents only).

b. Determine the number of parent bees in the 11th previous generation of a male honeybee.

**Determine whether each sequence is *convergent* or *divergent*.** (Example 4)

18. $a_1 = 4$, $1.5a_{n-1}$, $n \geq 2$
19. $a_n = \frac{5}{10^n}$
20. $a_n = -n^2 - 8n + 106$
21. $a_1 = -64$, $\frac{3}{4}a_{n-1}$, $n \geq 2$
22. $a_1 = 1$, $a_n = 4 - a_{n-1}$, $n \geq 2$
23. $a_n = n^2 - 3n + 1$
24. $a_n = \frac{n^2 + 4}{3 + n}$
25. $a_1 = 9$, $a_n = \frac{a_{n-1} + 3}{2}$, $n \geq 2$
26. $a_n = \frac{5n + 6}{n}$
27. $a_n = \frac{5n}{5^n} + 1$

**Find the indicated sum for each sequence.** (Example 5)

28. 5th partial sum of $a_n = n(n - 4)(n - 3)$
29. 6th partial sum of $a_n = \frac{-5n + 3}{n}$
30. $S_8$ of $a_1 = 1$, $a_n = a_{n-1} + (18 - n)$, $n \geq 2$
31. $S_4$ of $a_1 = 64$, $a_n = -\frac{3}{4}a_{n-1}$, $n \geq 2$
32. 11th partial sum of $a_1 = 4$, $a_n = (-1)^{n-1}\left(|a_{n-1}| + 3\right)$, $n \geq 2$
33. $S_9$ of $a_1 = -35$, $a_n = a_{n-1} + 8$, $n \geq 2$
34. 4th partial sum of $a_1 = 3$, $a_n = (a_{n-1} - 2)^3$, $n \geq 2$
35. $S_4$ of $a_n = \frac{(-3)^n}{10}$

**Find each sum.** (Example 6)

36. $\sum_{n=1}^{8} (6n - 11)$
37. $\sum_{n=4}^{11} (30 - 4n)$
38. $\sum_{n=1}^{7} [n^2(n - 5)]$
39. $\sum_{n=2}^{7} (n^2 - 6n + 1)$
40. $\sum_{n=8}^{15} \left(\frac{n}{4} - 7\right)$
41. $\sum_{n=1}^{10} [(n - 4)^2(n - 5)]$
42. $\sum_{n=0}^{6} [(-2)^n - 9]$
43. $\sum_{n=1}^{3} 7\left(\frac{1}{10}\right)^{2n}$
44. $\sum_{n=1}^{\infty} 5\left(\frac{1}{10^n}\right)$
45. $\sum_{n=1}^{\infty} \frac{8}{10^n}$

46. **FINANCIAL LITERACY** Jim's bank account had an initial deposit of $380, earning 3.5% interest per year compounded annually.

a. Find the balance each year for the first five years.

b. Write a recursive and an explicit formula defining his account balance.

c. For very large values of $n$, which formula gives a more accurate balance? Explain.

**47.** **INVESTING** Melissa invests \$200 every 3 months. The investment pays an annual percentage rate of 8%, and the interest is compounded quarterly. If Melissa makes each payment at the beginning of the quarter and the interest is posted at the end of the quarter, what will the total value of the investment be after 2 years?

**48.** **RIDES** The table shows the number of riders of the Mean Streak each year from 1998 to 2007. This ridership data can be approximated by $a_n = -\frac{1}{20}n + 1.3$, where $n = 1$ represents 1998, $n = 2$ represents 1999, and so on.

| Mean Streak Roller Coaster | | | |
|---|---|---|---|
| Year | Number of Riders (millions) | Year | Number of Riders (millions) |
| 1998 | 1.31 | 2003 | 0.99 |
| 1999 | 1.15 | 2004 | 0.95 |
| 2000 | 1.14 | 2005 | 0.89 |
| 2001 | 1.09 | 2006 | 0.81 |
| 2002 | 1.05 | 2007 | 0.82 |

**Source:** *Cedar Fair Entertainment Company*

**a.** Sketch a graph of the number of riders from 1998 to 2007. Then determine whether the sequence appears to be *convergent* or *divergent*. Does this make sense in the context of the situation? Explain your reasoning.

**b.** Use the table to find the total number of riders from 1998 to 2005. Then use the explicit sequence to find the 8th partial sum of $a_n$. Compare the results.

**c.** If the sequence continues, find $a_{14}$. What does this number represent?

**Copy and complete the table.**

| | Recursive Formula | Sequence | Explicit Formula |
|---|---|---|---|
| **49.** | | 6, 8, 10, 12, … | |
| **50.** | $a_1 = 15, a_n = a_{n-1} - 1, n \geq 2$ | | |
| **51.** | | 7, 21, 63, 189, … | |
| **52.** | | | $a_n = 10(-2)^n$ |
| **53.** | | | $a_n = 8n - 3$ |
| **54.** | $a_1 = 2, a_n = 4a_{n-1}, n \geq 2$ | | |
| **55.** | $a_1 = 3, a_n = a_{n-1} + 2n - 1, n \geq 2$ | | |
| **56.** | | | $a_n = n^2 + 1$ |

**Write each series in sigma notation. The lower bound is given.**

**57.** $-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5;\ n = 1$

**58.** $\frac{1}{20} + \frac{1}{25} + \frac{1}{30} + \frac{1}{35} + \frac{1}{40} + \frac{1}{45};\ n = 4$

**59** $8 + 27 + 64 + \cdots + 1000;\ n = 2$

**60.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128};\ n = 1$

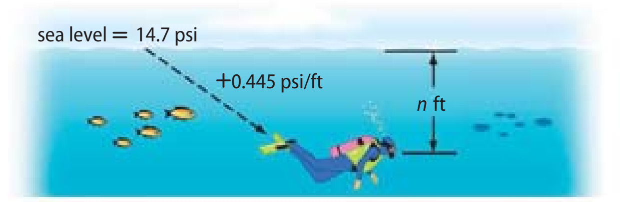
**61.** $-8 + 16 - 32 + 64 - 128 + 256 - 512;\ n = 3$

**62.** $8\left(-\frac{1}{3}\right) + 8\left(\frac{1}{9}\right) + 8\left(-\frac{1}{27}\right) + \cdots + 8\left(-\frac{1}{243}\right);\ n = 1$

**Determine whether each sequence is *convergent* or *divergent*. Then find the fifth partial sum of the sequence.**

**63.** $a_n = \sin \frac{n\pi}{2}$ **64.** $a_n = n \cos \pi$ **65.** $a_n = e^{-\frac{n}{2}} \cos \pi n$

**66.** **WATER PRESSURE** The pressure exerted on the human body at sea level is 14.7 pounds per square inch (psi). For each additional foot below sea level, the pressure is about 0.445 psi greater, as shown.



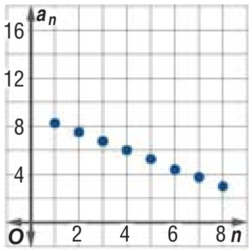

**a.** Write a recursive formula to represent $a_n$, the pressure at $n$ feet below sea level. (*Hint*: Let $a_0 = 14.7$.)

**b.** Write the first three terms of the sequence and describe what they represent.

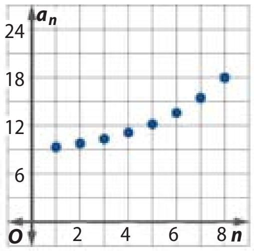
**c.** Scuba divers cannot safely dive deeper than 100 feet. Write an explicit formula to represent $a_n$. Then use the formula to find the water pressure at 100 feet below sea level.

**Match each sequence with its graph.**

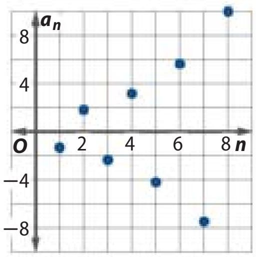
**67.**



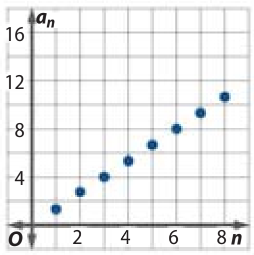
**68.**



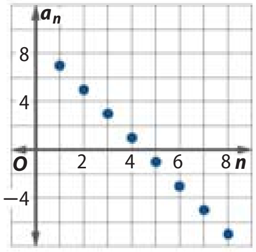
**69.**



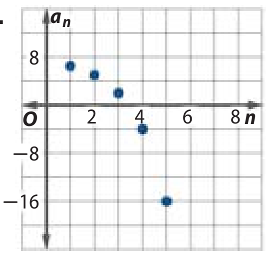
**70.**



**71.**



**72.**



**a.** $a_n = \frac{4}{3}n$ **b.** $a_n = -\frac{3}{4}n + 9$

**c.** $a_n = \left(-\frac{4}{3}\right)^n$ **d.** $a_n = 8 - \frac{3}{4}(2^n)$

**e.** $a_n = 9 - 2n$ **f.** $a_n = \left(\frac{4}{3}\right)^n + 8$

73. **GOLDEN RATIO** Consider the Fibonacci sequence 1, 1, 2, 3, …, $a_{n-2} + a_{n-1}$.

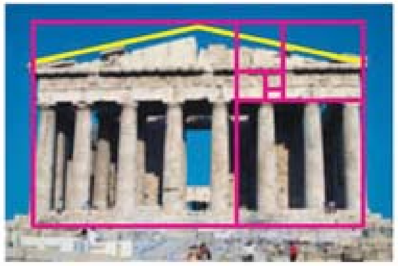
**a.** Find $\frac{a_n}{a_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

**b.** Sketch a graph of the terms found in part **a.** Let $n - 1$ be the $x$-coordinate and $\frac{a_n}{a_{n-1}}$ be the $y$-coordinate.

**c.** Based on the graph found in part **b,** does this sequence appear to be convergent? If so, describe the limit to three decimal places. If not, explain why not.

**d.** In a *golden rectangle*, the ratio of the length to the width is about 1.61803399. This is called the *golden ratio*. How does the limit of the sequence $\frac{a_n}{a_{n-1}}$ compare to the golden ratio?

**e.** Golden rectangles are common in art and architecture. The Parthenon, in Greece, is an example of how golden rectangles are used in architecture.

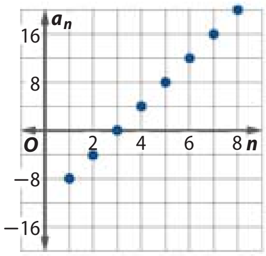


DAJ/Getty Images

Research golden rectangles and find two more examples of golden rectangles in art or architecture.

**Determine whether each sequence is *convergent* or *divergent*.**

74.

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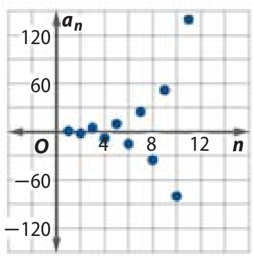


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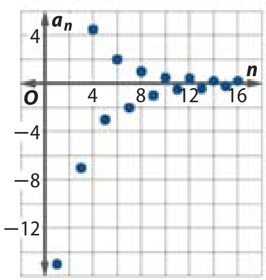


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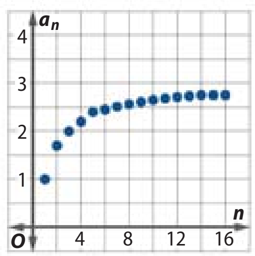



**Write an explicit formula for each recursively defined sequence.**

78. $a_1 = 10;\ a_n = a_{n-1} + 5$

79. $a_1 = 1.25;\ a_n = a_{n-1} - 0.5$

80. $a_1 = 128;\ a_n = 0.5a_{n-1}$

81. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate sums of infinite series.

**a. NUMERICAL** Calculate the first five terms of the infinite sequence $a_n = \frac{4}{10^n}$.

**b. GRAPHICAL** Use a graphing calculator to sketch $y = \frac{4}{10^x}$.

**c. VERBAL** Describe what is happening to the terms of the sequence as $n \to \infty$.

**d. NUMERICAL** Find the sum of the first 5 terms, 7 terms, and 9 terms of the series.

**e. VERBAL** Describe what is happening to the partial sums $S_n$ as $n$ increases.

**f. VERBAL** Predict the sum of the first $n$ terms of the series. Explain your reasoning.

## H.O.T. Problems Use Higher-Order Thinking Skills

82. **CHALLENGE** Consider the recursive sequence below.

$$a_n = a_{n - a_{n-1}} + a_{n - a_{n-2}} \text{ for } a_1 = 1, a_2 = 1, n \geq 3$$

**a.** Find the first eight terms of the sequence.

**b.** Describe the similarities and differences between this sequence and the other recursive sequences in this lesson.

83. **OPEN ENDED** Write a sequence either recursively or explicitly that has the following characteristics.

**a.** converges to 0

**b.** converges to 3

**c.** diverges

84. **WRITING IN MATH** Describe why an infinite sequence must not only converge, but converge to 0, in order for there to be a sum.

**REASONING** **Determine whether each statement is *true* or *false*. Explain your reasoning.**

85. $\sum_{n=1}^{5} (n^2 + 3n) = \sum_{n=1}^{5} n^2 + 3\sum_{n=1}^{5} n$

86. $\sum_{n=1}^{5} 3^n = \sum_{n=3}^{7} 3^{n-2}$

87. **CHALLENGE** Find the sum of the first 60 terms of the sequence below. Explain how you determined your answer.

$$15, 17, 2, -15, -17, \ldots,$$
$$\text{where } a_n = a_{n-1} - a_{n-2} \text{ for } n \geq 3$$

88. **WRITING IN MATH** Make an outline that could be used to describe the steps involved in finding the 300th partial sum of the infinite sequence $a_n = 2n - 3$. Then explain how to express the same sum using sigma notation.

## Spiral Review

**Graph each complex number on a polar grid. Then express it in rectangular form.** (Lesson 9-5)

**89.** $2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

**90.** $2.5(\cos 1 + i \sin 1)$

**91.** $5(\cos 0 + i \sin 0)$

**Determine the eccentricity, type of conic, and equation of the directrix given by each polar equation.** (Lesson 9-4)

**92.** $r = \frac{3}{2 - 0.5 \cos \theta}$

**93.** $r = \frac{6}{1.2 \sin \theta + 0.3}$

**94.** $r = \frac{1}{0.2 - 0.2 \sin \theta}$

**Determine whether the points are collinear. Write *yes* or *no*.** (Lesson 8-5)

**95.** $(-3, -1, 4), (3, 8, 1), (5, 12, 0)$

**96.** $(4, 8, 6), (0, 6, 12), (8, 10, 0)$

**97.** $(0, -4, 3), (8, -10, 5), (12, -13, 2)$

**98.** $(-7, 2, -1), (-9, 3, -4), (-5, 1, 2)$

**Find the length and the midpoint of the segment with the given endpoints.** (Lesson 8-4)

**99.** $(2, -15, 12), (1, -11, 15)$

**100.** $(-4, 2, 8), (9, 6, 0)$

**101.** $(7, 1, 5), (-2, -5, -11)$

**102. TIMING** The path traced by the tip of the hour-hand of a clock can be modeled by a circle with parametric equations $x = 6 \sin t$ and $y = 6 \cos t$. (Lesson 7-5)

**a.** Find an interval for $t$ in radians that can be used to describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon the next day.

**b.** Simulate the motion described in part **a** by graphing the equation in parametric mode on a graphing calculator.

**c.** Write an equation in rectangular form that models the motion of the hour-hand. Find the radius of the circle traced out by the hour-hand if $x$ and $y$ are given in inches.


**Find the exact value of each expression.** (Lesson 5-4)

**103.** $\tan \frac{\pi}{12}$

**104.** $\sin 75°$

**105.** $\cos 165°$

**Find the partial fraction decomposition of each rational expression.** (Lesson 6-4)

**106.** $\frac{10x^2 - 11x + 4}{2x^2 - 3x + 1}$

**107.** $\frac{1}{2x^2 + x}$

**108.** $\frac{x + 1}{x^3 + x}$

## Skills Review for Standardized Tests

**109. SAT/ACT** The first term in a sequence is $-5$, and each subsequent term is 6 more than the term that immediately precedes it. What is the value of the 104th term?

A 607

B 613

C 618

D 619

E 615

**110. REVIEW** Find the exact value of $\cos 2\theta$ if $\sin \theta = -\frac{\sqrt{5}}{3}$ and $180° < \theta < 270°$.

F $-\frac{\sqrt{6}}{6}$

G $-\frac{4\sqrt{5}}{9}$

H $-\frac{\sqrt{30}}{6}$

J $-\frac{1}{9}$

**111.** The first four terms of a sequence are 144, 72, 36, and 18. What is the tenth term in the sequence?

A 0

B $\frac{9}{64}$

C $\frac{9}{32}$

D $\frac{9}{16}$

**112. REVIEW** How many 5-inch cubes can be stacked inside a box that is 10 inches long, 15 inches wide, and 5 inches tall?

F 5

G 6

H 15

J 20

LESSON 10-2

# Arithmetic Sequences and Series

Enigma/Alamy

**Then**

- You found terms of sequences and sums of series. (Lesson 10-1)

**Now**

1. Find $n$th terms and arithmetic means of arithmetic sequences.
2. Find sums of $n$ terms of arithmetic series.

**Why?**

- With cross country season approaching, Meg decides to train every day until the first day of practice. She plans to run 1 mile the first day, 1.25 miles the second day, 1.5 miles the third day, and so on. Her goal is to run a total of 100 miles before the first day of practice.

**NewVocabulary**

arithmetic sequence
common difference
arithmetic means
first difference
second difference
arithmetic series

## 1 Arithmetic Sequences

A sequence in which the difference between successive terms is a constant is called an **arithmetic sequence**. The constant is referred to as the **common difference**, denoted $d$. To find the common difference of an arithmetic sequence, subtract any term from its succeeding term. To find the next term in the sequence, add the common difference to the given term.

### Example 1 Arithmetic Sequences

**Determine the common difference, and find the next four terms of the arithmetic sequence 17, 12, 7, … .**

First, find the common difference.

$a_2 - a_1 = 12 - 17$ or $-5$

$a_3 - a_2 = 7 - 12$ or $-5$

Find the difference between two pairs of consecutive terms to verify the common difference.

The common difference is $-5$. Add $-5$ to the third term to find the fourth term, and so on.

$a_4 = 7 + (-5)$ or $2$ $\quad a_5 = 2 + (-5)$ or $-3$ $\quad a_6 = -3 + (-5)$ or $-8$ $\quad a_7 = -8 + (-5)$ or $-13$

The next four terms are 2, $-3$, $-8$, and $-13$.

**GuidedPractice**

**Determine the common difference, and find the next four terms of each arithmetic sequence.**

**1A.** $-129, -98, -67, \ldots$

**1B.** $244, 187, 130, \ldots$

Each term in an arithmetic sequence is found by adding the common difference to the preceding term. Therefore, $a_n = a_{n-1} + d$. You can use this recursive formula to develop an explicit formula for generating an arithmetic sequence. Consider the arithmetic sequence in which $a_1 = 6$ and $d = 3$.

| | | | |
|---|---|---|---|
| first term | $a_1$ | $a_1$ | $6$ |
| second term | $a_2$ | $a_1 + d$ | $6 + 1(3) = 9$ |
| third term | $a_3$ | $a_1 + 2d$ | $6 + 2(3) = 12$ |
| fourth term | $a_4$ | $a_1 + 3d$ | $6 + 3(3) = 15$ |
| fifth term | $a_5$ | $a_1 + 4d$ | $6 + 4(3) = 18$ |
| $n$th term | $a_n$ | $a_1 + (n-1)d$ | $6 + (n-1)3$ |

The pattern formed leads to the following formula for finding the $n$th term of an arithmetic sequence.

**KeyConcept The $n$th Term of an Arithmetic Sequence**

**Words** The $n$th term of an arithmetic sequence with first term $a_1$ and common difference $d$ is given by $a_n = a_1 + (n-1)d$.

**Example** The 16th term of 2, 5, 8, … is $a_{16} = 2 + (16 - 1) \cdot 3$ or 47.

**StudyTip**

**Explicit Formulas** If a term other than $a_1$ is given, the explicit formula for finding the $n$th term of a sequence needs to be adjusted. This can be done by subtracting the number of the term given from $n$. For example, if $a_5$ is given, the equation would become $a_n = a_5 + (n - 5)d$, or if $a_0$ is given, then $a_n = a_0 + nd$.

## Example 2 Explicit and Recursive Formulas

**Find both an explicit formula and a recursive formula for the $n$th term of the arithmetic sequence 12, 21, 30, … .**

First, find the common difference.

$a_2 - a_1 = 21 - 12$ or 9 — Find the difference between two pairs of consecutive terms to verify the common difference.

$a_3 - a_2 = 30 - 21$ or 9

For an explicit formula, substitute $a_1 = 12$ and $d = 9$ in the formula for the $n$th term of an arithmetic sequence.

$a_n = a_1 + (n - 1)d$ — $n$th term of an arithmetic sequence

$= 12 + (n - 1)9$ — $a_1 = 12$ and $d = 9$

$= 12 + 9(n - 1)$ or $9n + 3$ — Simplify.

For a recursive formula, state the first term $a_1$ and then indicate that the next term is the sum of the previous term $a_{n-1}$ and $d$.

$a_1 = 12, a_n = a_{n-1} + 9$

**GuidedPractice**

**2.** Find both an explicit formula and a recursive formula for the $n$th term of the arithmetic sequence 35, 23, 11, … .

The formula for the $n$th term of an arithmetic sequence can be used to find a specific term in a sequence.

**StudyTip**

**Rate of Change** Arithmetic sequences have a constant rate of change which is equivalent to the common difference $d$.

## Example 3 $n$th Terms

**a. Find the 68th term of the arithmetic sequence 25, 17, 9, … .**

First, find the common difference.

$a_2 - a_1 = 17 - 25$ or $-8$ — Find the difference between two pairs of consecutive terms to verify the common difference.

$a_3 - a_2 = 9 - 17$ or $-8$

Use the formula for the $n$th term of an arithmetic sequence to find $a_{68}$.

$a_n = a_1 + (n - 1)d$ — $n$th term of an arithmetic sequence

$a_{68} = 25 + (68 - 1)(-8)$ — $n = 68$, $a_1 = 25$, and $d = -8$

$a_{68} = -511$ — Simplify.

**b. Find the first term of the arithmetic sequence for which $a_{25} = 139$ and $d = \frac{3}{4}$.**

Substitute $a_{25} = 139$, $n = 25$, and $d = \frac{3}{4}$ in the formula for the $n$th term of an arithmetic sequence to find $a_1$.

$a_n = a_1 + (n - 1)d$ — $n$th term of an arithmetic sequence

$139 = a_1 + (25 - 1)\frac{3}{4}$ — $n = 25$, $a_n = 139$, and $d = \frac{3}{4}$

$a_1 = 121$ — Simplify.

**GuidedPractice**

**3A.** Find the 38th term of the arithmetic sequence $-29, -2, 25, \ldots$ .

**3B.** Find $d$ of the arithmetic sequence for which $a_1 = 75$ and $a_{38} = 56.5$.

If two nonconsecutive terms of an arithmetic sequence are known, the terms between them can be calculated. These terms are called **arithmetic means**. In the sequence below, 17, 27, and 37 are the arithmetic means between 7 and 47.

$$-3, 7, 17, 27, 37, 47, 57$$

**StudyTip**

**Alternative Method** An alternative method to find $d$ would be to subtract the first term from the last term and divide by the total number of terms minus 1.

## Example 4 Arithmetic Means

**Write an arithmetic sequence that has four arithmetic means between 4.3 and 12.8.**

The sequence will resemble 4.3, __?__, __?__, __?__, __?__, 12.8. Note that 12.8 is the sixth term of the sequence or $a_6$.

First, find the common difference using $a_6 = 12.8$, $a_1 = 4.3$, and $n = 6$.

| | |
|---|---|
| $a_n = a_1 + (n - 1)d$ | $n$th term of an arithmetic sequence |
| $12.8 = 4.3 + (6 - 1)d$ | $a_n = 12.8$, $a_1 = 4.3$, and $n = 6$ |
| $12.8 = 4.3 + 5d$ | Simplify. |
| $d = 1.7$ | Solve for $d$. |

Then determine the arithmetic means by using $d = 1.7$.

$a_2 = 4.3 + 1.7$ or 6
$a_3 = 6 + 1.7$ or 7.7
$a_4 = 7.7 + 1.7$ or 9.4
$a_5 = 9.4 + 1.7$ or 11.1

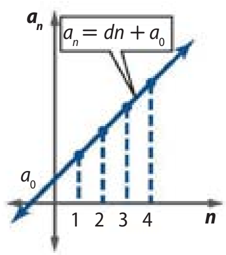
The sequence is 4.3, 6, 7.7, 9.4, 11.1, 12.8.

### GuidedPractice

**4.** Write a sequence that has six arithmetic means between 12.4 and −24.7.

The **first differences** of a sequence are found by subtracting each term from its successive term.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Sequence** | $a_1$, | | $a_2$, | | $a_3$, | | $a_4$, | | $a_5$, … |
| **1st differences** | | $d$ | | $d$ | | $d$ | | $d$ | |




When the first differences are all the same, the sequence is arithmetic and the $n$th term can be modeled by a linear function of the form $a_n = dn + a_0$, as shown.

If the first differences are not the same, the sequence is not arithmetic. However, the differences may still help to identify the type of function that can be used to model the sequence. Consecutive first differences may be subtracted from one another, thus producing **second differences**.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Sequence** | 12, | | 20, | | 30, | | 42, | | 56, … |
| **1st differences** | | 8 | | 10 | | 12 | | 14 | |
| **2nd differences** | | | 2 | | 2 | | 2 | | |

If the second differences are constant, then the $n$th term of the sequence can be modeled by a quadratic function. This function can be found by solving a system of equations, as demonstrated in Example 5.

## Example 5 Second Differences

**Find a quadratic model for the sequence 12, 20, 30, 42, 56, 72, … .**

The $n$th term can be represented by a quadratic equation of the form $a_n = an^2 + bn + c$. Substitute values for $a_n$ and $n$ into the equation.

$12 = a(1)^2 + b(1) + c$ — $a_n = 12$ and $n = 1$

$20 = a(2)^2 + b(2) + c$ — $a_n = 20$ and $n = 2$

$30 = a(3)^2 + b(3) + c$ — $a_n = 30$ and $n = 3$

This yields a system of linear equations in three variables.

$12 = a + b + c$ — Simplified first equation

$20 = 4a + 2b + c$ — Simplified second equation

$30 = 9a + 3b + c$ — Simplified third equation

Solving for $a$, $b$, and $c$ gives $a = 1$, $b = 5$, and $c = 6$. Substituting these values in the equation for $a_n$, the model for the sequence is $a_n = n^2 + 5n + 6$, as shown in Figure 10.2.1.

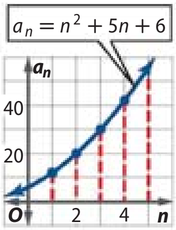



**Figure 10.2.1**

### GuidedPractice

**5.** Find a quadratic model for the sequence −14, −8, 0, 10, 22, 36, … .

**StudyTip**

**Higher Differences** The number of variables in the standard form of the equation dictates the number of equations needed in the system formed.

If calculating second differences does not result in a constant difference, higher differences may be found. This process is similar to the process needed for finding a quadratic equation. The function that will model a sequence is dependent upon how many computed differences are necessary before finding a constant difference.

Higher differences may never result in constant differences. In this case, there may not be a polynomial model that can be used to describe the sequence.

| Differences | Model |
| --- | --- |
| first | linear |
| second | quadratic |
| third | cubic |
| fourth | quartic |
| fifth | quintic |

## 2 Arithmetic Series

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence.

| Arithmetic Sequence | Arithmetic Series |
| --- | --- |
| −6, −3, 0, 3, 6 | −6 + (−3) + 0 + 3 + 6 |
| 4.25, 4, 3.75, 3.5, 3.25 | 4.25 + 4 + 3.75 + 3.5 + 3.25 |
| $a_1, a_2, a_3, a_4, \ldots, a_n$ | $a_1 + a_2 + a_3 + a_4 + \cdots + a_n$ |

To develop a formula for finding the sum of a finite arithmetic series, start by looking at the series $S_n$ that has terms created by adding multiples of $d$ to $a_1$. If we combine this with the same series written in reverse order, we can find a formula for calculating the sum of a finite arithmetic series.

$$\begin{array}{rl} S_n = & a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n \\ (+)\ S_n = & a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \\ \hline 2S_n = & (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \\ 2S_n = & n(a_1 + a_n) \end{array}$$

There are $n$ terms in the series, all of which are $(a_1 + a_n)$.

Therefore, $S_n = \frac{n}{2}(a_1 + a_n)$. When the value of the last term is unknown, you can still determine the $n$th partial sum of the series by combining the $n$th term of an arithmetic sequence formula and the sum of a finite arithmetic series formula.

$S_n = \frac{n}{2}(a_1 + a_n)$ — Sum of a finite arithmetic series formula

$S_n = \frac{n}{2}\{a_1 + [a_1 + (n - 1)d]\}$ — $a_n = a_1 + (n - 1)d$, $n$th term of an arithmetic sequence formula

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ — Simplify.

## KeyConcept Sum of a Finite Arithmetic Series

The sum of a finite arithmetic series with $n$ terms or the $n$th partial sum of an arithmetic series can be found using one of two related formulas.

**Formula 1** $S_n = \frac{n}{2}(a_1 + a_n)$

**Formula 2** $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

**StudyTip**

**Arithmetic Series** All infinite arithmetic sequences diverge except for those in which $d = 0$. As a result, only a finite arithmetic series or the $n$th partial sum of an infinite arithmetic series can be calculated.

## Example 6 Sum of Arithmetic Series

**Find the indicated sum of each arithmetic series.**

**a.** $-5 + 2 + 9 + \cdots + 317$

In this sequence, $a_1 = -5$, $a_n = 317$, and $d = 2 - (-5)$ or 7. Use the $n$th term formula to find the number of terms in the sequence $n$.

| | |
|---|---|
| $a_n = a_1 + (n - 1)d$ | $n$th term of an arithmetic sequence |
| $317 = -5 + (n - 1)7$ | $a_n = 317$, $a_1 = -5$, and $d = 7$ |
| $47 = n$ | Simplify. |

Now use Formula 1 to find the sum of the series.

| | |
|---|---|
| $S_n = \frac{n}{2}(a_1 + a_n)$ | Formula 1 |
| $S_{47} = \frac{47}{2}(-5 + 317)$ | $n = 47$, $a_1 = -5$, and $a_n = 317$ |
| $= 7332$ | Simplify. |

**b.** **the 28th partial sum of $27 + 14 + 1 + \cdots$**

In this sequence, $a_1 = -5$ and $d = 14 - 27$ or $-13$. Use Formula 2 to find the 28th partial sum.

| | |
|---|---|
| $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ | Formula 2 |
| $S_{28} = \frac{28}{2}[2(27) + (28 - 1)(-13)]$ | $n = 28$, $a_1 = 27$, and $d = -13$ |
| $= -4158$ | Simplify. |

**c.** $\sum_{n=6}^{28} (5n - 17)$

$$\sum_{n=6}^{28} (5n - 17) = [5(6) - 17] + [5(7) - 17] + \cdots + [5(28) - 17]$$

$$= 13 + 18 + \cdots + 123$$

The first term of this series is 13 and the last term is 123. The number of terms is equal to the upper bound minus the lower bound plus one, which is $28 - 6 + 1$ or 23. Therefore $a_1 = 13$, $a_n = 123$, and $n = 23$. Use Formula 1 to find the sum of the series.

| | |
|---|---|
| $S_n = \frac{n}{2}(a_1 + a_n)$ | Formula 1 |
| $S_{23} = \frac{23}{2}(13 + 123)$ | $n = 23$, $a_1 = 13$, and $a_n = 123$ |
| $= 1564$ | Simplify. |

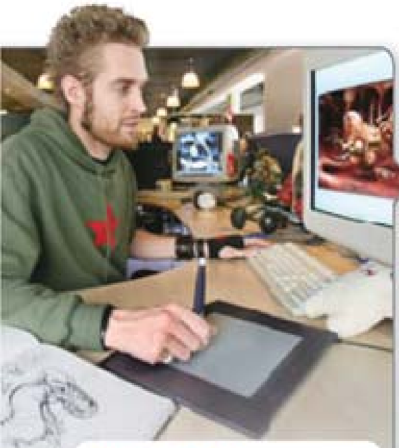
### GuidedPractice

**6A.** $211 + 193 + 175 + \cdots + (-455)$

**6B.** the 19th partial sum of $-19 + 23 + 65 + \cdots$

**6C.** $\sum_{n=23}^{37} (2n + 3)$

**6D.** $\sum_{n=12}^{18} (-2n + 57)$



**Real-World Career**

**Software Engineer** Most video game programmers are software engineers who plan and write game software. Most programmers have a bachelor's degree in computer science, information systems, or mathematics. Some also obtain technical or professional certification.

Arithmetic series have many useful real-life applications.

### Real-World Example 7 Sum of an Arithmetic Series

**VIDEO GAMES A video game tournament, in which gamers compete in multiple games and accumulate an overall score, pays the top 20 finishers. First place receives $5000, second place receives $4800, third place receives $4600, and so on. How much total prize money is awarded?**

| | |
|---|---|
| $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ | Formula 2 |
| $S_{20} = \frac{20}{2}[2(5000) + (20 - 1)(-200)]$ | $n = 20$, $a_1 = 5000$, and $d = -200$ |
| $= 62{,}000$ | Simplify. |

The total prize money awarded is $62,000

**Guided Practice**

**7. VIDEO GAMES** Selma is playing a video game. She scores 50 points if she clears the first level. Each following level is worth 50 more points than the previous level. Thus, she scores 100 points for clearing the second level, 150 for the third, and so on. What is the total amount of points Selma will score after she clears the ninth level?

The formula for the sum of a finite arithmetic series can also be used to solve for values of $n$.

### Real-World Example 8 Sum of an Arithmetic Series

**BASEBALL Carter has been collecting baseball cards since his father gave him a 20-card collection. During each month, Carter's father gives him 5 more cards than the previous month. In how many months will Carter reach 1000 cards?**

| | |
|---|---|
| $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ | Formula 2 |
| $1000 = \frac{n}{2}[2(20) + (n - 1)5]$ | $S_n = 1000$, $a_1 = 20$, and $d = 5$ |
| $2000 = n(5n + 35)$ | Multiply each side by 2 and simplify. |
| $0 = 5n^2 + 35n - 2000$ | Distribute and subtract 2000 from each side. |
| $0 = n^2 + 7n - 400$ | Divide each side by 5. |
| $n = \frac{-7 \pm \sqrt{7^2 - 4(1)(-400)}}{2(1)}$ | Use the Quadratic Formula. |
| $n \approx 16.8$ and $-23.8$ | Simplify. |

Because time cannot be negative, Carter will reach 1000 cards in 17 months.

**CHECK** $20 + 25 + \cdots + 100 = \frac{17}{2}(20 + 100)$

$= 1020$ ✓

In seventeen months, Carter will have 1020 baseball cards, which is more than 1000.

**Guided Practice**

**8. LAWN SERVICE** Kevin runs a lawn mowing service. He currently has 14 clients. He has gained 2 new clients at the beginning of each of the past three years. Each year, he mows each client's lawn an average of 15 times. Starting now, if Kevin continues to gain 2 clients each year and if he charges $30 per lawn, after how many years will he earn a total of $51,300?

Ton Koene/Alamy

## Exercises

= Step-by-Step Solutions begin on page R29.

**Determine the common difference, and find the next four terms of each arithmetic sequence.** (Example 1)

**1.** 20, 17, 14, …

**2.** 3, 16, 29, …

**3.** 117, 108, 99, …

**4.** −83, −61, −39, …

**5.** −3, 1, 5, …

**6.** 4, 21, 38, …

**7.** −4.5, −9.5, −14.5, …

**8.** −97, −29, 39, …

**9. MARCHING BAND** A marching band begins its performance in a pyramid formation. The first row has 1 band member, the second row has 3 band members, the third row has 5 band members, and so on. (Examples 1 and 2)

**a.** Find the number of band members in the 8th row.

**b.** Write an explicit formula and a recursive formula for finding the number of band members in the $n$th row.

**Find both an explicit formula and a recursive formula for the $n$th term of each arithmetic sequence.** (Example 2)

**10.** 2, 5, 8, …

**11.** −6, 5, 16, …

**12.** −9, −16, −23, …

**13.** 4, 19, 34, …

**14.** 25, 11, −3, …

**15.** 7, −3.5, −14, …

**16.** −18, 4, 26, …

**17.** 1, 37, 73, …

**Find the specified value for the arithmetic sequence with the given characteristics.** (Example 3)

**18.** If $a_{14} = 85$ and $d = 9$, find $a_1$.

**19.** Find $d$ for −24, −31, −38, ….

**20.** If $a_n = 14$, $a_1 = -36$, and $d = 5$, find $n$.

**21.** If $a_1 = 47$ and $d = -5$, find $a_{12}$.

**22.** If $a_{22} = 95$ and $a_1 = 11$, find $d$.

**23.** Find $a_6$ for 84, 5, −74, ….

**24.** If $a_n = -20$, $a_1 = 46$, and $d = -11$, find $n$.

**25.** If $a_{35} = -63$ and $a_1 = 39$, find $d$.

**26. CONSTRUCTION** Each 8-foot section of a wooden fence contains 14 pickets. Let $a_n$ represent the number of pickets in $n$ sections. (Example 3)

**a.** Find the first 5 terms of the sequence.

**b.** Write a recursive formula for the sequence in part **a**.

**c.** If 448 pickets were used to fence in the customer's backyard, how many feet of fencing was used?

**Find the indicated arithmetic means for each set of nonconsecutive terms.** (Example 4)

**27.** 3 means; 19 and −5

**28.** 5 means; −62 and −8

**29.** 4 means; 3 and 88

**30.** 8 means; −5.5 and 23.75

**31.** 7 means; −4.5 and 7.5

**32.** 10 means; 6 and 259

**Find a quadratic model for each sequence.** (Example 5)

**33.** 12, 19, 28, 39, 52, 67, …

**34.** −11, −9, −5, 1, 9, 19, …

**35.** 8, 3, −6, −19, −36, −57, …

**36.** −7, −2, 9, 26, 49, 78, …

**37.** 6, −2, −12, −24, −38, −54, …

**38.** −3, 1, 13, 33, 61, 97, …

**Find the indicated sum of each arithmetic series.** (Example 6)

**39.** 26th partial sum of $3 + 15 + 27 + \cdots + 303$

**40.** $-28 + (-19) + (-10) + \cdots + 242$

**41.** 42nd partial sum of $120 + 114 + 108 + \cdots$

**42.** 54th partial sum of $213 + 205 + 197 + \cdots$

**43.** $-17 + 1 + 19 + \cdots + 649$

**44.** $89 + 58 + 27 + \cdots + (-562)$

**45. RUNNING** Refer to the beginning of the lesson. (Example 6)

**a.** Determine the number of miles Meg will run on her 12th day of training.

**b.** During which day of training will Meg reach her goal of 100 total miles?

**Find the indicated sum of each arithmetic series.** (Example 6)

**46.** $\sum_{n=1}^{20} (3 + 2n)$

**47.** $\sum_{n=1}^{28} (100 - 4n)$

**48.** $\sum_{n=7}^{18} (-9n - 26)$

**49.** $\sum_{n=6}^{52} (7n + 1)$

**50.** $\sum_{n=7}^{42} (84 - 3n)$

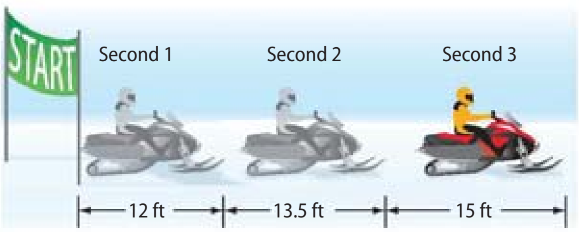
**51.** $\sum_{n=1}^{13} [32 + 4(n - 1)]$

**52.** $\sum_{n=20}^{24} \left(\frac{n}{2} - 9\right)$

**53.** $\sum_{n=2}^{9} (-15n - 12)$

**54. CONSTRUCTION** A crew is tiling a hotel lobby with a trapezoidal mosaic pattern. The shorter base of the trapezoid begins with a row of 8 tiles. Each row has two additional tiles until the 20th row. Determine the number of tiles needed to create the mosaic design. (Example 7)

**55. SNOWMOBILING** A snowmobiling competitor travels 12 feet in the first second of a race. If the competitor travels 1.5 additional feet each subsequent second, how many feet did the competitor travel in 64 seconds? (Example 7)



56. **FUNDRAISING** Lalana organized a charity walk. In the first year, the walk generated \$3000. She hopes to increase this amount by \$900 each year for the next several years. If her goal is met, in how many years will the walk have generated a total of at least \$65,000? (Example 8)

57. Find $a_n$ if $S_n = 490$, $a_1 = -5$, and $n = 100$.

58. If $S_n = 51.7$, $n = 22$, $a_n = -11.3$, find $a_1$.

59. Find $n$ for $-7 + (-5.5) + (-4) + \cdots$ if $S_n = -14$ and $a_n = 3.5$.

60. Find $a_1$ if $S_n = 1287$, $n = 22$, and $d = 5$.

61. If $S_{26} = 1456$, and $a_1 = -19$, find $d$.

62. If $S_{12} = 174$, $a_{12} = 39$, find $d$.

**Write each arithmetic series in sigma notation. The lower bound is given.**

63. $6 + 12 + 18 + \cdots + 66$; $n = 1$

64. $-1 + 0 + 1 + \cdots + 7$; $n = 1$

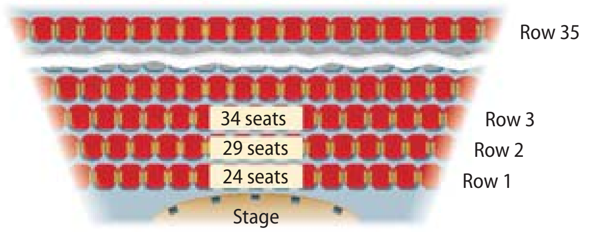
65. $17 + 21 + 25 + \cdots + 61$; $n = 4$

66. $1 + 0 + (-1) + (-2) + \cdots + (-13)$; $n = 6$

67. $-\frac{13}{5} + \left(-\frac{12}{5}\right) + \left(-\frac{11}{5}\right) + \cdots + \left(-\frac{3}{5}\right)$; $n = 2$

68. $9.25 + 8.5 + 7.75 + \cdots - 2$; $n = 1$

69. **CONCERTS** The seating in a concert auditorium is arranged as shown below.




a. Write a series in sigma notation to represent the number of seats in the auditorium, if the seating pattern shown in the first 3 rows continues for each successive row.

b. Find the total number of seats in the auditorium.

c. Another auditorium has 32 rows with 18 seats in the first row and 4 more seats in each of the successive rows. How many seats are there in this auditorium?

**Write a function that can be used to model the *n*th term of each sequence.**

70. $2, 5, 8, 11, 14, 17, \ldots$

71. $8, 13, 20, 29, 40, 53, \ldots$

72. $2, 2, 4, 8, 14, 22, \ldots$

73. $5, 31, 97, 221, 421, 715, \ldots$

74. $-6, -8, -6, 6, 34, 84, \ldots$

75. $0, 23, 134, 447, 1124, 2375, \ldots$

**Find each common difference.**

76. $\sum_{n=1}^{100} (6n + 2)$

77. $\sum_{n=21}^{65} \left(8 - \frac{2n}{3}\right)$

78. $a_{12} = 63$, $a_{19} = 7$

79. $a_8 = -4$, $a_{27} = \frac{7}{3}$

80. **CALCULUS** The area between the graph of a continuous function and the $x$-axis can be approximated using sequences. Consider $f(x) = x^2$ on the interval $[1, 3]$.

a. Write the sequence $x_n$ formed when there are 5 arithmetic means between 1 and 3.

b. Write the sequence $y_n$ formed when $y_n = f(x_n)$.

c. Write the sequence $p_n$ defined by $d \cdot y_n$.

d. The *left-hand* approximation of the area is given by $L_n = \sum_{k=1}^{n} p_k$. Find $L_6$.

e. The *right-hand* approximation of the area is given by $R_n = \sum_{k=2}^{n+1} p_k$. Find $R_6$.

## H.O.T. Problems Use Higher-Order Thinking Skills

81. **ERROR ANALYSIS** Peter and Candace are given the arithmetic sequence $2, 9, 16, \ldots$. Peter wrote the explicit formula $a_n = 2 + 7(n - 1)$ for the sequence. Candace's formula is $a_n = 7n - 5$. Is either of them correct? Explain.

82. **OPEN ENDED** You have learned that the $n$th term of an arithmetic sequence can be modeled by a linear function. Can the sequence of partial sums of an arithmetic series also be modeled by a linear function? If yes, provide an example. If no, how can the sequence be modeled? Explain.

83. **CHALLENGE** Prove that for an arithmetic sequence, $a_n = a_k + (n - k)d$ for integers $k$ in the domain of the sequence.

**REASONING Determine whether each statement is *true* or *false* for finite arithmetic series. Explain.**

84. If you know the sum and $d$, you can solve for $a_1$.

85. If you only know the first and last terms, then you can find the sum.

86. If the first three terms of a sequence are positive, then all of the terms of the sequence are positive or the sum of the series is positive.

87. **CHALLENGE** Consider the arithmetic sequence of odd natural numbers.

a. Find $S_7$ and $S_9$.

b. Make a conjecture about the pattern that results from the sums of the corresponding arithmetic series.

c. Write an algebraic proof verifying the conjecture that you made in part **b**.

88. **WRITING IN MATH** Explain why the arithmetic series $25 + 20 + 15 + \ldots$ does not have a sum.

## Spiral Review

**Find the next four terms of each sequence.** (Lesson 10-1)

**89.** 12, 16, 20, …

**90.** 3, 1, −1, …

**91.** 31, 24, 17, …

**Find each product or quotient and express it in rectangular form.** (Lesson 9-5)

**92.** $6\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \cdot 3\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

**93.** $3\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right) \div \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

**Find the dot product of u and v. Then determine if u and v are orthogonal.** (Lesson 8-3)

**94.** $\mathbf{u} = \langle 4, -1\rangle$, $\mathbf{v} = \langle 1, 5\rangle$

**95.** $\mathbf{u} = \langle 8, -3\rangle$, $\mathbf{v} = \langle 4, 2\rangle$

**96.** $\mathbf{u} = \langle 4, 6\rangle$, $\mathbf{v} = \langle 9, -5\rangle$

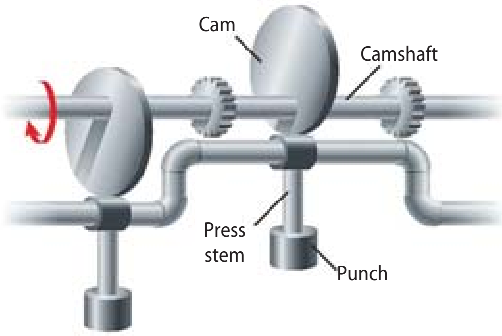
**Find the direction angle of each vector to the nearest tenth of a degree.** (Lesson 8-2)

**97.** $-\mathbf{i} - 3\mathbf{j}$

**98.** $\langle -9, 5\rangle$

**99.** $\langle -7, 7\rangle$

**100. MANUFACTURING** A cam in a punch press is shaped like an ellipse with the equation $\frac{x^2}{81} + \frac{y^2}{36} = 1$. The camshaft goes through the focus on the positive axis. (Lesson 7-4)




**a.** Graph a model of the cam.

**b.** Find an equation that translates the model so that the camshaft is at the origin.

**c.** Find the equation of the model in part **b** when the cam is rotated to an upright position.

**101.** Use the graph of $f(x) = \ln x$ to describe the transformation that results in the graph of $g(x) = 3\ln(x - 1)$. Then sketch the graphs of $f$ and $g$. (Lesson 3-2)

## Skills Review for Standardized Tests

**102. SAT/ACT** What is the units digit of $3^{36}$?

**A** 0

**B** 1

**C** 3

**D** 7

**E** 9

**103.** Using the table, which formula can be used to determine the $n$th term of the sequence?

| $n$ | $a_n$ |
|---|---|
| 1 | 6 |
| 2 | 10 |
| 3 | 14 |
| 4 | 18 |

**F** $a_n = 6n$

**G** $a_n = n + 5$

**H** $a_n = 2n + 1$

**J** $a_n = 4n + 2$

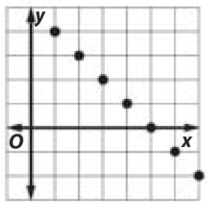
**104. REVIEW** If $a_1 = 3$, $a_2 = 5$, and $a_n = a_{n-2} + 3n$, find $a_{10}$.

**A** 59 **C** 89

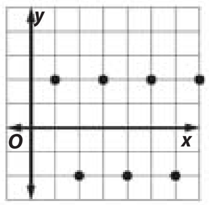
**B** 75 **D** 125

**105. REVIEW** Which of the sequences shown below is convergent?

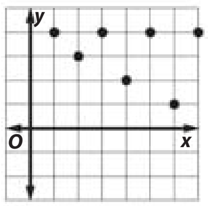
**F**



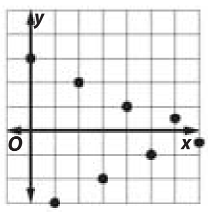
**H**



**G**



**J**



# LESSON 10-3 Geometric Sequences and Series

## Then

- You found terms and means of arithmetic sequences and sums of arithmetic series. (Lesson 10-2)

## Now

1. Find *n*th terms and geometric means of geometric sequences.
2. Find sums of *n* terms of geometric series and the sums of infinite geometric series.

## Why?

- The first summer X Games took place in Rhode Island in 1995 and included 27 events. Due to their growing popularity, the winter X Games were introduced at Big Bear Lake, California, in 1997. With an immense fan base, the annual X Games now receive live 24-hour network coverage. Since its inaugural year, the event has seen an average growth of 13% in revenue each year.

**NewVocabulary**
geometric sequence
common ratio
geometric means
geometric series

## 1 Geometric Sequences

A sequence in which the ratio between successive terms is a constant is called a **geometric sequence**. The constant is referred to as the **common ratio**, denoted $r$. To find the common ratio of a geometric sequence, divide any term following the first term by the preceding term. Given a term of the sequence, to find the next term of the sequence, multiply the given term by the common ratio. While the rate of change of an arithmetic sequence is constant, the rate of change of a geometric sequence can either increase or decrease.

### Example 1 Geometric Sequences

**Determine the common ratio, and find the next three terms of each geometric sequence.**

**a.** $8, -2, \frac{1}{2}, \ldots$

First, find the common ratio.

$a_2 \div a_1 = -2 \div 8 \text{ or } -\frac{1}{4}$

$a_3 \div a_2 = \frac{1}{2} \div -2 \text{ or } -\frac{1}{4}$

Find the ratio between two pairs of consecutive terms to verify the common ratio.

The common ratio is $-\frac{1}{4}$. Multiply the third term by $-\frac{1}{4}$ to find the fourth term, and so on.

$a_4 = \frac{1}{2}\left(-\frac{1}{4}\right) \text{ or } -\frac{1}{8}$ $\quad a_5 = -\frac{1}{8}\left(-\frac{1}{4}\right) \text{ or } \frac{1}{32}$ $\quad a_6 = \frac{1}{32}\left(-\frac{1}{4}\right) \text{ or } -\frac{1}{128}$

The next three terms are $-\frac{1}{8}, \frac{1}{32}$, and $-\frac{1}{128}$.

**b.** $w + 3, 2w + 6, 4w + 12, \ldots$

First, find the common ratio.

$a_2 \div a_1 = \frac{2w + 6}{w + 3}$ — $a_2 = 2w + 6$ and $a_1 = w + 3$

$= \frac{2(w + 3)}{w + 3}$ — Factor.

$= 2$ — Simplify.

$a_3 \div a_2 = \frac{4w + 12}{2w + 6}$ — $a_3 = 4w + 12$ and $a_2 = 2w + 6$

$= \frac{4(w + 3)}{2(w + 3)}$ — Factor.

$= 2$ — Simplify.

The common ratio is 2. Multiply the third term by 2 to find the fourth term, and so on.

$a_4 = 2(4w + 12) \text{ or } 8w + 24$

$a_5 = 2(8w + 24) \text{ or } 16w + 48$

$a_6 = 2(16w + 48) \text{ or } 32w + 96$

The next three terms are $8w + 24$, $16w + 48$, and $32w + 96$.

**GuidedPractice**

**1A.** 4, 11, 30.25, …

**1B.** $64r - 128, -16r + 32, 4r - 8, \ldots$

**WatchOut!**

**Type of Sequence** Remember that if a sequence is not arithmetic, it does not necessarily mean that the sequence is geometric. Test several terms for a common ratio before determining that the sequence is indeed geometric.

In Lesson 10-2, you learned that arithmetic sequences can be defined both recursively and explicitly. This also applies to geometric sequences. A geometric sequence can be expressed recursively, where a term $a_n$ is found by taking the product of the previous term $a_{n-1}$ and $r$, or $a_n = a_{n-1}r$, as illustrated by the previous example. To develop an explicit formula for a geometric sequence, consider the pattern created by the geometric sequence for which $a_1 = 3$ and $r = 4$.

| | Term | Expanded Form | Exponential Form | Example |
|---|---|---|---|---|
| first term | $a_1$ | $a_1$ | $a_1$ | 3 |
| second term | $a_2$ | $a_1 \cdot r$ | $a_1r^1$ | $3 \cdot 4 = 12$ |
| third term | $a_3$ | $a_1 \cdot r \cdot r$ | $a_1r^2$ | $3 \cdot 4^2 = 48$ |
| fourth term | $a_4$ | $a_1 \cdot r \cdot r \cdot r$ | $a_1r^3$ | $3 \cdot 4^3 = 192$ |
| fifth term | $a_5$ | $a_1 \cdot r \cdot r \cdot r \cdot r$ | $a_1r^4$ | $3 \cdot 4^4 = 768$ |
| $n$th term | $a_n$ | $\underbrace{a_1 \cdot r \cdot r \cdot r \cdot \ldots \cdot r}_{n-1 \text{ factors}}$ | $a_1r^{n-1}$ | $3 \cdot 4^{n-1}$ |

## KeyConcept The $n$th Term of a Geometric Sequence

**Words** The $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ is given by $a_n = a_1r^{n-1}$.

**Example** The 9th term of 2, 10, 50, … is $a_9 = 2 \cdot 5^{9-1}$ or 781,250.

## Example 2 Explicit and Recursive Formulas

**Write an explicit formula and a recursive formula for finding the $n$th term of the geometric sequence given in Example 1a.**

For an explicit formula, substitute $a_1 = 8$ and $r = -0.25$ in the $n$th term formula.

$a_n = a_1r^{n-1}$ — $n$th term of a geometric sequence

$= 8(-0.25)^{n-1}$ — $a_1 = 8$ and $r = -0.25$

For a recursive formula, state the first term $a_1$. Then indicate that the next term is the product of the previous term $a_{n-1}$ and $r$.

$a_1 = 8, a_n = (-0.25)a_{n-1}$

### GuidedPractice

**2.** Write an explicit formula and a recursive formula for finding the $n$th term in the sequence 2, 25, 312.5, … .

Finding the $n$th term of a geometric sequence is simplified by explicit formulas.

## Example 3 $n$th Terms

**Find the 27th term of the geometric sequence 189, 151.2, 120.96, … .**

First, find the common ratio.

$a_2 \div a_1 = 151.2 \div 189$ or 0.8

$a_3 \div a_2 = 120.96 \div 151.2$ or 0.8

Find the ratio between two pairs of consecutive terms to verify the common ratio.

Use the formula for the $n$th term of a geometric sequence.

$a_n = a_1r^{n-1}$ — $n$th term of a geometric sequence

$a_{27} = 189(0.8)^{27-1}$ — $n = 27$, $a_1 = 189$, and $r = 0.8$

$a_{27} \approx 0.57$ — Simplify.

## **Guided**Practice

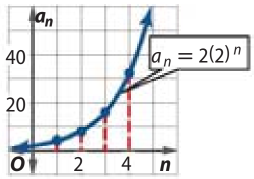
**Find the specified term of each geometric sequence or sequence with the given characteristics.**

**3A.** $a_9$ for 4, 14, 49, …

**3B.** $a_{12}$ if $a_3 = 32$ and $r = -4$

Just as arithmetic sequences are linear functions with restricted domains, geometric sequences are also functions. Consider the exponential function $f(x) = 2(2)^x$ and the explicit formula for the geometric sequence $a_n = 2(2)^n$.

Notice that the graphs of the terms of the geometric sequence lie on a curve, as shown. A geometric sequence can be modeled by an exponential function in which the domain is restricted to the natural numbers.




## **Real-World**Link

The value of a newly purchased vehicle can depreciate by as much as 30–35% in its first year. Each year after, the value continues to depreciate by 7–12%, depending on the make and model. After a five-year period, on average, cars are worth 35% of the original sticker price, making car buying costly.

**Source:** *Kelly Blue Book*

## Real-World Example 4 *n*th Term of a Geometric Sequence

**AUTOMOBILE** **Damian purchased a late-model car for $15,000. At the end of each year, the value of the car depreciates 11%.**

**a. Write an explicit formula for the value of Damian's car after *n* years.**

If the car's value depreciates at a rate of 11% per year, it retains 100% − 11% or 89% of its original value. Note that the original value given represents the $a_0$ and not the $a_1$ term, so we need to use an adjusted formula for the *n*th term of this geometric sequence.

first term $\quad a_1 = a_0r$

second term $\quad a_2 = a_0r^2$

⋮ $\quad$ ⋮

*n*th term $\quad a_n = a_0r^n$

Use this adjusted formula to find an explicit formula for the value of the car after *n* years.

$a_n = a_0r^n$ $\quad$ Adjusted *n*th term of a geometric sequence

$a_n = 15{,}000(0.89)^n$ $\quad$ $a_0 = 15{,}000$, $r = 0.89$

**b. What is the value of Damian's car at the end of the seventh year?**

Evaluate the formula found in part **a** for $n = 7$.

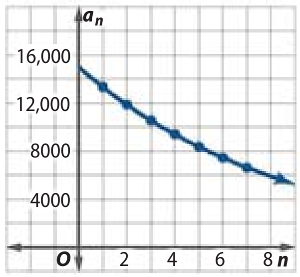
$a_n = 15{,}000(0.89)^n$ $\quad$ Original equation

$= 15{,}000(0.89)^7$ $\quad$ $n = 7$

$\approx 6634.70$ $\quad$ Simplify.

The value of the car at the beginning of the seventh year is about $6634.70.

The value of the car at each year is shown as a point on the graph. The function connecting the points represents exponential decay.




## **Guided**Practice

**4.** **WATERCRAFT** Rohan purchased a personal watercraft for $9000. Assume that by the end of each year, the value of the watercraft depreciates 30%.

**A.** Write an explicit formula for finding the value of Rohan's watercraft after *n* years.

**B.** What is the value of Rohan's watercraft after 5 years?

Stan Honda/AFP/Getty Images

Similar to arithmetic sequences, if two nonconsecutive terms of a geometric sequence are known, the terms between them can be calculated. These terms are called **geometric means**.

### Example 5 Geometric Means

**Write a sequence that has two geometric means between 480 and −7.5.**

The sequence will resemble 480, $\underline{\ ?\ }$, $\underline{\ ?\ }$, −7.5.

Note that $a_1 = 480$, $n = 4$, and $a_4 = -7.5$. Find the common ratio using the $n$th term for a geometric sequence formula.

| | |
|---|---|
| $a_4 = a_1 r^{n-1}$ | $n$th term of a geometric sequence |
| $-7.5 = 480r^{4-1}$ | $a_4 = -7.5$, $a_1 = 480$, and $n = 4$ |
| $-\frac{1}{64} = r^3$ | Simplify and divide each side by 480. |
| $-\frac{1}{4} = r$ | Take the cube root of each side. |

The common ratio is $-\frac{1}{4}$. Use $r$ to find the geometric means.

$a_2 = 480(-0.25)$ or $-120$ $\qquad$ $a_3 = -120(-0.25)$ or $30$

Therefore, a sequence with two geometric means between 480 and −7.5, is 480, −120, 30, −7.5.

**Study Tip**

**Geometric Means** Sometimes, more than one set of geometric means are possible. For example, the three geometric means between 3 and 48 can be 6, 12, and 24 *or* −6, 12, and −24.

**Guided Practice**

**Find the indicated geometric means for each pair of nonconsecutive terms.**

**5A.** −4 and 13.5; 2 means

**5B.** 10 and 0.016; 3 means

## 2 Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence.

| Geometric Sequence | Geometric Series |
|---|---|
| 2, 4, 8, 16, 32 | 2 + 4 + 8 + 16 + 32 |
| $27, 9, 3, 1, \frac{1}{3}$ | $27 + 9 + 3 + 1 + \frac{1}{3}$ |
| $a_1, a_2, a_3, a_4, \ldots, a_n$ | $a_1 + a_2 + a_3 + a_4 + \cdots + a_n$ |

A formula for the sum $S_n$ of the first $n$ terms of a finite geometric series can be developed by looking at the series $S_n$ and $rS_n$. To create the terms for $rS_n$, each term in $S_n$ is multiplied by $r$. These series are then aligned so that similar terms are grouped together and then $rS_n$ is subtracted from $S_n$.

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} \\ (-)\ rS_n &= \phantom{a_1 + {}} a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \end{aligned}$$

| | |
|---|---|
| $S_n - rS_n = a_1 - a_1r^n$ | Subtract. |
| $S_n(1 - r) = a_1 - a_1r^n$ | Factor. |
| $S_n = \frac{a_1 - a_1r^n}{1 - r}$ | Divide each side by $1 - r$. |
| $S_n = \frac{a_1(1 - r^n)}{1 - r}$ | Factor. |

Therefore, $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

If the value of $n$ is not provided, the sum of a finite geometric series can still be found. If we take a look at the next-to-last step of the proof, we can substitute for $a_1r^n$.

| | |
|---|---|
| $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$ | Sum of a finite geometric series formula |
| $S_n = \frac{a_1 - a_1r^n}{1-r}$ | Multiply. |
| $= \frac{a_1 - a_1r^{n-1} \cdot r}{1-r}$ | Factor one $r$ from $a_1r^n$. |
| $= \frac{a_1 - a_nr}{1-r}$ | $a_1r^{n-1} = a_n$, $n$th term of a geometric sequence formula |

## KeyConcept Sum of a Finite Geometric Series

The sum of a finite geometric series with $n$ terms or the $n$th partial sum of a geometric series can be found using one of two related formulas.

**Formula 1** $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

**Formula 2** $S_n = \frac{a_1 - a_nr}{1-r}$

## Example 6 Sums of Geometric Series

**a. Find the sum of the first six terms of the geometric series 8 + 14 + 24.5 + … .**

First, find the common ratio.

$a_2 \div a_1 = 14 \div 8$ or 1.75

$a_3 \div a_2 = 24.5 \div 14$ or 1.75

Find the ratio between two pairs of consecutive terms to verify the common ratio.

The common ratio is 1.75. Use Formula 1 to find the sum of the series.

| | |
|---|---|
| $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$ | Formula 1 |
| $S_6 = 8\left(\frac{1-1.75^6}{1-1.75}\right)$ | $n = 6$, $a_1 = 8$, and $r = 1.75$ |
| $S_6 \approx 295.71$ | Simplify. |

The sum of the first six terms of the geometric series is about 295.71.

**CHECK** The next three terms of the related sequence are 42.875, 75.03125, and 131.3046875.

$8 + 14 + 24.5 + 42.875 + 75.03125 + 131.3046875 \approx 295.71$ ✔

**b. Find the sum of the first $n$ terms of a geometric series with $a_1 = 3$, $a_n = 768$, and $r = -2$.**

Use Formula 2 for the sum of a finite geometric series.

| | |
|---|---|
| $S_n = \frac{a_1 - a_nr}{1-r}$ | Formula 2 |
| $= \frac{3 - 768(-2)}{1-(-2)}$ | $a_1 = 3$, $a_n = 768$, and $r = -2$ |
| $= 513$ | Simplify. |

The sum of the first $n$ terms of the geometric series is 513.

### GuidedPractice

**6A.** Find the sum of the first 11 terms of the geometric series $7 + (-24.5) + 85.75 + \dots$ .

**6B.** Find the sum of the first $n$ terms of a geometric series with $a_1 = -8$, $a_n = 131{,}072$, and $r = -4$.

**StudyTip**

**Infinite vs. Finite Geometric Series** Notice that the series given in Example 6a is an infinite geometric series. Because you are asked to find the sum of the *first six terms* of the series, you are actually finding the sum of a finite series.

Geometric series may also be represented in sigma notation.

## Example 7 Geometric Sum in Sigma Notation

**Find** $\sum_{n=2}^{7} 3(5)^{n-1}$.

Find $n$, $a_1$, and $r$.

$n = 7 - 2 + 1$ or $6$ — Upper bound minus lower bound plus 1

$a_1 = 3(5)^2 - 1$ or $15$ — $n = 2$

$r = 5$ — $r$ is the base of the exponential function.

**Method 1** Substitute $n = 6$, $a_1 = 15$, and $r = 5$ into Formula 1.

$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$ — Formula 1

$S_6 = 15\left(\frac{1 - 5^6}{1 - 5}\right)$ — $a_1 = 15$, $r = 5$, and $n = 6$

$S_6 = 58,590$ — Simplify.

**Method 2** Find $a_n$.

$a_n = a_1 r^{n-1}$ — $n$th term of a geometric sequence

$= 15(5)^{6-1}$ — $a_1 = 15$, $r = 5$, and $n = 6$

$= 46,875$ — Simplify.

Substitute $a_1 = 15$, $a_n = 46,875$, and $r = 5$ into Formula 2.

$S_n = \frac{a_1 - a_n r}{1 - r}$ — Formula 2

$S_6 = \frac{15 - (46,875)(5)}{1 - 5}$ — $a_1 = 15$, $a_n = 46,875$, and $r = 5$

$S_6 = 58,590$ — Simplify.

Therefore, $\sum_{n=2}^{7} 3(5)^{n-1} = 58,590$.

### Guided Practice

**7A.** $\sum_{n=16}^{31} 0.5(2)^{n-1}$

**7B.** $\sum_{n=4}^{11} 120(0.5)^{n-1}$

**Study Tip**

**Infinite Series** If the sequence of partial sums $S_n$ has a limit, then the corresponding infinite series has a sum and the $n$th term $a_n$ of the series approaches 0 as $n \to \infty$. However, if the $n$th term of the series approaches 0, the infinite series does not necessarily have a sum. For example, the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ does not have a sum.

In Lesson 10-1, you learned that calculating the sums of infinite series may be possible if the sequence of terms converges to 0. For this reason, the sums of infinite arithmetic series cannot be found.

The formula for the sum of a finite geometric series can be used to develop a formula for the sum of an infinite geometric series. If $|r| > 1$, then $|r^n|$ increases without limit as $n \to \infty$. However, when $|r| < 1$, $r^n$ approaches 0 as $n \to \infty$. Thus,

$S = a_1\left(\frac{1 - r^n}{1 - r}\right)$ — Sum of a finite geometric series formula

$= a_1\left(\frac{1 - 0}{1 - r}\right)$ — $r^n$ approaches 0 as $n \to \infty$.

$= \frac{a_1}{1 - r}$ — Simplify and multiply.

### Key Concept The Sum of an Infinite Geometric Series

The sum $S$ of an infinite geometric series for which $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

The formula for the sum of an infinite geometric series involves three variables: $S$, $a_1$, and $r$. If any two of the three variables are known, you can solve for the third.

## Example 8 Sums of Infinite Geometric Series

**If possible, find the sum of each infinite geometric series.**

**a.** $9 + 3 + 1 + \ldots$

First, find the common ratio.

$a_2 \div a_1 = 3 \div 9 \text{ or } \frac{1}{3}$ — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 1 \div 3 \text{ or } \frac{1}{3}$

The common ratio $r$ is $\frac{1}{3}$, and $\left|\frac{1}{3}\right| < 1$. This infinite geometric series has a sum. Use the formula for the sum of an infinite geometric series.

$S = \frac{a_1}{1 - r}$ — Sum of an infinite geometric series formula

$= \frac{9}{1 - \frac{1}{3}}$ — $a_1 = 9$ and $r = \frac{1}{3}$

$= 13.5$ — Simplify.

The sum of the infinite series is 13.5.

**StudyTip**

**Common Ratio** Recall that $|r| < 1$ is equivalent to $-1 < r < 1$.

**b.** $0.25 + (-1.25) + 6.25 + \ldots$

First, find the common ratio.

$a_2 \div a_1 = -1.25 \div 0.25 \text{ or } -5$ — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 6.25 \div (-1.25) \text{ or } -5$

The common ratio $r$ is $-5$, and $|-5| > 1$. Therefore, this infinite geometric series has no sum.

**c.** $\sum_{n=4}^{\infty} 4(0.2)^{n-1}$

The common ratio $r$ is 0.2, and $|0.2| < 1$. Therefore, this infinite geometric series has a sum. Find $a_1$.

$a_1 = 4(0.2)^{4-1}$ — Lower bound = 4

$= 0.032$ — Simplify.

Use the formula for the sum of an infinite geometric series to find the sum.

$S = \frac{a_1}{1 - r}$ — Sum of an infinite geometric series formula

$= \frac{0.032}{1 - 0.2}$ — $a_1 = 0.032$ and $r = 0.2$

$= 0.04$ — Simplify.

The sum of the infinite series is 0.04.

### GuidedPractice

**8A.** $10 + (-5) + 2.5 + \ldots$

**8B.** $20 + 15 + 10 + \ldots$

**8C.** $\sum_{n=1}^{\infty} 120(0.8)^{n-1}$

## Exercises

= Step-by-Step Solutions begin on page R29.

**Determine the common ratio, and find the next three terms of each geometric sequence.** (Example 1)

**1.** $-\frac{1}{4}, \frac{1}{2}, -1, \ldots$

**2.** $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$

**3.** $0.5, 0.75, 1.125, \ldots$

**4.** $8, 20, 50, \ldots$

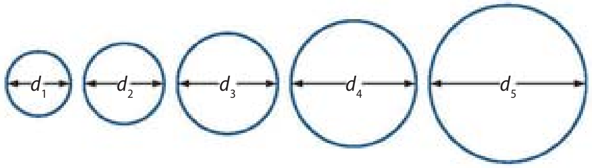
**5.** $2x, 10x, 50x, \ldots$

**6.** $64x, 16x, 4x, \ldots$

**7.** $x + 5, 3x + 15, 9x + 45, \ldots$

**8.** $-9 - y, 27 + 3y, -81 - 9y, \ldots$

**9.** **GEOMETRY** Consider a sequence of circles with diameters that form a geometric sequence: $d_1, d_2, d_3, d_4, d_5$.




**a.** Show that the sequence of circumferences of the circles is also geometric. Identify $r$.

**b.** Show that the sequence of areas of the circles is also geometric. Identify the common ratio.

**Write an explicit formula and a recursive formula for finding the *n*th term of each geometric sequence.** (Example 2)

**10.** $36, 12, 4, \ldots$

**11.** $64, 16, 4, \ldots$

**12.** $-2, 10, -50, \ldots$

**13.** $4, -12, 36, \ldots$

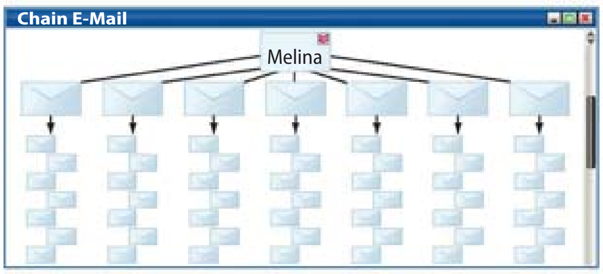
**14.** $4, 8, 16, \ldots$

**15.** $20, 30, 45, \ldots$

**16.** $15, 5, \frac{5}{3}, \ldots$

**17.** $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$

**18.** **CHAIN E-MAIL** Melina receives a chain e-mail that she forwards to 7 of her friends. Each of her friends forwards it to 7 of their friends. (Example 2)




**a.** Write an explicit formula for the pattern.

**b.** How many will receive the e-mail after 6 forwards?

**19.** **BIOLOGY** A certain bacteria divides every 15 minutes to produce two complete bacteria. (Example 2)

**a.** If an initial colony contains a population of $b_0$ bacteria, write an equation that will determine the number of bacteria $b_t$ present after $t$ hours.

**b.** Suppose a Petri dish contains 12 bacteria. Use the equation found in part **a** to determine the number of bacteria present 4 hours later.

**Find the specified term for each geometric sequence or sequence with the given characteristics.** (Example 3)

**20.** $a_9$ for $60, 30, 15, \ldots$

**21.** $a_4$ for $7, 14, 28, \ldots$

**22.** $a_5$ for $3, 1, \frac{1}{3}, \ldots$

**23.** $a_6$ for $540, 90, 15, \ldots$

**24.** $a_7$ if $a_3 = 24$ and $r = 0.5$

**25.** $a_6$ if $a_3 = 32$ and $r = -0.5$

**26.** $a_6$ if $a_1 = 16{,}807$ and $r = \frac{3}{7}$

**27.** $a_8$ if $a_1 = 4096$ and $r = \frac{1}{4}$

**28.** **ACCOUNTING** Julian Rockman is an accountant for a small company. On January 1, 2009, the company purchased \$50,000 worth of computers, printers, scanners, and hardware. Because this equipment is a company asset, Mr. Rockman needs to determine how much the computer equipment is presently worth. He estimates that the computer equipment depreciates at a rate of 45% per year. What value should Mr. Rockman assign the equipment in his 2014 year-end accounting report? (Example 4)

**29.** Find the sixth term of a geometric sequence with a first term of 9 and a common ratio of 2. (Example 4)

**30.** If $r = 4$ and $a_8 = 100$, what is the first term of the geometric sequence? (Example 4)

**31.** **X GAMES** Refer to the beginning of the lesson. The X Games netted approximately \$40 million in revenue in 2002. If the X Games continue to generate 13% more revenue each year, how much revenue will the X Games generate in 2020? (Example 4)

**Find the indicated geometric means for each pair of nonconsecutive terms.** (Example 5)

**32.** 4 and 256; 2 means

**33.** 256 and 81; 3 means

**34.** $\frac{4}{7}$ and 7; 1 mean

**35.** $-2$ and 54; 2 means

**36.** 1 and 27; 2 means

**37.** 48 and $-750$; 2 means

**38.** $i$ and $-1$; 4 means

**39.** $t^8$ and $t^{-7}$; 4 means

**Find the sum of each geometric series described.** (Example 6)

**40.** first six terms of $3 + 9 + 27 + \cdots$

**41.** first nine terms of $0.5 + (-1) + 2 + \cdots$

**42.** first eight terms of $2 + 2\sqrt{3} + 6 + \cdots$

**43.** first $n$ terms of $a_1 = 4, a_n = 2000, r = -3$

**44.** first $n$ terms of $a_1 = 5, a_n = 1{,}747{,}625, r = 4$

**45.** first $n$ terms of $a_1 = 3, a_n = 46{,}875, r = -5$

**46.** first $n$ terms of $a_1 = -8, a_n = -256, r = 2$

**47.** first $n$ terms of $a_1 = -36, a_n = 972, r = 7$

**Find each sum.** (Example 7)

**48.** $\sum_{n=1}^{6} 5(2)^{n-1}$

**49.** $\sum_{n=1}^{5} -4(3)^{n-1}$

**50.** $\sum_{n=1}^{5} (-3)^{n-1}$

**51.** $\sum_{n=1}^{6} 2(1.4)^{n-1}$

**52.** $\sum_{n=1}^{6} 100\left(\frac{1}{2}\right)^{n-1}$

**53.** $\sum_{n=1}^{9} \frac{1}{27}(-3)^{n-1}$

**54.** $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

**55.** $\sum_{n=1}^{20} 3(2)^{n-1}$

**If possible, find the sum of each infinite geometric series.** (Example 8)

**56.** $\frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \cdots$

**57.** $\frac{2}{7} + \frac{4}{7} + \frac{8}{7} + \cdots$

**58.** $18 + (-27) + 40.5 + \cdots$

**59.** $12 + (-7.2) + 4.32 + \cdots$

**60.** $\sum_{n=1}^{\infty} 6(-0.4)^{n-1}$

**61.** $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

**62.** $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{8}\right)^{n-1}$

**63.** $\sum_{n=1}^{\infty} 35\left(-\frac{3}{4}\right)^{n-1}$

**64. BUNGEE JUMPING** A bungee jumper falls 35 meters before his cord causes him to spring back up. He rebounds $\frac{2}{5}$ of the distance after each fall. (Example 8)

**a.** Find the first five terms of the infinite sequence representing the vertical distance traveled by the bungee jumper. Include each drop and rebound distance as separate terms.

**b.** What is the total vertical distance the jumper travels before coming to rest? (*Hint*: Rewrite the infinite sequence suggested by part **a** as two infinite geometric sequences.)

**Find the missing quantity for the geometric sequence with the given characteristics.**

**65.** Find $a_1$ if $S_{12} = 1365$ and $r = 2$.

**66.** If $S_6 = 196.875$, $a_1 = 100$, $r = 0.5$, find $a_6$.

**67.** Find $r$ if $a_1 = 0.12$, $S_n = 590.52$, and $a_n = 787.32$.

**68.** Find $n$ for $4.1 + 8.2 + 16.4 + \cdots$ if $S_n = 61.5$.

**69.** If $15 - 18 + 21.6 - \cdots$, $S_n = 23.784$, find $a_n$.

**70.** If $r = -0.4$, $S_5 = 144.32$, and $a_1 = 200$, find $a_5$.

**71.** Find $a_1$ if $S_n = 468$, $a_n = 375$, and $r = 5$.

**72.** If $S_n = \frac{61}{40}$, $\frac{5}{8} + \frac{1}{2} + \frac{2}{5} + \cdots$, find $n$.

**73. LOANS** Marc is making monthly payments on a loan. Suppose instead of the same monthly payment, the bank requires a low initial payment that grows at an exponential rate each month. The total cost of the loan is represented by $\sum_{n=1}^{k} 5(1.1)^{n-1}$.

**a.** What is Marc's initial payment and at what rate is this payment increasing?

**b.** If the sum of Marc's payments at the end of the loan is \$7052, how many payments did Marc make?

**Find the common ratio for the geometric sequence with the given terms.**

**74.** $a_3 = 12$, $a_6 = 187.5$

**75.** $a_2 = -6$, $a_7 = -192$

**76.** $a_4 = -28$, $a_6 = -1372$

**77.** $a_5 = 6$, $a_8 = -0.048$

**78. ADVERTISING** Word-of-mouth advertising can be an effective form of marketing, or it can be very harmful. Consider a new restaurant that serves 27 customers on its opening night.

**a.** Of the 27 customers, 25 found the experience enjoyable and each told 3 friends over the next month. This group each told 3 friends over the next month, and so on, for 6 months. Assuming that no one heard twice, how many people have had a positive experience or heard positive reviews of the restaurant?

**b.** Suppose the 2 unhappy customers each told 6 friends over the next month about the experience. This group then each told 6 friends, and so on, for 6 months. Assuming that no one heard a review twice, how many people have had a negative experience or heard a negative review?

**Write the first 3 terms of the infinite geometric series with the given characteristics.**

**79.** $S = 12$, $r = \frac{1}{2}$

**80.** $S = -25$, $r = 0.2$

**81.** $S = 44.8$, $a_1 = 56$

**82.** $S = \frac{2}{3}$, $a_1 = \frac{8}{9}$

**83.** $S = -60$, $r = 0.4$

**84.** $S = -126.25$, $a_1 = -50.5$

**85.** $S = -115$, $a_1 = -138$

**86.** $S = \frac{891}{20}$, $r = -\frac{1}{9}$

**87.** $\sum_{n=1}^{\infty} 12\left(-\frac{1}{4}\right)^{n-1}$

**88.** $\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{3}\right)^{n-1}$

**Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Then find the next three terms of the sequence.**

**89.** $\frac{1}{4}, \frac{2}{6}, \frac{3}{8}, \frac{4}{10}, \ldots$

**90.** $\frac{9}{2}, \frac{17}{4}, 4, \frac{15}{4}, \ldots$

**91.** $12, 24, 36, 48, \ldots$

**92.** $128, 96, 72, 54, \ldots$

**93.** $36k, 49k, 64k, 81k, \ldots$

**94.** $7.2y, 9.1y, 11y, 12.9y, \ldots$

**95.** $3\sqrt{5}, 15, 15\sqrt{5}, 75, \ldots$

**96.** $2\sqrt{3}, 2\sqrt{6}, 2\sqrt{9}, 2\sqrt{12}, \ldots$

**Write each geometric series in sigma notation.**

**97.** $3 + 12 + 48 + \cdots + 3072$

**98.** $9 + 18 + 36 + \cdots + 1152$

**99.** $50 + 85 + 144.5 + \cdots + 417.605$

**100.** $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \cdots + 8$

**101.** $0.2 - 1 + 5 - \cdots - 625$

**102. HORSES** For each of the first few months after a horse is born, the amount of weight that it gains is about 120% of the previous month's weight gain. In the first month, a horse has gained 30 pounds.

**a.** Write a geometric series in sigma notation that can be used to model the horse's weight gain for the first five months.

**b.** About how much weight did the horse gain in the fourth month?

**c.** If the horse weighed 150 pounds at birth, about how much did it weigh after 5 months?

**d.** Will the horse continue to grow at this rate indefinitely? Explain.

**103. MEDICINE** A newly developed and researched medicine has a half-life of about 1.5 hours after it is administered. The medicine is given to patients in doses of $d$ milligrams every 6 hours.

**a.** What fraction of the first dose will be left in the patient's system when the second dose is taken?

**b.** Find the first four terms of the sequence that represents the amount of medicine in the patient's system after the first 4 doses.

**c.** Write a recursive formula that can be used to determine the amount of medicine in the patient's system after the $n$th dose.

**104. MULTIPLE REPRESENTATIONS** In this problem, you will investigate the limits of $\frac{1 - r^n}{1 - r}$.

**a. GRAPHICAL** Graph $S_n = \frac{1 - r^n}{1 - r}$ for $r = 0.2$, 0.5, and 0.9 on the same graph.

**b. TABULAR** Copy and complete the table shown below.

| $n$ | $S_n$, $r = 0.2$ | $S_n$, $r = 0.5$ | $S_n$, $r = 0.9$ |
|---|---|---|---|
| 0 | | | |
| 4 | | | |
| 8 | | | |
| 12 | | | |
| 16 | | | |
| 20 | | | |
| 24 | | | |

**c. ANALYTICAL** For each graph in part **a**, describe the values of $S_n$ as $n \to \infty$.

**d. GRAPHICAL** Graph $S_n = \frac{1 - r^n}{1 - r}$ for $r = 1.2$, 2.5, and 4 on the same graph.

**e. ANALYTICAL** For each graph in part **d**, describe the values of $S_n$ as $n \to \infty$.

**f. ANALYTICAL** Make a conjecture about what happens to $S_n$ as $n \to \infty$ for $S_n = \frac{1 - 8.6^n}{1 - 8.6}$.

## H.O.T. Problems Use Higher-Order Thinking Skills

**105. ERROR ANALYSIS** Emilio believes that the sum of the infinite geometric series $16 + 4 + 1 + 0.25 + \ldots$ can be calculated. Annie disagrees. Is either of them correct? Explain your reasoning.

**106. CHALLENGE** A ball is dropped from a height of 5 meters. On each bounce, the ball rises to 65% of the height it reached on the previous bounce.


**a.** Approximate the total vertical distance the ball travels, until it stops bouncing.

**b.** The ball makes its first complete bounce in 2 seconds, that is, from the moment it first touches the ground until it next touches the ground. Each complete bounce that follows takes 0.8 times as long as the preceding bounce. Estimate the total amount of time that the ball bounces.

**107. WRITING IN MATH** Explain why an infinite geometric series will not have a sum if $|r| > 1$.

**REASONING** **Determine whether each statement is *true* or *false*. Explain your reasoning.**

**108.** If the first two terms of a geometric sequence are positive, then the third term is positive.

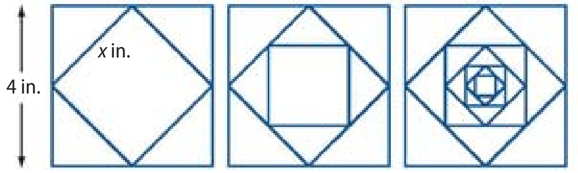
**109.** If you know $r$ and the sum of a finite geometric series, you can find the last term.

**110.** If $r$ is negative, then the geometric sequence converges.

**111. REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

*If all of the terms of an infinite geometric series are negative, then the series has a sum that is a negative number.*

**112. CHALLENGE** The midpoints of the sides of a square are connected so that a new square is formed. Suppose this process is repeated indefinitely.




**a.** What is the perimeter of the square with side lengths of $x$ inches?

**b.** What is the sum of the perimeters of all the squares?

**c.** What is the sum of the areas of all the squares?

## Spiral Review

**Find each sum.** (Lesson 10-2)

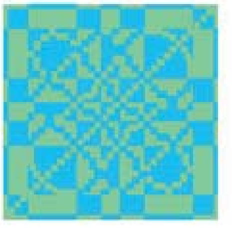
**113.** $\sum_{n=1}^{7} (2n + 1)$

**114.** $\sum_{n=3}^{7} (3n + 4)$

**115.** $\sum_{n=1}^{150} (11 + 2n)$

**116. TOURIST ATTRACTIONS** To prove that objects of different weights fall at the same rate, Marlene dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. If this pattern continues, how many feet would an object fall in the sixth second? (Lesson 10-1)

**117. TEXTILES** Patterns in fabric can often be created by modifying a mathematical graph. The pattern at the right can be modeled by a lemniscate. (Lesson 9-2)



**a.** Suppose the designer wanted to begin with a lemniscate that was 6 units from end to end. What polar equation could have been used?

**b.** What polar equation could have been used to generate a lemniscate that was 8 units from end to end?

**Graph each polar equation on a polar grid.** (Lesson 9-1)

**118.** $\theta = -\frac{\pi}{4}$

**119.** $r = 1.5$

**120.** $\theta = -150°$

**Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.** (Lesson 8-5)

**121.** $\mathbf{u} = \langle 1, 9, -1 \rangle$, $\mathbf{v} = \langle -2, 6, -4 \rangle$

**122.** $\mathbf{u} = \langle -3, 8, 2 \rangle$, $\mathbf{v} = \langle 1, -5, -7 \rangle$

**123.** $\mathbf{u} = \langle 9, 0, -4 \rangle$, $\mathbf{v} = \langle -6, 2, 5 \rangle$

**Find the component form and magnitude of $\overline{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overline{AB}$.** (Lesson 8-4)

**124.** $A(6, 7, 9)$, $B(18, 21, 18)$

**125.** $A(24, -6, 16)$, $B(8, 12, -4)$

**126.** $A(3, -5, 9)$, $B(-1, 15, -7)$

## Skills Review for Standardized Tests

**127. SAT/ACT** In the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$, each term after the first is equal to the previous term times a constant. What is the value of the 13th term?

**A** $2^7$

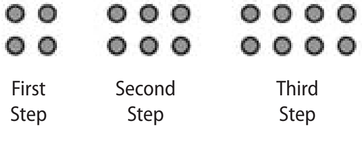
**B** $2^8$

**C** $2^9$

**D** $2^{10}$

**E** $2^{11}$

**128. REVIEW** The pattern of dots shown below continues infinitely, with more dots being added at each step.




Which expression can be used to determine the number of dots at the $n$th step?

**F** $2n$

**G** $n(n + 2)$

**H** $n(n + 1)$

**J** $2(n + 1)$

**129.** The first term of a geometric series is $-1$, and the common ratio is $-3$. How many terms are in the series if its sum is 182?

**A** 6

**B** 7

**C** 8

**D** 9

**130. REVIEW** Cora begins a phone tree to notify her friends about a party. In step 1, she calls 3 friends. In step 2, each of those friends calls 3 new friends. In step 3, each of those friends calls 3 more new friends. After step 3, how many people know about the party, including Cora?

**F** 12

**G** 13

**H** 39

**J** 40

# Graphing Technology Lab
# Continued Fractions

## Objective

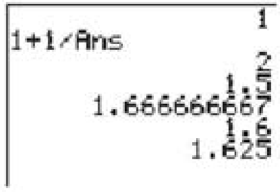
- Use a graphing calculator to represent continued fractions.

An expression of the following form is called a *continued fraction.* Continued fractions can be used to write sequences that approach limits.

$$a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ldots}}}$$

### Activity 1 $a = b = 1$

Evaluate $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$.




**Step 1** On the home screen, press 1 and ENTER.

**Step 2** Then type in 1 + 1 ÷ 2nd [ANS] ENTER.

**Step 3** Press ENTER four more times.

**StudyTip**

**Memory** You may need to clear the calculator's memory to eliminate any previously stored values.

## Analyze the Results

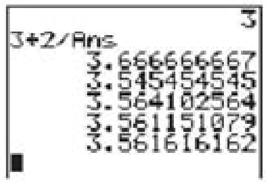
**1A.** Write the expression that was evaluated to generate each of the values obtained in Steps 2–3.

**1B.** Press ENTER 20 more times and record the result to approximate the limit of this sequence. This number is an approximation for the *golden ratio*.

**1C.** Solve $x = 1 + \frac{1}{x}$. How do you think this relates to the golden ratio?

### Activity 2 $a = 3$ and $b = 2$

Evaluate $3 + \cfrac{2}{3 + \cfrac{2}{3 + \cfrac{2}{3 + \ldots}}}$.




**Step 1** On the home screen, press 3 and ENTER.

**Step 2** Then type in 3 + 2 ÷ 2nd [ANS] ENTER.

**Step 3** Press ENTER four more times.

## Analyze the Results

**2A.** Write the expression that was evaluated to generate each of the values obtained in Steps 2–3.

**2B.** Press ENTER 20 more times. What is the approximation for the limit of this sequence?

**2C.** **MAKE A CONJECTURE** Use $a = 3$ and the approximation found in Exercise 2B to develop an expression for both $a$ and $b$. (*Hint*: Solve for the radicand, and determine how $b$ can be used to make it equal.) This is the general expression for the limit of the continued fraction sequence for any values of $a$ and $b$.

**StudyTip**

**Finding Expressions** When developing the expression in 2C, think of different ways in which $b$ can make the expression equivalent to the limit of the sequence. Use different values of $a$ and $b$ to confirm your answer.

## Exercises

**Approximate the value of the continued fraction with the given values for $a$ and $b$.**

**1.** $a = 4$ and $b = 3$ **2.** $a = 5$ and $b = 2$ **3.** $a = 3$ and $b = 1$

CHAPTER 10

# Mid-Chapter Quiz

## Lessons 10-1 through 10-3

Find the next four terms of each sequence. (Lesson 10-1)

1. 109, 97, 85, 73, …
2. 2, 6, 14, 30, …
3. 0, 1, 5, 14, …
4. $-2187, 729, -243, 81, \ldots$

5. **NATURE** A petting zoo starts a population of rabbits with one newborn male and one newborn female. Assuming that each adult pair will produce one male and one female offspring per month starting at two months, how many rabbits will there be after 6 months? (Lesson 10-1)

Determine whether each sequence is *convergent* or *divergent*. (Lesson 10-1)

6. 3, 5, 8, 12, …
7. $a_1 = 15, a_n = \frac{a_{n-1} - 1}{3}$
8. 48, 24, 12, 6, …
9. $a_n = n^2 + 5n$

Find each sum. (Lesson 10-1)

10. $\sum_{n=0}^{9} \frac{n^2}{4}$

11. $\sum_{n=-5}^{0} (n^3 + 7)$

12. $\sum_{n=1}^{6} (2^n - 4)$

13. $\sum_{n=8}^{13} (4n - 10)$

14. **GOLF** In a charity golf tournament, each of the top ten finishers wins a donation to the charity of his or her choice. The amount of the donation follows the arithmetic sequence shown below. What is the total amount of money donated to charity as a result of the tournament? (Lesson 10-2)


Write an explicit formula and a recursive formula for finding the $n$th term of each arithmetic sequence. (Lesson 10-2)

15. $-11, -15, -19, -23, \ldots$
16. $-96, -84, -72, -60, \ldots$
17. 7, 10, 13, 16, …
18. 32, 30, 28, 26, …

19. **JEWELRY** Mary Anne is hosting a jewelry party. For each guest who buys an item of jewelry, she gets a hostess bonus in the amount shown. She receives a larger amount for each guest making a purchase. (Lesson 10-2)

| Guests Purchasing Jewelry | Amount Mary Anne Receives ($) |
|---|---|
| first | 10 |
| second | 15 |
| third | 20 |

a. How much will Mary Anne receive for the 12th guest who makes a purchase?

b. If she wants a total hostess bonus of $100, how many guests need to make a purchase?

Write an explicit formula and a recursive formula for finding the $n$th term of each geometric sequence. (Lesson 10-3)

20. $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \ldots$
21. $9, -3, 1, -\frac{1}{3}, \ldots$
22. 3, 18, 108, 648, …
23. $-4, 20, -100, 500, \ldots$

24. **POPULATION** The population of Sandy Shores is currently 55,000 and is decreasing at a rate of 3% annually. (Lesson 10-3)

a. Write an explicit formula for finding the population of Sandy Shores during the $n$th year.

b. What do you predict will be the population of Sandy Shores after 10 years?

c. After how many years do you predict the population of Sandy shores will reach 37,000?

25. **MULTIPLE CHOICE** If possible, find the sum of the geometric series $12 + 3 + \frac{3}{4}, + \frac{3}{16} + \ldots$. (Lesson 10-3)

**A** 13.5

**B** 16

**C** 18

**D** not possible

# LESSON 10-4 Mathematical Induction

## Then

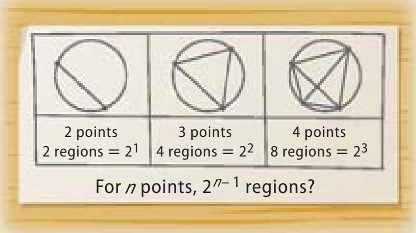
- You found the next term in a sequence or series. (Lesson 10-1)

## Now

1. Use mathematical induction to prove summation formulas and properties of divisibility involving a positive integer $n$.
2. Use extended mathematical induction.

## Why?

- Raini draws points on a circle and connects every pair of points with a chord, dividing the circle into regions. After drawing circles with 2, 3, and 4 points, Raini conjectures that if there are $n$ points on a circle, then connecting each pair of points will divide the circle into $2^{n-1}$ regions. While his conjecture holds for $n = 2$, 3, and 4, are these three examples sufficient to prove that his conjecture is true?




## NewVocabulary

principle of mathematical induction
anchor step
inductive hypothesis
inductive step
extended principle of mathematical induction

## 1 Mathematical Induction

When looking for patterns and making conjectures, it is often tempting to assume that if a conjecture holds for several cases, then it is true in all cases. In the situation above, Raini may be convinced that his conjecture is true once he shows that it holds for $n = 5$ because connecting 5 points does form 16 or $2^4$ regions. "Proof by example," however, is *not* a logically valid method of proof because it does not show that a conjecture is true for *all* cases. In fact, you can show that Raini's conjecture fails when $n = 6$.

While all that is required to prove mathematical conjectures false is a counterexample, proving one true requires a more formal method. One such method uses the **principle of mathematical induction.** The essential idea behind the principle of mathematical induction is that a conjecture can be proven true if you can:

1. show that something works for the first case (base or **anchor step**),
2. assume that it works for any particular case (**inductive hypothesis**), and then
3. show that it works for the next case (**inductive step**).

This principle, described more formally below, is a powerful tool for proving many conjectures about positive integers.

### KeyConcept The Principle of Mathematical Induction

Let $P_n$ be a statement about a positive integer $n$. Then $P_n$ is true for all positive integers $n$ if and only if

- $P_1$ is true, and
- for every positive integer $k$, if $P_k$ is true, then $P_{k+1}$ is true.

To understand why the principle of mathematical induction works, imagine a ladder with an infinite number of rungs (Figure 10.4.1). If you can get on the ladder (anchor step) and then move from one rung to the next (inductive hypothesis and step), you can climb the whole ladder. Similarly, imagine an unending line of dominos (Figure 10.4.2) arranged so that if any $k$th domino falls, the $(k + 1)$th domino will also fall (inductive hypothesis and step). By pushing over the first domino (anchor step), you start a chain reaction that knocks down the whole line.

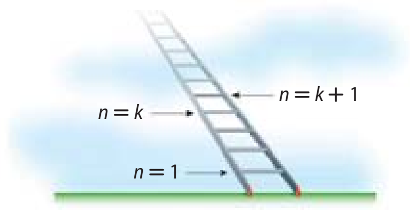


Figure 10.4.1

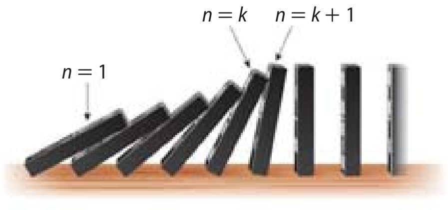


Figure 10.4.2

To apply the principle of mathematical induction, follow these steps.

**Step 1** Verify that a conjecture $P_n$ is true for $n = 1$. (Anchor Step)

**Step 2** Assume that $P_n$ is true for $n = k$. (Inductive Hypothesis)

**Step 3** Use this assumption to prove that $P_n$ is also true for $n = k + 1$. (Inductive Step)

## Example 1 Prove a Summation Formula

**Use mathematical induction to prove that the sum of the first $n$ even positive integers is $n^2 + n$. That is, prove that $2 + 4 + 6 + \cdots + 2n = n^2 + n$ is true for all positive integers $n$.**

**Conjecture** Let $P_n$ be the statement that $2 + 4 + 6 + \cdots + 2n = n^2 + n$.

**Anchor Step** Verify that $P_n$ is true for $n = 1$.

$P_n$: $2 + 4 + 6 + \cdots + 2n = n^2 + n$ — Original statement $P_n$

$P_1$: $2 = \mathbf{1}^2 + 1$ — $P_n$ for $n = 1$, the first partial sum

Because $2 = 1^2 + 1$ is a true statement, $P_n$ is true for $n = 1$.

**Inductive Hypothesis** Assume that $P_n$ is true for $n = k$.

To write the inductive hypothesis, replace $n$ with $k$ in $P_n$. That is, assume that $P_k$: $2 + 4 + 6 + \cdots + 2k = k^2 + k$ is true.

**Inductive Step** Use the inductive hypothesis to prove that $P_n$ is true for $n = k + 1$.

To prove that $P_n$ is true for $n = k + 1$, we need to show that $P_{k+1}$ must be true. Start with your inductive hypothesis and then add the next term, the $(k + 1)$th term, to each side.

| | |
|---|---|
| $2 + 4 + 6 + \cdots + 2k = k^2 + k$ | Inductive hypothesis |
| $2 + 4 + 6 + \cdots + 2k + \mathbf{2(k + 1)} = k^2 + k + \mathbf{2(k + 1)}$ | Add the $(k + 1)$th term to each side. |
| $2 + 4 + 6 + \cdots + 2k + 2(k + 1) = k^2 + k + 2k + 2$ | Simplify the right-hand side. |
| $2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (k^2 + 2k + 1) + (k + 1)$ | Rewrite 2 as 1 + 1 and regroup. |
| $2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (k + 1)^2 + (k + 1)$ | Factor $k^2 + 2k + 1$. |

This final statement is exactly the statement for $P_{k+1}$, so $P_{k+1}$ is true. It follows that if $P_n$ is true for $n = k$, then $P_n$ is also true for $n = k + 1$.

**Conclusion** Because $P_n$ is true for $n = 1$ and $P_k$ implies $P_{k+1}$, $P_n$ is true for $n = 2$, $n = 3$, and so on. That is, by the principle of mathematical induction, $P_n$: $2 + 4 + 6 + \cdots + 2n = n^2 + n$ is true for all positive integers $n$.

### GuidedPractice

**1.** Use mathematical induction to prove that the sum of the first $n$ even positive integers is $n^2$. That is, prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is true for all positive integers $n$.

**StudyTip**

**Representing the Next Term** To determine the $(k + 1)$th term, substitute the quantity $k + 1$ for $k$ in the expression for the general form of the next term in the series. In Example 1, because $2k$ represents the $k$th term, $2(k + 1)$ represents the $(k + 1)$th term.

**WatchOut!**

**Use the Inductive Hypothesis** To show that $P_n$ is true for $n = k + 1$, you do not substitute $k + 1$ for $n$ on each side of the equation for $P_n$. Doing so would give you nothing to prove. To complete the inductive step, you *must* use the inductive hypothesis.

In Example 1, notice that you are *not* trying to prove that $P_n$ is true for $n = k$. Instead, you *assume* that $P_n$ is true for $n = k$ and use that assumption to show that $P_n$ is true for the next number, $n = k + 1$. If while trying to complete either the anchor step or the inductive step you arrive at a contradiction, then the assumption you made in your inductive hypothesis *may be* false. For example, if there is a contradiction in the anchor step, then you know it does not work for *that value only*. The inductive hypothesis may still be true.

Mathematical induction can be used to prove divisibility. Recall that an integer $p$ is *divisible* by an integer $q$ if $p = qr$ for some integer $r$.

## Example 2 Prove Divisibility

**Prove that $3^n - 1$ is divisible by 2 for all positive integers $n$.**

**Conjecture and Anchor Step** Let $P_n$ be the statement that $3^n - 1$ is divisible by 2. $P_1$ is the statement that $3^1 - 1$ is divisible by 2. $P_1$ is true because $3^1 - 1$ is 2, which is divisible by 2.

**Inductive Hypothesis and Step** Assume that $3^k - 1$ is divisible by 2. That is, assume that $3^k - 1 = 2r$ for some integer $r$. Use this hypothesis to show that $3^{k+1} - 1$ is divisible by 2.

| | |
|---|---|
| $3^k - 1 = 2r$ | Inductive hypothesis |
| $3^k = 2r + 1$ | Add 1 to each side. |
| $3 \cdot 3^k = 3(2r + 1)$ | Multiply each side by 3. |
| $3^{k+1} = 6r + 3$ | Simplify. |
| $3^{k+1} - 1 = 6r + 2$ | Subtract 1 from each side. |
| $3^{k+1} - 1 = 2(3r + 1)$ | Factor. |

Because $r$ is an integer, $3r + 1$ is an integer and $2(3r + 1)$ is divisible by 2. Therefore, $3^{k+1} - 1$ is divisible by 2.

**Conclusion** Because $P_n$ is true for $n = 1$ and $P_k$ implies $P_{k+1}$, $P_n$ is true for $n = 2$, $n = 3$, and so on. By the principle of mathematical induction, $3^n - 1$ is divisible by 2 for all positive integers $n$.

### Guided Practice

**2.** Prove that $4^n - 1$ is divisible by 3 for all positive integers $n$.

**StudyTip**

**Inductive Step** There is no "fixed" way of completing the inductive step for a proof by mathematical induction. Each problem has its own special characteristics that require a different technique to complete the proof.

You can also prove statements of inequality using mathematical induction.

## Example 3 Prove Statements of Inequality

**Prove that $n < 2^n$ for all positive integers $n$.**

**Conjecture and Anchor Step** Let $P_n$ be the statement $n < 2^n$. $P_1$ and $P_2$ are true, since $1 < 2^1$ and $2 < 2^2$ are true inequalities. Showing $P_2$ to be true provides the anchor for the second part of our inductive hypothesis below.

**Inductive Hypothesis and Step** Assume that $k < 2^k$ is true for a positive $k > 1$. Use both parts of this inductive hypothesis to show that $k + 1 < 2^{k+1}$ is true.

| | | |
|---|---|---|
| $k < 2^k$ | Inductive hypothesis | $k > 1$ |
| $2 \cdot k < 2 \cdot 2^k$ | | $k - 1 > 0$ |
| $2k < 2^{k+1}$ | | $2k - k - 1 > 0$ |
| | | $2k - (k + 1) > 0$ |
| | | $2k > k + 1$ |
| | | $k + 1 < 2k$ |

By the Transitive Property of Inequality, if $k + 1 < 2k$ and $2k < 2^{k+1}$, then $k + 1 < 2^{k+1}$.

**Conclusion** Because $P_n$ is true for $n = 1$ and 2 and $P_k$ implies $P_{k+1}$ for $k \geq 2$, $P_n$ is true for $n = 3$, $n = 4$, and so on. By the principle of mathematical induction, $n < 2^n$ is true for all positive integers $n$.

### Guided Practice

**3.** Prove that $2n < 3^n$ for all positive integers $n$.

**StudyTip**

**Proofs of Inequalities** The approach of showing that the difference between some quantity and $k + 1$ is greater than or less than zero, along with the Transitive Property of Inequality, is used in many inequality proofs by mathematical induction.

## 2 Extended Mathematical Induction

Sometimes you will be asked to prove a statement that is true for an arbitrary value greater than 1. In situations like this, you can use a variation on the principle of mathematical induction called the **extended principle of mathematical induction.** Instead of verifying that $P_n$ is true for $n = 1$, you can instead verify that $P_n$ is true for the first possible case.

### Example 4 Use Extended Mathematical Induction

**Prove that $n! > 2^n$ for integer values of $n \geq 4$.**

**Conjecture and Anchor Step** Let $P_n$ be the statement $n! > 2^n$. $P_4$ is true since $4! > 2^4$ or $24 > 16$ is a true statement.

**Inductive Hypothesis and Step** Assume that $k! > 2^k$ is true for a positive integer $k \geq 4$. Show that $(k + 1)! > 2^{k+1}$ is true. Use this inductive hypothesis and its restriction that $k \geq 4$.

| | |
|---|---|
| $k! > 2^k$ | Inductive hypothesis |
| $(k + 1) \cdot k! > (k + 1) \cdot 2^k$ | Multiply each side by $k + 1$. |
| $(k + 1)! > (k + 1) \cdot 2^k$ | $(k + 1) \cdot k! = (k + 1)!$ |
| $(k + 1)! > (k + 1) \cdot 2^k > 2 \cdot 2^k$ | $k + 1 > 2$ is true for $k \geq 4$; therefore by the Multiplication Property of Inequality $(k + 1) \cdot 2^k > 2 \cdot 2^k$. |
| $(k + 1)! > 2 \cdot 2^k$ | Transitive Property of Inequality |
| $(k + 1)! > 2^{k+1}$ | Simplify. |

Therefore, $(k + 1)! > 2^{k+1}$ is true.

**Conclusion** Because $P_n$ is true for $n = 4$ and $P_k$ implies $P_{k+1}$ for $k \geq 4$, $P_n$ is true for $n = 5$, $n = 6$, and so on. That is, by the extended principle of mathematical induction, $n! > 2^n$ is true for integer values of $n \geq 4$.

**GuidedPractice**

**4.** Prove that $n! > 3^n$ for integer values of $n \geq 7$.

**Real-WorldLink**

Most automatic teller machines dispense withdrawals in \$10 or \$20 increments, but a few can go as low as \$5.

**Source:** *The Economist*

### Real-World Example 5 Apply Extended Mathematical Induction

**MONEY** **Prove that all multiples of \$10 greater than \$40 can be formed using just \$20 and \$50 bills.**

**Conjecture and Anchor Step** $P_n$: There exists a set of \$20 and \$50 bills that adds to \$10$n$ for $n > 4$. For $n = 5$, the first possible case, the conjecture is true because \$10(5) = \$20(0) + \$50(1).

**Inductive Hypothesis and Step** Assume that there exists a set of \$20 and/or \$50 bills that adds to \$10$k$. Show that this implies the existence of a set of \$20 and/or \$50 bills that adds to \$10$(k + 1)$.

**Case 1** The set contains at least one \$50 bill. Replace one \$50 bill in the set with three \$20 bills and the value of the set is increased by \$10 to \$10$k$ + 10 or \$10$(k + 1)$, which is exactly $P_{k+1}$.

**Case 2** The set contains no \$50 bills. The set must contain at least three \$20 bills because the value of the set must be greater than \$40. Replace two of the \$20 bills with a \$50 bill and the value of the set is increased by \$10, to \$10$k$ + 10 or \$10$(k + 1)$, which is exactly $P_{k+1}$.

**Conclusion** In both cases, $P_n$ is true for $n = k + 1$. Because $P_n$ is true for $n = 5$ and $P_k$ implies $P_{k+1}$ for $k \geq 5$, $P_n$ is true for $n = 6$, $n = 7$, and so on. That is, by the extended principle of mathematical induction, all multiples of \$10 greater than \$40 can be formed using just \$20 and \$50 bills.

**GuidedPractice**

**5.** **AMUSEMENT** Prove that all rides at the fair requiring more than 7 tickets can be paid for using just 3-ticket and 5-ticket vouchers offered by the school for donations of canned goods.

Stockbyte/Photolibraryv

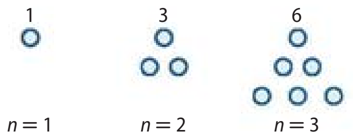
## Exercises

= Step-by-Step Solutions begin on page R29.

**Use mathematical induction to prove that each conjecture is true for all positive integers $n$.** (Example 1)

1. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$
2. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$
3. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$
4. $3 + 7 + 11 + \cdots + (4n - 1) = 2n^2 + n$
5. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$
6. $1 + 8 + 27 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
7. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$
8. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$
9. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$
10. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

11. **TRIANGULAR NUMBERS** Triangular numbers are numbers that can be represented by a triangular array of dots, with $n$ dots on each side. The first three triangular numbers are 1, 3, and 6. (Example 1)




a. Find the next five triangular numbers.

b. Write a general formula for the $n$th term of this sequence.

c. Prove that the sum of first $n$ triangular numbers can be found using $S_n = \dfrac{n(n+1)(n+2)}{6}$.

12. **ICEBREAKER** At freshman orientation, students are separated into groups to play an icebreaker game. The game requires each student in a group to have one individual interaction with every other student in the group. (Example 1)

a. Develop a formula to calculate the total number of interactions taking place during the icebreaker for a group of $n$ students.

b. Prove that the formula is true for all positive integer values of $n$.

c. Determine the length of the icebreaker game in minutes for a group of 12 students if each interaction is allotted 30 seconds.

**Prove that each conjecture is true for all positive integers $n$.** (Examples 2)

13. $9^n - 1$ is divisible by 8.
14. $6^n + 4$ is divisible by 5.
15. $2^{3n} - 1$ is divisible by 7.
16. $5^n - 2^n$ is divisible by 3.

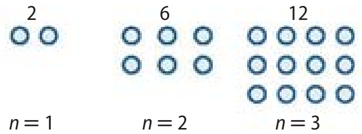
**Prove that each inequality is true for the indicated values of $n$.** (Examples 3 and 4)

17. $3^n \geq 3n,\ n \geq 1$
18. $n! > 4^n,\ n \geq 9$
19. $2^n > 2n,\ n \geq 3$
20. $9n < 3^n,\ n \geq 4$
21. $3n < 4^n,\ n \geq 1$
22. $2^n > 10n + 7,\ n \geq 10$
23. $2n < 1.5^n,\ n \geq 7$
24. $1.5^n > 10 + 0.5n,\ n \geq 7$

25. **POSTAGE** Prove that all postage greater than 20¢ can be formed using just 5¢ and 6¢ stamps. (Example 5)

26. **ENTERTAINMENT** All of the activities at the Family Fun Entertainment Center, such as video games, paintball, and go-kart racing, require tokens worth 25¢, 50¢, 75¢, and so on. Prove that all of the activities costing more than \$1.50 can be paid for using just 50¢ and 75¢ tokens. (Example 5)

27. **OBLONG NUMBERS** Oblong numbers are numbers that can be represented by a rectangular array having one more column than rows.




Prove that the sum of the first $n$ oblong numbers is given by $S_n = \dfrac{n^3 + 3n^2 + 2n}{3}$.

**Use mathematical induction to prove that each conjecture is true for all positive integers $n$.**

28. $\displaystyle\sum_{a=1}^{n} (4a - 3) = n(2n - 1)$
29. $\displaystyle\sum_{a=1}^{n} (3a - 2) = \frac{n}{2}(3n - 1)$
30. $\displaystyle\sum_{a=1}^{n} (a^2 + a) = \frac{n(n+1)(n+2)}{3}$
31. $\displaystyle\sum_{a=1}^{n} \frac{1}{4a^2 - 1} = \frac{n}{2n+1}$
32. $\displaystyle\sum_{a=1}^{n} \frac{1}{2a(a+1)} = \frac{n}{2(n+1)}$
33. $\displaystyle\sum_{a=1}^{n} \frac{1}{(a+1)(a+2)} = \frac{n}{2(n+2)}$

**Prove that each statement is true for all positive integers $n$ or find a counterexample.**

**34.** $1 + 6 + 11 + \cdots + (5n - 4) = n(2n - 1)$

**35.** $\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \cdots + \frac{1}{2n(2n+2)} = \frac{n}{2(2n+2)}$

**36.** $n^2 - n + 5$ is prime.

**37.** $3^n + 4n + 1$ is divisible by 4.

**38.** $4^n + 6n - 1$ is divisible by 9.

**39.** $2^{2n+1} + 3^{2n+1}$ is divisible by 5.

**Prove the inequality for the indicated integer values of $n$ and indicated values of $a$, $b$, and $x$.**

**40.** $\left(\frac{a}{b}\right)^n > \left(\frac{a}{b}\right)^{n+1}$, $n \geq 1$ and $0 < a < b$

**41.** $(x + 1)^n \geq nx$, $n \geq 1$ and $x \geq 1$

**42.** $(1 + a)^n > 1 + na$, $n \geq 2$ and $a > 0$

**Find a formula for the sum $S_n$ of the first $n$ terms of each sequence. Then prove that your formula is true using mathematical induction.**

**43.** $2, 6, 10, 14, \ldots, (4n - 2)$

**44.** $2, 7, 12, 17, \ldots, (5n - 3)$

**45.** $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \ldots, -\frac{1}{2^n}$

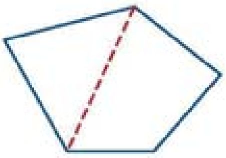
**46.** $\frac{1}{6}, \frac{1}{18}, \frac{1}{36}, \frac{1}{60}, \ldots, \frac{1}{3n(n+1)}$

**47.** **FIBONACCI NUMBERS** In the Fibonacci sequence, 1, 1, 2, 3, 5, 8, …, each element after the first two is found by adding the previous two terms. Numbers in the Fibonacci sequence occur throughout nature. For example, the number of scales in the clockwise and counterclockwise spirals on a pinecone are Fibonacci numbers. If $f_n$ represents the $n$th Fibonacci number, prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

**48.** **COMPLEX NUMBERS** Prove that DeMoivre's Theorem for finding the power of a complex number written in polar form is true for any positive integer $n$.

$$[r(\cos\theta + \boldsymbol{i}\sin\theta)]^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$$

**49.** **GEOMETRY** According to the Interior Angle Sum Formula, the sum of the measures of the interior angles of a convex polygon with $n$ sides is $180(n - 2)$ degrees. Use extended mathematical induction to prove this formula for $n \geq 3$.



**Use mathematical induction to prove that each conjecture is true for all positive integers $n$.**

**50.** $(xy)^n = x^n y^n$

**51.** $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

**52.** $x^{-n} = \frac{1}{x^n}$

**53.** $\cos n\pi = (-1)^n$

**Use mathematical induction to prove each formula for the sum of the first $n$ terms in a series.**

**54.** $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

**55.** $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

## H.O.T. Problems Use Higher-Order Thinking Skills

**56.** **CHALLENGE** Prove that $n! < n^n$ when $n > 1$.

**57** **OPEN ENDED** Consider the following statement.

$$a^n + b \text{ is divisible by } c.$$

**a.** Make a divisibility conjecture by replacing $a$, $b$, and $c$ with positive integers.

**b.** Use mathematical induction to prove that the conjecture you found in part **a** is true. If it is not true, find a counterexample.

**58.** **REASONING** Describe the error in the proof by mathematical induction shown below.

**Conjecture and Anchor Step**
Let $P_n$ be the statement that in a room with $n$ students, all of the students have the same birthday. When $n = 1$, $P_1$ is true since one student has only one birthday.

**Inductive Hypothesis and Step**
Assume that in a room with $k$ students, all of the students have the same birthday. Suppose $k + 1$ students are in a room. If one student leaves, then the remaining $k$ students must have the same birthday, according to the inductive hypothesis. Then if the first student returns and another student leaves, then that group (one) of $k$ students must have the same birthday. So, $P_n$ is true for $n = k + 1$. Therefore, $P_n$ is true for all positive integers $n$. That is, in a room with $n$ students, all of the students have the same birthday.

**59.** **CHALLENGE** If $a_n$ is represented by the sequence

$$\sqrt{6}, \sqrt{6 + \sqrt{6}}, \sqrt{6 + \sqrt{6 + \sqrt{6}}}, \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}, \ldots,$$

prove that the $n$th term in this sequence is always less than 3.

**60.** **WRITING IN MATH** In the inductive hypothesis step of mathematical induction, you assume that $P_n$ is true for $n = k$. Explain why you cannot simply assume that $P_n$ is true for $n$.

## Spiral Review

**Find the specified $n$th term of each geometric sequence.** (Lesson 10-3)

**61.** $a_9$ for $a_1 = \frac{1}{5}, 1, 5, \ldots$

**62.** $a_4 = 16, r = 0.5, n = 8$

**63.** $a_6 = 3, r = 2, n = 12$

**64.** **GAMES** In a game, the first person in a group states his or her name and an interesting fact about himself or herself. The next person must repeat the first person's name and fact and then say his or her own information. Each person must repeat the information for all those who preceded him or her. If there are 20 people in a group, what is the total number of times the names and facts will be stated? (Lesson 10-2)

**Graph each number in the complex plane, and find its absolute value.** (Lesson 9-5)

**65.** $z = 5 - 3i$

**66.** $z = -9 - 8i$

**67.** $z = 2 + 6i$

**Find rectangular coordinates for each point with the given polar coordinates.** (Lesson 9-3)

**68.** $\left(3, -\frac{5\pi}{4}\right)$

**69.** $\left(2, \frac{7\pi}{6}\right)$

**70.** $(-4, 1.4)$

**Write the component form of each vector.** (Lesson 8-5)

**71.** $\mathbf{w}$ lies in the $xz$ plane, has a magnitude of 2, and makes a 45° angle to the left of the positive $z$-axis.

**72.** $\mathbf{u}$ lies in the $yz$ plane, has a magnitude of 9, and makes a 30° angle to the right of the negative $z$-axis.

**For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.** (Lesson 7-1)

**73.** $(x - 4)^2 = -(y + 7)$

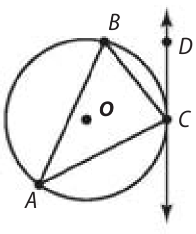
**74.** $6(x + 6) = (y - 4)^2$

**75.** $(y - 6)^2 = 4x$

**76.** **BUSINESS** A factory is making skirts and dresses from the same fabric. Each skirt requires 1 hour of cutting and 1 hour of sewing. Each dress requires 2 hours of cutting and 3 hours of sewing. The cutting and sewing departments can work up to 120 and 150 hours each week, respectively. If profits are \$12 for each skirt and \$18 for each dress, how many of each item should the factory make for maximum profit? (Lesson 6-5)

## Skills Review for Standardized Tests

**77.** **SAT/ACT** Triangle $ABC$ is inscribed in circle $O$. $\overleftrightarrow{CD}$ is tangent to circle $O$ at point $C$. If $m\angle BCD = 40°$, find $m\angle A$.




**A** 60°
**B** 50°
**C** 40°
**D** 30°
**E** 25°

**78.** Which of the following is divisible by 2 for all positive integers $n$?

**F** $1^n - 1$
**G** $2^n - 1$
**H** $3^n - 1$
**J** $4^n - 1$

**79.** **REVIEW** What is the first term in the arithmetic sequence?

$$\_\_\_, 8\tfrac{1}{3}, 7, 5\tfrac{2}{3}, 4\tfrac{1}{3}, \ldots$$

**A** 3
**B** $9\frac{2}{3}$
**C** $10\frac{1}{3}$
**D** 11

**80.** **REVIEW** What is the tenth term in the arithmetic sequence that begins 10, 5.6, 1.2, −3.2, …?

**F** −39.6
**G** −29.6
**H** 29.6
**J** 39.6

LESSON 10-5

# The Binomial Theorem

**Then**

- You represented infinite series using sigma notation. (Lesson 10-1)

**Now**

1. Use Pascal's triangle to write binomial expansions.
2. Use the Binomial Theorem to write and find the coefficients of specified terms in binomial expansions.

**Why?**

- Suppose a biologist studying an endangered species of gibbon has found that on average 80% of gibbon offspring are female and 20% are male. Zoo workers anticipate that their captive gibbons will produce $n$ offspring and want to know the probability that none of these offspring will be male. The biologist can use a term from the binomial expansion of $(0.8 + 0.2)^n$ to solve this problem.

**NewVocabulary**
binomial coefficients
Pascal's triangle
Binomial Theorem

## 1 Pascal's Triangle

Recall that a *binomial* is an algebraic expression involving the sum of two unlike terms. An important series is generated by the expansion of a binomial raised to an integer power. Examine this series generated by the expansion of $(a + b)^n$ for several nonnegative integer values of $n$.

$$(a+b)^0 = 1a^0b^0$$
$$(a+b)^1 = 1a^1b^0 + 1a^0b^1$$
$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$
$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$
$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

Observe the following patterns in the expansions of $(a + b)^n$ above.

- Each expansion has $n + 1$ terms.
- The first term is $a^n$, and the last term is $b^n$.
- In successive terms, the powers of $a$ decrease by 1 and the powers of $b$ increase by 1.
- The sum of the exponents in each term is $n$.
- The coefficients, in red above, increase and then decrease in a symmetric pattern.

If just the coefficients of these expansions, called the **binomial coefficients**, are extracted and arranged in a triangular array, they form a pattern called **Pascal's triangle**, named after the French mathematician Blaise Pascal. The top row in this triangle is called the *zeroth row* because it corresponds to the binomial expansion of $(a + b)^0$.

```
                  1                   0th row
               1     1                1st row
            1     2     1             2nd row
         1     3     3     1          3rd row
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

Notice that the first and last numbers in each row are 1 and every other number is formed by adding the two numbers immediately above that number in the previous row. Pascal's triangle can be extended indefinitely using the recursive relationship that the coefficients in the $(n - 1)$th row can be used to determine the coefficients in the $n$th row.

By extending Pascal's triangle and using the patterns observed in the first 5 expansions of $(a + b)^n$, you can expand a binomial raised to any whole number power.

## Example 1 Power of a Binomial Sum

**Use Pascal's triangle to expand each binomial.**

**a.** $(a + b)^7$

**Step 1** Write the series for $(a + b)^7$, omitting the coefficients. Because the power is 7, this series should have 7 + 1 or 8 terms. Use the pattern of increasing and decreasing exponents to complete the series.

$$a^7b^0 + a^6b^1 + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + a^1b^6 + a^0b^7$$

Exponents of $a$ decrease from 7 to 0.
Exponents of $b$ increase from 0 to 7.

**Step 2** Use the numbers in the seventh row of Pascal's triangle as the coefficients of the terms. To find these numbers, extend Pascal's triangle to the 7th row.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | | | 6th row |
| | | + | | + | | + | | + | | + | | + | | | | |
| **1** | | **7** | | **21** | | **35** | | **35** | | **21** | | **7** | | **1** | | 7th row |

$$(a + b)^7 = \mathbf{1}a^7b^0 + \mathbf{7}a^6b^1 + \mathbf{21}a^5b^2 + \mathbf{35}a^4b^3 + \mathbf{35}a^3b^4 + \mathbf{21}a^2b^5 + \mathbf{7}a^1b^6 + \mathbf{1}a^0b^7$$

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Simplify.

**StudyTip**

**Finding the Correct Row** The second number in any row of Pascal's triangle is always the same as the power to which the binomial is raised. For example, the second number in the 7th row of Pascal's triangle is 7.

**b.** $(3x + 2)^4$

**Step 1** Write the series for $(a + b)^4$, omitting the coefficients and replacing $a$ with $3x$ and $b$ with 2. The series has 4 + 1 or 5 terms.

$$(3x)^4(2)^0 + (3x)^3(2)^1 + (3x)^2(2)^2 + (3x)^1(2)^3 + (3x)^0(2)^4$$

Exponents of $3x$ decrease from 4 to 0.
Exponents of 2 increase from 0 to 4.

**Step 2** The numbers in the 4th row of Pascal's triangle are 1, 4, 6, 4, and 1. Use these numbers as the coefficients of the terms in the series. Then simplify.

$$(3x + 2)^4 = \mathbf{1}(3x)^4(2)^0 + \mathbf{4}(3x)^3(2)^1 + \mathbf{6}(3x)^2(2)^2 + \mathbf{4}(3x)^1(2)^3 + \mathbf{1}(3x)^0(2)^4$$

$$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$$

### GuidedPractice

**1A.** $(a + b)^8$

**1B.** $(2x + 3y)^5$

To expand a binomial difference, first rewrite the expression as a binomial sum.

## Example 2 Power of a Binomial Difference

**Use Pascal's triangle to expand $(x - 4y)^5$.**

Because $(x - 4y)^5 = [x + (-4y)]^5$, write the series for $(a + b)^5$, replacing $a$ with $x$ and $b$ with $-4y$. Use the numbers in the 5th row of Pascal's triangle, 1, 5, 10, 10, 5, and 1, as the binomial coefficients. Then simplify.

$$(x - 4y)^5 = \mathbf{1}x^5(-4y)^0 + \mathbf{5}x^4(-4y)^1 + \mathbf{10}x^3(-4y)^2 + \mathbf{10}x^2(-4y)^3 + \mathbf{5}x^1(-4y)^4 + \mathbf{1}x^0(-4y)^5$$

$$= x^5 - 20x^4y + 160x^3y^2 - 640x^2y^3 + 1280xy^4 - 1024y^5$$

**StudyTip**

**Alternating Signs** Notice that when expanding a power of a binomial difference, the signs of the terms in the series alternate.

### GuidedPractice

**Use Pascal's triangle to expand each binomial.**

**2A.** $(2x - 7)^3$

**2B.** $(2x - 3y)^4$

## 2 The Binomial Theorem

While it is possible to expand any binomial using Pascal's triangle, the recursive method of computing the binomial coefficients makes expansions of $(a+b)^n$ for large values of $n$ time consuming. An explicit formula for computing each binomial coefficient is developed by considering $(a+b)^n$ as the product of $n$ factors in which each factor contributes an $a$ or a $b$ to each product in the expansion.

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\,(a+b)\cdots(a+b)}_{n \text{ factors}} \quad = \quad \ldots + \underbrace{a^{n-r}b^r}_{\text{If there are } r \text{ letter } b\text{'s, then there are } (n-r) \text{ letter } a\text{'s.}} + \ldots$$

Consider $(a+b)^3$. There are three ways to choose 1 $a$ and 2 $b$'s from each of its three factors to form the product $ab^2$, and 3 is the binomial coefficient of the $ab^2$ term in the expansion.

$$\left.\begin{aligned}(a+b)(a+b)(a+b) &= \ldots + ab^2 + \ldots\\ (a+b)(a+b)(a+b) &= \ldots + ab^2 + \ldots\\ (a+b)(a+b)(a+b) &= \ldots + ab^2 + \ldots\end{aligned}\right\} \xrightarrow{\text{3 ways}} (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

**ReadingMath**

**Combinations** The notations ${}_nC_r$ and $\binom{n}{r}$ are both read as *the combination of n things taken r at a time*. To review computing combinations, see Lesson 0-7.

Because those factors that do not contribute a $b$ will by default contribute an $a$, the number of ways to form the product $a^{n-r}b^r$ can be more simply thought of as the number of ways to choose $r$ factors to contribute a $b$ to the product from the $n$ factors available. This is the combination given by ${}_nC_r = \frac{n!}{(n-r)!\,r!}$, also written $\binom{n}{r}$.

### KeyConcept Formula for the Binomial Coefficients of $(a+b)^n$

**Words** The binomial coefficient of the $a^{n-r}b^r$ term in the expansion of $(a+b)^n$ is given by ${}_nC_r = \frac{n!}{(n-r)!\,r!}$.

**Example**

$$\begin{aligned}(a+b)^3 &= {}_3C_0a^3b^0 + {}_3C_1a^2b^1 + {}_3C_2a^1b^2 + {}_3C_3a^0b^3\\ &= \frac{3!}{(3-0)!\,0!}a^3 + \frac{3!}{(3-1)!\,1!}a^2b + \frac{3!}{(3-2)!\,2!}ab^2 + \frac{3!}{(3-3)!\,3!}b^3\\ &= 1a^3 + 3a^2b + 3ab^2 + 1b^3\end{aligned}$$

In the example above, notice that for the first term $r = 0$, for the second term $r = 1$, for the third term $r = 2$, and so on. In general, to find the binomial coefficient of the $k$th term in an expansion of the form $(a+b)^n$, use the formula ${}_nC_r$ and let $r = k - 1$.

### Example 3 Find Binomial Coefficients

**Find the coefficient of the 5th term in the expansion of $(a+b)^7$.**

To find the coefficient of the 5th term, evaluate ${}_nC_r$ for $n = 7$ and $r = 5 - 1$ or 4.

$$\begin{aligned}{}_7C_4 &= \frac{7!}{(7-4)!\,4!} && {}_nC_r = \frac{n!}{(n-r)!\,r!}\\ &= \frac{7!}{3!\,4!} && \text{Subtract.}\\ &= \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{3!\,\cancel{4!}} && \text{Rewrite 7! as } 7\cdot 6\cdot 5\cdot 4! \text{ and divide out common factorials.}\\ &= \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} \text{ or } 35 && \text{Simplify.}\end{aligned}$$

The coefficient of the 5th term in the expansion of $(a+b)^7$ is 35.

**CHECK** From Example 1a, you know the 5th term in the expansion of $(a+b)^7$ is $35a^3b^4$. ✔

**GuidedPractice**

**Find the coefficient of the indicated term in each expansion.**

**3A.** $(x+y)^9$, 6th term

**3B.** $(a-b)^{13}$, 3rd term

## Example 4 Binomials with Coefficients Other than 1

**Find the coefficient of the $x^7y^2$ term in the expansion of $(4x - 3y)^9$.**

For $(4x - 3y)^9$ to have the form $(a + b)^n$, let $a = 4x$ and $b = -3y$. The coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a + b)^n$ is given by ${}_nC_r$. So, to find the binomial coefficient of the term containing $a^7b^2$ in the expansion of $(a + b)^9$, evaluate ${}_nC_r$ for $n = 9$ and $r = 2$.

$${}_9C_2 = \frac{9!}{(9-2)!\,2!} \qquad {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{9!}{7!\,2!} \qquad \text{Subtract.}$$

$$= \frac{9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}\;2!} \qquad \text{Rewrite 9! as } 9 \cdot 8 \cdot 7! \text{ and divide out common factorials.}$$

$$= \frac{9 \cdot 8}{2 \cdot 1} \text{ or } 36 \qquad \text{Simplify.}$$

Thus, the binomial coefficient of the $a^7b^2$ term in $(a + b)^9$ is 36. Substitute $4x$ for $a$ and $-3y$ for $b$ to find the coefficient of the $x^7y^2$ term in the original binomial expansion.

$$36a^7b^2 = 36(4x)^7(-3y)^2 \qquad a = 4x \text{ and } b = -3y$$

$$= 5{,}308{,}416x^7y^2 \qquad \text{Simplify.}$$

Therefore, the coefficient of the $x^7y^2$ term in the expansion of $(4x - 3y)^9$ is 5,308,416.

### Guided Practice

**Find the coefficient of the indicated term in each binomial expansion.**

**4A.** $(2x - 3y)^8$, $x^3y^5$ term

**4B.** $(2p + 1)^{15}$, 11th term

### Real-World Link

An at bat in baseball is any time that a batter faces a pitcher except when the player "(i) hits a sacrifice bunt or sacrifice fly; (ii) is awarded first base on four called balls; (iii) is hit by a pitched ball; or (iv) is awarded first base because of interference or obstruction."

**Source:** Major League Baseball

MEB-Photography/Alamy

You can use the coefficients of binomial expansions to solve real-world problems in which there are only two outcomes for an event. Problems that can be solved using a binomial expansion are called *binomial experiments*. Such experiments occur if and only if: (1) the experiment consists of $n$ identical trials, (2) each trial results in *one* of two outcomes, and (3) the trials are independent.

For $n$ independent trials of an experiment, if the probability of a success is $p$ and the probability of a failure is $q = 1 - p$, then the term ${}_nC_x\, p^xq^{n-x}$ in the expansion of $(p + q)^n$ gives the probability of $x$ successes for those $n$ trials.

## Real-World Example 5 Use Binomial Coefficients

**BASEBALL** **The probability that Andres gets a hit when at bat is $\frac{1}{5}$. What is the probability that Andres gets exactly 4 hits during his next 10 at bats?**

A success in this situation is Andres getting a hit, so $p = \frac{1}{5}$ and $q = 1 - \frac{1}{5}$ or $\frac{4}{5}$. Each at bat represents a trial, so $n = 10$. You want to find the probability that Andres succeeds 4 times out of those 10 trials, so let $x = 4$. To find this probability, find the value of the term ${}_nC_x\, p^xq^{n-x}$ in the expansion of $(p + q)^n$.

$${}_nC_x\, p^xq^{n-x} = {}_{10}C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^{10-4} \qquad p = \frac{1}{5},\ q = \frac{4}{5},\ n = 10, \text{ and } x = 4$$

$$= \frac{10!}{(10-4)!\,4!}\left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6 \qquad {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\approx 0.088 \qquad \text{Use a calculator.}$$

So, the probability that Andres gets 4 hits during his next 10 at bats is about 0.088 or 8.8%.

### Guided Practice

**5. COIN TOSS** A fair coin is flipped 8 times. Find the probability of each outcome.

**A.** exactly 3 heads

**B.** exactly 6 tails

The formula for finding the coefficients of a binomial expansion leads us to a theorem about expanding powers of binomials called the **Binomial Theorem**.

### KeyConcept Binomial Theorem

For any positive integer $n$, the expansion of $(a + b)^n$ is given by

$$(a+b)^n = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + {}_nC_2\, a^{n-2} b^2 + \cdots + {}_nC_r\, a^{n-r} b^r + \cdots + {}_nC_n\, a^0 b^n,$$

where $r = 0, 1, 2, \ldots, n$.

You will prove the Binomial Theorem in Exercise 75.

**TechnologyTip**

**Combinations** To evaluate ${}_{10}C_4$ using a calculator, enter 10, select nCr from the MATH ▸ PRB menu, and then enter 4.

### Example 6 Expand a Binomial Using the Binomial Theorem

**Use the Binomial Theorem to expand each binomial.**

**a.** $(3x - y)^4$

Apply the Binomial Theorem to expand $(a + b)^4$, where $a = 3x$ and $b = -y$.

$$\begin{aligned}(3x-y)^4 &= {}_4C_0\,(3x)^4(-y)^0 + {}_4C_1\,(3x)^3(-y)^1 + {}_4C_2\,(3x)^2(-y)^2 + \\ &\quad {}_4C_3\,(3x)^1(-y)^3 + {}_4C_4\,(3x)^0(-y)^4 \\ &= 1(81x^4)(1) + 4(27x^3)(-y) + 6(9x^2)(y^2) + 4(3x)(-y^3) + 1(1)(y^4) \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4\end{aligned}$$

**b.** $(2p + q^2)^5$

Apply the Binomial Theorem to expand $(a + b)^5$, where $a = 2p$ and $b = q^2$.

$$\begin{aligned}(2p+q^2)^5 &= {}_5C_0\,(2p)^5(q^2)^0 + {}_5C_1\,(2p)^4(q^2)^1 + {}_5C_2\,(2p)^3(q^2)^2 + {}_5C_3\,(2p)^2(q^2)^3 + {}_5C_4\,(2p)^1(q^2)^4 + \\ &\quad {}_5C_5\,(2p)^0(q^2)^5 \\ &= 1(32p^5)(1) + 5(16p^4)(q^2) + 10(8p^3)(q^4) + 10(4p^2)(q^6) + 5(2p)(q^8) + 1(1)(q^{10}) \\ &= 32p^5 + 80p^4q^2 + 80p^3q^4 + 40p^2q^6 + 10pq^8 + q^{10}\end{aligned}$$

**GuidedPractice**

**6A.** $(5m + 4)^3$

**6B.** $(8x^2 - 2y)^6$

Because a binomial expansion is a sum, the Binomial Theorem is often written using sigma notation. In addition, the notation ${}_nC_r$ is usually replaced by $\binom{n}{r}$.

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

### Example 7 Write a Binomial Expansion Using Sigma Notation

**Represent the expansion of $(5x - 7y)^{20}$ using sigma notation.**

Apply the Binomial Theorem to represent the expansion of $(a + b)^{20}$ using sigma notation, where $a = 5x$ and $b = -7y$.

$$(5x - 7y)^{20} = \sum_{r=0}^{20} \binom{20}{r} (5x)^{20-r}(-7y)^r$$

**GuidedPractice**

**7.** Represent the expansion of $(3a + 12b)^{30}$ using sigma notation.

## Exercises

= Step-by-Step Solutions begin on page R29.

**Use Pascal's triangle to expand each binomial.**
(Examples 1 and 2)

**1.** $(2 + x)^4$

**2.** $(n + m)^5$

**3.** $(4a - b)^3$

**4.** $(x + y)^6$

**5.** $(3x + 2y)^7$

**6.** $(n - 4)^6$

**7.** $(3c - d)^4$

**8.** $(m - a)^5$

**9.** $(a - b)^3$

**10.** $(3p - 2q)^4$

**Find the coefficient of the indicated term in each expansion.**
(Examples 3 and 4)

**11.** $(x - 2)^{10}$, 5th term

**12.** $(4m + 1)^8$, 3rd term

**13.** $(x + 3y)^{10}$, 8th term

**14.** $(2c - d)^{12}$, 6th term

**15.** $(a + b)^8$, 4th term

**16.** $(2a + 3b)^{10}$, 5th term

**17.** $(x - y)^9$, 6th term

**18.** $(x + y)^{12}$, 7th term

**19.** $(x + 2)^7$, 4th term

**20.** $(a - 3)^8$, 5th term

**21.** $(2a + 3b)^{10}$, $a^6b^4$ term

**22.** $(2x + 3y)^9$, $x^6y^3$ term

**23.** $\left(x + \frac{1}{3}\right)^7$, 4th term

**24.** $\left(x - \frac{1}{2}\right)^{10}$, 6th term

**25.** $(x + 4y)^7$, $x^2y^5$ term

**26.** $(3x + 5y)^{10}$, $x^6y^4$ term

**27. TESTING** Alfonso is taking a test that contains a section of 16 true-false questions. (Example 3)

**a.** How many of the possible sets of answers to these questions have exactly 12 correct answers of false?

**b.** How many of the possible sets of answers to these questions have exactly 8 correct answers of true?

**28. BUSINESS** The probability of a certain sales representative successfully making a sale is $\frac{1}{5}$. The sales representative has 12 appointments this week. (Example 5)

**a.** Find the probability that the sales representative makes no sales this week.

**b.** What is the probability that the sales representative makes exactly 3 sales this week?

**c.** Find the probability that the sales representative will make 10 sales this week.

**29. BIOLOGY** Refer to the beginning of the lesson. Assume that the zoo workers expect 30 gibbon offspring this year. (Example 5)

**a.** What is the probability that there will be no male gibbon offspring this year?

**b.** Find the probability that there will be exactly 2 male gibbon offspring this year.

**c.** What is the probability that there will be 23 female gibbon offspring this year?

**30. BOWLING** Carol averages 2 strikes every 10 frames. What is the probability that Carol will get exactly 4 strikes in the next 10 frames? (Example 5)

**Bowling Castle Scorecard**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Carol | 7 / | X | 5 4 | 9 / | 8 / | X | X | 7 0 | X | 8 1 | 165 |
| | 20 | 39 | 48 | 66 | 86 | 113 | 130 | 137 | 156 | 165 | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
| | | | | | | | | | | | |

**Use the Binomial Theorem to expand each binomial.**
(Example 6)

**31.** $(4t + 3)^5$

**32.** $(8 - 5y)^3$

**33.** $(2m - n)^6$

**34.** $(9h + 2j)^4$

**35.** $(3p + q)^7$

**36.** $(a^2 - 2b)^8$

**37.** $(7c^2 + 3d)^5$

**38.** $(2w - 4x^3)^7$

**Represent the expansion of each expression using sigma notation.** (Example 7)

**39.** $(2q + 3)^{15}$

**40.** $(m - 8n)^{25}$

**41.** $(11x + y)^{31}$

**42.** $(4a + 7b)^{19}$

**43.** $\left(3f - \frac{3}{4}g\right)^{22}$

**44.** $\left(\frac{1}{2}s - 5t\right)^{36}$

**45. COMPUTER GAMES** In a computer game, when a treasure chest is opened, it contains either gold coins or rocks. The probability that it contains gold coins is $\frac{5}{6}$.

**a.** The treasure chest is opened 15 times per game. In one game, how many different ways is it possible to open the chest and find gold coins exactly 9 times?

**b.** What is the probability that a person playing the game will find gold in the chest more than 12 times?

**46. COMMUNITY OUTREACH** At a food bank, canned goods are received and distributed to people in the community who are in need. Volunteers check the quality of the food before distribution. The probability that a canned good received at a food bank is distributed to the needy is $\frac{4}{5}$.

**a.** A volunteer checks 30 canned goods per hour. In one hour, how many different ways is it possible to check a canned good and donate it exactly 23 times?

**b.** What is the probability that a volunteer checking canned goods will find themselves throwing out items less than 4 times?

**Use the Binomial Theorem to expand and simplify each expression.**

**47.** $(2d + \sqrt{5})^4$

**48.** $(\sqrt{a} - \sqrt{b})^5$

**49.** $\left(4s + \frac{1}{2}t\right)^5$

**50.** $\left(\frac{1}{y} - 3z\right)^6$

**Find the coefficient of the indicated term in each expansion.**

**51.** $(k - \sqrt{5})^9$, 5th term

**52.** $(\sqrt{2} + 2c)^{10}$, middle term

**53.** $\left(\frac{1}{4}p + q\right)^{11}$, 7th term

**54.** $\left(\sqrt{h} - 3\sqrt{j}\right)^{11}$, 6th term

**Use the Binomial Theorem to expand and simplify each power of a complex number.**

**55.** $(i + 2)^4$

**56.** $(i - 3)^3$

**57.** $(1 - 4i)^5$

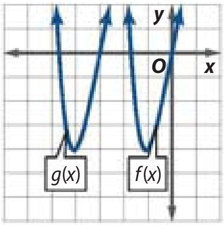
**58.** $(2 + \sqrt{7}i)^4$

**59.** $\left(\frac{\sqrt{2}}{2}i - \frac{1}{2}\right)^3$

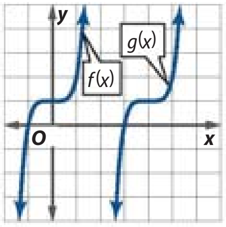
**60.** $(\sqrt{-16}i + 3)^5$

**The graph of $g(x)$ is a translation of the graph of $f(x)$. Use the Binomial Theorem to find the polynomial function for $g(x)$ in standard form.**

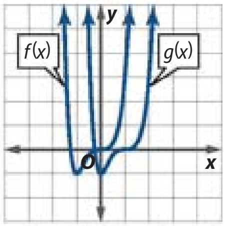
**61.** $f(x) = x^4 + 5x$
$g(x) = f(x + 3)$



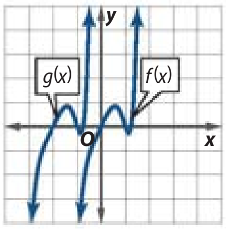

**62.** $f(x) = x^5 + 1$
$g(x) = f(x - 4)$




**63.** $f(x) = x^6 + 2x^3$
$g(x) = f(x - 1)$




**64.** $f(x) = x^7 - 3x^4 + 2x$
$g(x) = f(x + 2)$




**65.** **MULTIPLE REPRESENTATIONS** In this problem, you will use the Binomial Theorem to investigate the difference quotient $\frac{f(x+h) - f(x)}{h}$ for power functions.

**a.** **ANALYTICAL** Use the Binomial Theorem to expand and simplify the difference quotient for $f(x) = x^3$, $f(x) = x^4$, $f(x) = x^5$, $f(x) = x^6$, and $f(x) = x^7$. Use the pattern to simplify the difference quotient for $f(x) = x^n$.

**b.** **TABULAR** Evaluate each expression in part **a** for $h = 0.1, 0.01, 0.001$, and $0.0001$ and record your results in a table. What do you observe?

**c.** **GRAPHICAL** Graph the set of resulting functions from part **b** for $f(x) = x^3$ on the same coordinate plane. What do you observe?

**d.** **ANALYTICAL** As $h$ approaches 0, write an expression for the difference quotient when $f(x) = x^n$, where $n$ is a positive integer.

**66.** In the expansion of $(ax + b)^5$ the numerical coefficient of the second term is 400 and the numerical coefficient of the third term is 2000. Find the values of $a$ and $b$.

**67.** **RESEARCH** Although Pascal's triangle is named for Blaise Pascal, other mathematicians applied their knowledge of the triangle hundreds of years before Pascal. Use the Internet or another source to research at least one other person who used the properties of the triangle before Pascal was born. Then describe other patterns found in Pascal's triangle that are not described in this lesson.

## H.O.T. Problems Use Higher-Order Thinking Skills

**68.** **ERROR ANALYSIS** Jena and Gil are finding the 6th term of the expansion of $(x + y)^{14}$. Jena says that the coefficient of the term is 3003. Gil thinks that it is 2002. Is either of them correct? Explain your reasoning.

**69.** **CHALLENGE** Describe a strategy that uses the Binomial Theorem to expand $(x + y + z)^n$. Then write and simplify an expansion for the expression.

**70.** **PROOF** The sums of the coefficients in the first five rows of Pascal's triangle are shown below.

| | | |
|---|---|---|
| row 0 | $1$ | $= 2^0$ |
| row 1 | $1 + 1$ | $= 2^1$ |
| row 2 | $1 + 2 + 1$ | $= 2^2$ |
| row 3 | $1 + 3 + 3 + 1$ | $= 2^3$ |
| row 4 | $1 + 4 + 6 + 4 + 1$ | $= 2^4$ |
| row 5 | $1 + 5 + 10 + 10 + 5 + 1$ | $= 2^5$ |

Prove that the sum of the coefficients in the $n$th row of Pascal's triangle is $2^n$. (*Hint*: Write $2^n$ as $(1 + 1)^n$. Then use the Binomial Theorem to expand.)

**71.** **WRITING IN MATH** Describe how to find the numbers in each row of Pascal's triangle. Then write a few sentences to describe how the expansions of $(a + b)^{n-1}$ and $(a - b)^n$ are different from the expansion of $(a + b)^n$.

**72.** **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Justify your reasoning.

*If a binomial is raised to the power 5, the two middle terms of the expansion have the same coefficients.*

**73.** **CHALLENGE** Explain how you could find a term in the expansion of $\left(\frac{1}{2v} + 6v^7\right)^8$ that does not contain the variable $v$. Then find the term.

**74.** **PROOF** Use the principle of mathematical induction to prove the Binomial Theorem.

## Spiral Review

**Prove that each statement is true for all positive integers $n$ or find a counterexample.** (Lesson 10-4)

**75.** $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(3n-1)}{2}$

**76.** $10^{2n-1} + 1$ is divisible by 11.

**77. GENEALOGY** In the book *Roots*, author Alex Haley traced his family history back many generations. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be? (Lesson 10-3)

**78.** Let $\overrightarrow{DE}$ be the vector with initial point $D(5, -12)$ and terminal point $E(8, -17)$. Write $\overrightarrow{DE}$ as a linear combination of the vectors **i** and **j**. (Lesson 8-2)

**Write each equation in standard form. Identify the related conic.** (Lesson 7-2)

**79.** $x^2 + y^2 - 16x + 10y + 64 = 0$

**80.** $y^2 + 16x - 10y + 57 = 0$

**81.** $x^2 + y^2 + 2x + 24y + 141 = 0$

**Find the partial fraction decomposition of each rational expression.** (Lesson 6-4)

**82.** $\frac{5x^2 - 14}{(x^2-2)^2}$

**83.** $\frac{x^3 - 8x^2 + 21x - 22}{x^2 - 8x + 15}$

**84.** $\frac{3x}{(x-3)^2}$

**Determine whether each matrix is in row-echelon form.** (Lesson 6-1)

**85.** $\left[\begin{array}{ccc|c} 1 & 10 & -5 & 3 \\ 0 & 1 & 14 & -2 \\ 0 & 1 & 9 & 6 \end{array}\right]$

**86.** $\left[\begin{array}{ccc|c} 1 & 3 & -7 & 11 \\ 0 & 1 & -13 & 18 \\ 0 & 0 & 0 & 1 \end{array}\right]$

**87.** $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{array}\right]$

**Find the exact value of each trigonometric expression.** (Lesson 5-4)

**88.** $\tan 195°$

**89.** $\csc \frac{5\pi}{12}$

**90.** $\sin \frac{\pi}{12}$

**91. SAVINGS** Janet's father deposited \$30 into a bank account for her. They forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years. (Lesson 3-5)

**a.** Make a scatter plot of the data.

**b.** Find an exponential function to model the data.

**c.** Use the function to predict the balance of the account in 41 years.

| Elapsed Time (yr) | Balance (\$) |
|---|---|
| 0 | 30.00 |
| 5 | 41.10 |
| 10 | 56.31 |
| 15 | 77.16 |
| 20 | 105.71 |
| 25 | 144.83 |
| 30 | 198.43 |

## Skills Review for Standardized Tests

**92. SAT/ACT** In the figure below, $ABCD$ is a parallelogram. What are the coordinates of point $C$?

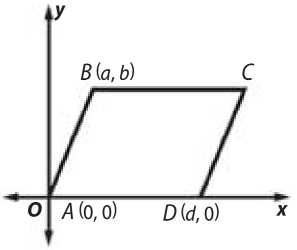
**A** $(d + a, y)$

**B** $(d - a, b)$

**C** $(d + x, b)$

**D** $(d + a, b)$

**E** $(d + b, a)$




**93. REVIEW** What is the value of $\frac{12!}{8!\,4!}$?

**F** 495

**G** 500

**H** 660

**J** 710

**94.** Mrs. Thomas is giving a four-question multiple-choice quiz. Each question can be answered A, B, C, or D. How many ways could a student answer the questions using each answer A, B, C, or D once?

**A** 20

**B** 22

**C** 24

**D** 26

**95. REVIEW** Which expression is equivalent to $(2x - 2)^4$?

**F** $16x^4 + 64x^3 - 96x^2 - 64x + 16$

**G** $16x^4 - 32x^3 - 192x^2 - 64x + 16$

**H** $16x^4 - 64x^3 + 96x^2 - 64x + 16$

**J** $16x^4 + 32x^3 - 192x^2 - 64x + 16$

LESSON

# 10-6 Functions as Infinite Series

## Then

- You found the $n$th term of an infinite series expressed using sigma notation. (Lesson 10-1)

## Now

- **1** Use a power series to represent a rational function.
- **2** Use power series representations to approximate values of transcendental functions.

## Why?

- The music to which you listen on a digital audio player is first performed by an artist. The waveform of each sound in that performance is then broken down into its component parts and stored digitally. These parts are then retrieved and combined to reproduce each original sound of the performance. The analysis of a special series is an essential ingredient in this process.

### NewVocabulary

power series
exponential series
Euler's Formula

## 1 Power Series

Earlier in this chapter, you saw how some series of numbers can be expressed as functions. In this lesson, you will see that some functions can be broken down into infinite series of component functions.

In Lesson 10-3, you learned that the sum of an infinite geometric series,

$$1 + r + r^2 + \cdots + r^n + \cdots, a_1 = 1$$

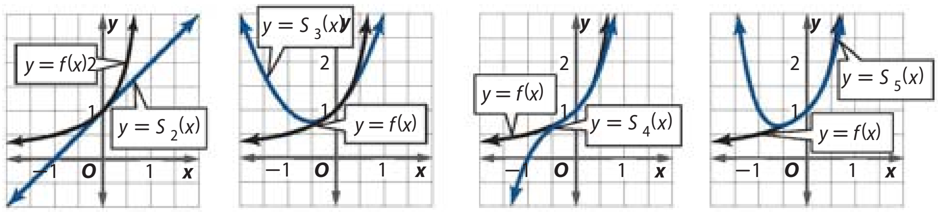
with common ratio $r$, converges to a sum of $\frac{a_1}{1 - r}$ if $|r| < 1$. Replacing $r$ with $x$,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1 - x}, \text{ for } |x| < 1.$$

It follows that $f(x) = \frac{1}{1 - x}$ can be expressed as an infinite series. That is,

$$f(x) = \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \text{ or } 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } |x| < 1.$$

The figures below show the graph of $f(x) = \frac{1}{1 - x}$ and the second through fifth partial sums $S_n(x)$ of the series: $S_2(x) = 1 + x$, $S_3(x) = 1 + x + x^2$, $S_4(x) = 1 + x + x^2 + x^3$, and $S_5(x) = 1 + x + x^2 + x^3 + x^4$.




Notice that as $n$ increases, the graph of $S_n(x)$ appears to come closer and closer to the graph of $f(x)$ on the interval $(-1, 1)$ or $|x| < 1$. Notice too that each of the partial sums of the series is a polynomial function, so the series can be thought of as an "infinite" polynomial. An infinite series of this type is called a **power series**.

### KeyConcept Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where $x$ and $a_n$ can take on any values for $n = 0, 1, 2, \ldots$, is called a power series in $x$.

If you know the power series representation of one function, you can use it to find the power series representations of other related functions.

## Example 1 Power Series Representation of a Rational Function

**Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{1}{3-x}$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.**

To find the transformation that relates $f(x)$ to $g(x)$, use $u$-substitution. Substitute $u$ for $x$ in $f$, equate the two functions, and solve for $u$ as shown.

$$g(x) = f(u)$$
$$\frac{1}{3-x} = \frac{1}{1-u}$$
$$1 - u = 3 - x$$
$$-u = 2 - x$$
$$u = x - 2$$

### WatchOut!

When finding the $k$th partial sum of a series where the lower bound starts at 0 use the series $\sum_{n=0}^{k-1}$. For instance in Example 1, the sixth partial sum is called for, but since the lower bound is 0, the upper bound is 6 − 1 or 5, not 6.

Therefore, $g(x) = f(x-2)$. Replacing $x$ with $x - 2$ in $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ yields

$$f(x-2) = \sum_{n=0}^{\infty} (x-2)^n \text{ for } |x-2| < 1.$$

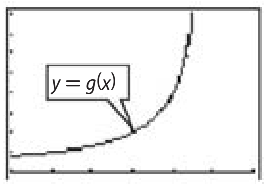
Therefore, $g(x) = \frac{1}{3-x}$ can be represented by the power series $\sum_{n=0}^{\infty} (x-2)^n$.

This series converges for $|x - 2| < 1$, which is equivalent to $-1 < x - 2 < 1$ or $1 < x < 3$.

The sixth partial sum $S_6(x)$ of this series is

$$\sum_{n=0}^{5} (x-2)^n \text{ or } 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5.$$

The graphs of $g(x) = \frac{1}{3-x}$ and $S_6(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5$ are shown. Notice that on the interval (1, 3), the graph of $S_6(x)$ comes close to the graph of $g(x)$.

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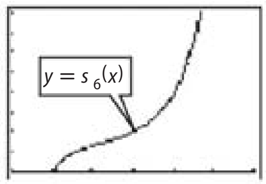


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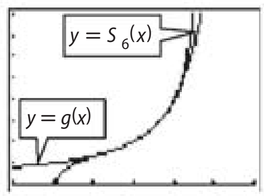


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### StudyTip

**Graphs of Series** Notice that the graphs of $f(x)$ and $S_n(x)$ only converge on an interval. The graphs may differ greatly outside of that interval.

### GuidedPractice

**Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.**

**1A.** $g(x) = \frac{1}{1-2x}$

**1B.** $g(x) = \frac{2}{1-x}$

In calculus, power series representations are often easier to use in calculations than other representations of functions when determining functions called *derivatives* and *integrals*. A more immediate application can be seen by looking at the power series representations of transcendental functions such as $f(x) = e^x$, $f(x) = \sin x$, and $f(x) = \cos x$.

**ReadingMath**

**Euler Number** The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number, called $e$, the Euler number.

## 2 Transcendental Functions as Power Series

In Lesson 3-1, you learned that the transcendental number $e$ is given by $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Thus, $e^x = \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{nx}$. We can use this definition along with the Binomial Theorem to derive a power series representation for $f(x) = e^x$.

If we let $u = \frac{1}{n}$ and $k = nx$, then $\left(1+\frac{1}{n}\right)^{nx}$ becomes $(1+u)^k$. Applying the Binomial Theorem,

$$\begin{aligned}(1+u)^k &= {}_kC_0\,(1)^k u^0 + {}_kC_1\,(1)^{k-1}u + {}_kC_2\,(1)^{k-2}u^2 + {}_kC_3\,(1)^{k-3}u^3 + \cdots\\ &= \frac{k!}{(k-0)!\,0!}(1) + \frac{k!}{(k-1)!\,1!}(1)u + \frac{k!}{(k-2)!\,2!}(1)u^2 + \frac{k!}{(k-3)!\,3!}(1)u^3 + \cdots\\ &= 1 + \frac{k(k-1)!}{(k-1)!}u + \frac{k(k-1)(k-2)!}{(k-2)!\,2!}u^2 + \frac{k(k-1)(k-2)(k-3)!}{(k-3)!\,3!}u^3 + \cdots\\ &= 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \cdots\end{aligned}$$

Now replace $u$ with $\frac{1}{n}$ and $k$ with $nx$ and find the limit as $n$ approaches infinity. Use the fact that as $n$ approaches infinity, the fraction $\frac{1}{n}$ gets increasingly smaller, so $\lim_{n\to\infty}\frac{1}{n} = 0$.

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{nx} &= 1 + (nx)\frac{1}{n} + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots\\ &= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \cdots\\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

This series is often called the **exponential series**.

**StudyTip**

**Defining $e$** The exponential series provides yet another way to define $e$. When $x = 1$, $e^1 = \sum_{n=0}^{\infty}\frac{1^n}{n!}$ or $\sum_{n=0}^{\infty}\frac{1}{n!}$.

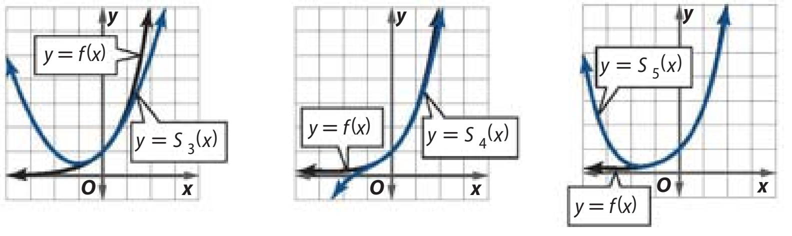
**KeyConcept Exponential Series**

The power series representing $e^x$ is called the exponential series and is given by

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

which is convergent for all $x$.

The graph of $f(x) = e^x$ and the partial sums $S_3(x)$, $S_4(x)$, and $S_5(x)$ of the exponential series are shown below.




You can see from the graphs that the partial sums of the exponential series approximate the graph of $f(x) = e^x$ on increasingly wider intervals of the domain for increasingly greater values of $n$.

Notice that the calculations involved in the exponential series are relatively simple: multiplications (for powers and factorials), divisions, and additions. Because of this, calculators and computer programs use partial sums of the exponential series to evaluate $e^x$ to desired degrees of accuracy.

**WatchOut!**

**Evaluating $e^x$** The fifth partial sum of the exponential series only gives reasonably good approximations of $e^x$ for $x$ on $[-1.5, 2.5]$. Subsequent partial sums, such as the sixth and seventh partial sums, are more accurate for wider intervals of $x$-values.

## Example 2 Exponential Series

**Use the fifth partial sum of the exponential series to approximate the value of $e^{1.5}$. Round to three decimal places.**

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad e^x \approx \sum_{n=0}^{4} \frac{x^n}{n!}$$

$$e^{1.5} \approx 1 + 1.5 + \frac{1.5^2}{2!} + \frac{1.5^3}{3!} + \frac{1.5^4}{4!} \qquad x = 1.5$$

$$\approx 4.398 \qquad \text{Simplify.}$$

**CHECK** A calculator, using a partial sum of the exponential series with many more terms, returns an approximation of 4.48 for $e^{1.5}$. Therefore, an approximation of 4.398 is reasonable. ✔

**GuidedPractice**

**Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.**

**2A.** $e^{-0.75}$

**2B.** $e^{0.25}$


**Math HistoryLink**

**Mádhava of Sangamagramma** (**1340–1425**)
An Indian mathematician born near Cochin, Mádhava discovered the series equivalent to the expansions of sin $x$, cos $x$, and arctan $x$ around 1400, two hundred years before their discovery in Europe.

Other transcendental functions have power series representations as well. Calculators and computers use **power series** to approximate the values of cosine and sine functions.

### KeyConcept Power Series for Cosine and Sine

The power series representations for cos $x$ and sin $x$ are given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \text{ and}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots,$$

which are convergent for all $x$.

By replacing $x$ with any angle measure expressed in radians and carrying out the computations, approximate values of the cosine and sine functions can be found to any desired degree of accuracy.

## Example 3 Trigonometric Series

**a. Use the fifth partial sum of the power series for cosine to approximate the value of $\cos \frac{\pi}{7}$. Round to three decimal places.**

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad \cos x \approx \sum_{n=0}^{4} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos \frac{\pi}{7} \approx 1 - \frac{(0.449)^2}{2!} + \frac{(0.449)^4}{4!} - \frac{(0.449)^6}{6!} + \frac{(0.449)^8}{8!} \qquad x = \frac{\pi}{7} \text{ or about } 0.449$$

$$\approx 0.901 \qquad \text{Simplify.}$$

**CHECK** A calculator, using a partial sum of the power series for cosine with many more terms, returns an approximation of 0.901, to three decimal places, for $\cos \frac{\pi}{7}$. Therefore, an approximation of 0.901 is reasonable. ✔

**StudyTip**

**Fifth Partial Sum** While additional partial sums provide a better approximation, the fifth partial sum typically is accurate to three decimal places.

**b. Use the fifth partial sum of the power series for sine to approximate the value of $\sin \frac{\pi}{5}$. Round to three decimal places.**

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$ — $\sin x \approx \sum_{n=0}^{5} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin \frac{\pi}{5} \approx 0.628 - \frac{(0.628)^3}{3!} + \frac{(0.628)^5}{5!} - \frac{(0.628)^7}{7!} + \frac{(0.628)^9}{9!}$ — $x = \frac{\pi}{5}$ or about 0.628

$\approx 0.588$ — Simplify.

**CHECK** Using a calculator, $\sin \frac{\pi}{5} \approx 0.588$. Therefore, an approximation of 0.588 is reasonable. ✔

**GuidedPractice**

**Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.**

**3A.** $\sin \frac{\pi}{11}$ **3B.** $\cos \frac{2\pi}{17}$

You may have noticed similarities in the power series representations of $f(x) = e^x$ and the power series representations of $f(x) = \sin x$ and $f(x) = \cos x$. A relationship is derived by replacing $x$ by $i\theta$ in the exponential series, where $i$ is the imaginary unit and $\theta$ is the measure of an angle in radians.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \qquad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \qquad i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + \cdots\right) \qquad \text{Group real and imaginary terms.}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \text{Distributive Property}$$

$$= \cos\theta + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$= \cos\theta + i\sin\theta \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

This relationship is called **Euler's Formula**.

**KeyConcept Euler's Formula**

For any real number $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$.

From your work in Lesson 9-5, you should recognize the right-hand side of this equation as being part of the polar form of a complex number. Applying Euler's Formula to the polar form of a complex number yields the following result.

$$a + bi = r(\cos\theta + i\sin\theta) \qquad \text{Polar form of a complex number}$$

$$= re^{i\theta} \qquad \text{Euler's Formula}$$

Therefore, Euler's Formula gives us a way of expressing a complex number in exponential form.

**KeyConcept Exponential Form of a Complex Number**

The exponential form of a complex number $a + bi$ is given by

$$a + bi = re^{i\theta},$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$ for $a > 0$ and $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$.

## Example 4 Write a Complex Number in Exponential Form

**Write $-\sqrt{3} + i$ in exponential form.**

Write the polar form of $-\sqrt{3} + i$. In this expression, $a = -\sqrt{3}$, $b = 1$, and $a < 0$. Find $r$.

$r = \sqrt{(-\sqrt{3})^2 + 1^2}$ — $r = \sqrt{a^2 + b^2}$

$= \sqrt{4}$ or 2 — Simplify.

Now find $\theta$.

$\theta = \tan^{-1}\frac{1}{-\sqrt{3}} + \pi$ — $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$

$= -\frac{\pi}{6} + \pi$ or $\frac{5\pi}{6}$ — Simplify.

Therefore, because $a + bi = re^{i\theta}$, the exponential form of $-\sqrt{3} + i$ is $2e^{i\frac{5\pi}{6}}$.

### GuidedPractice

**Write each complex number in exponential form.**

**4A.** $1 + \sqrt{3}i$

**4B.** $\sqrt{2} + \sqrt{2}i$

From your study of logarithms in Chapter 3, you know that no *real* number can be the logarithm of a negative number. We can use Euler's Formula to show that the natural logarithm of a negative number does exist in the *complex* number system.

| | |
|---|---|
| $e^{i\theta} = \cos\theta + i\sin\theta$ | Euler's Formula |
| $e^{i\pi} = \cos\pi + i\sin\pi$ | Let $\theta = \pi$. |
| $e^{i\pi} = -1 + i(0)$ | $\cos\pi = -1$ and $\sin\pi = 0$ |
| $e^{i\pi} = -1$ | Simplify. |
| $\ln e^{i\pi} = \ln(-1)$ | Take the natural logarithm of each side. |
| $i\pi = \ln(-1)$ | Power Property of Logarithms |

This result indicates that the natural logarithm of $-1$ exists and is the complex number $i\pi$. You can use this result to find the natural logarithm of any negative number $-k$, for $k > 0$.

| | |
|---|---|
| $\ln(-k) = \ln[(-1)k]$ | $-k = (-1)k$ |
| $= \ln(-1) + \ln k$ | Product Property of Logarithms |
| $= i\pi + \ln k$ | $\ln(-1) = i\pi$ |
| $= \ln k + i\pi$ | Write in the form $a + bi$. |

**TechnologyTip**

**Complex Numbers** You can use your calculator to evaluate the natural logarithm of a negative number by changing from REAL to $a + bi$ under MODE.

## Example 5 Natural Logarithm of a Negative Number

**Find the value of ln (−5) in the complex number system.**

$\ln(-5) = \ln 5 + i\pi$ — $\ln(-k) = \ln k + i\pi$

$\approx 1.609 + i\pi$ — Use a calculator to compute ln 5.

### GuidedPractice

**Find the value of each natural logarithm in the complex number system.**

**5A.** $\ln(-8)$

**5B.** $\ln(-6.24)$

## Exercises

= Step-by-Step Solutions begin on page R29.

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. **Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.** (Example 1)

**1.** $g(x) = \frac{4}{1-x}$

**2.** $g(x) = \frac{3}{1-2x}$

**3.** $g(x) = \frac{2}{1-x^2}$

**4.** $g(x) = \frac{3}{2-x}$

**5.** $g(x) = \frac{2}{5-3x}$

**6.** $g(x) = \frac{4}{3-2x^2}$

**Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.** (Example 2)

**7.** $e^{0.5}$

**8.** $e^{-0.25}$

**9.** $e^{-2.5}$

**10.** $e^{0.8}$

**11.** $e^{-0.3}$

**12.** $e^{3.5}$

**13.** **ECOLOGY** The population density $P$ per square meter of zebra mussels in the Upper Mississippi River can be modeled by $P = 3.5e^{0.08t}$, where $t$ is measured in weeks. Use the fifth partial sum of the exponential series to estimate the zebra mussel population density after 4 weeks, 12 weeks, and 1 year. (Example 2)

**Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.** (Example 3)

**14.** $\sin \frac{\pi}{9}$

**15.** $\cos \frac{2\pi}{13}$

**16.** $\sin \frac{5\pi}{13}$

**17.** $\cos \frac{3\pi}{10}$

**18.** $\cos \frac{2\pi}{9}$

**19.** $\sin \frac{3\pi}{19}$

**20.** **AMUSEMENT PARK** A ride at an amusement park is in the shape of a giant pendulum that swings riders back and forth in a 240° arc to a maximum height of 137 feet. The pendulum is supported by a tower that is 85 feet tall and dips below ground-level into a pit when swinging below the tower. Use the fifth partial sum of the power series for cosine or sine to approximate the length of the pendulum. (Example 3)

**Write each complex number in exponential form.** (Example 4)

**21.** $\sqrt{3} + i$

**22.** $\sqrt{3} - i$

**23.** $\sqrt{2} - \sqrt{2}i$

**24.** $-\sqrt{3} - i$

**25.** $1 - \sqrt{3}i$

**26.** $-1 + \sqrt{3}i$

**27.** $-\sqrt{2} + \sqrt{2}i$

**28.** $-1 - \sqrt{3}i$

**Find the value of each natural logarithm in the complex number system.** (Example 5)

**29.** $\ln(-6)$

**30.** $\ln(-3.5)$

**31.** $\ln(-2.45)$

**32.** $\ln(-7)$

**33.** $\ln(-4.36)$

**34.** $\ln(-9.12)$

**35.** **POWER SERIES** Use the power series representations of $\sin x$ and $\cos x$ to answer each of the following questions.

**a.** Graph $f(x) = \sin x$ and the third partial sum of the power series representing $\sin x$. Repeat for the fourth and fifth partial sums. Describe the interval of convergence for each.

**b.** Repeat part **a** for $f(x) = \cos x$ and the third, fourth, and fifth partial sums of the power series representing $\cos x$. Describe the interval of convergence for each.

**c.** Describe how the interval of convergence changes as $n$ increases. Then make a conjecture as to the relationship between each trigonometric function and its related power series as $n \to \infty$.

**Solve for $z$ over the complex numbers. Round to three decimal places.**

**36.** $2e^z + 5 = 0$

**37.** $e^{2z} + 12 = 0$

**38.** $4e^{2z} + 7 = 6$

**39.** $3(e^z - 1) + 5 = -2$

**40.** $e^{2z} - e^z = 2$

**41.** $10e^{2z} + 17e^z = -3$

**42.** **ECONOMICS** The total value of an investment of $P$ dollars compounded continuously at an annual interest rate of $r$ over $t$ years is $Pe^{rt}$. Use the first five terms of the exponential series to approximate the value of an investment of \$10,000 compounded continuously at 5.25% for 5 years.

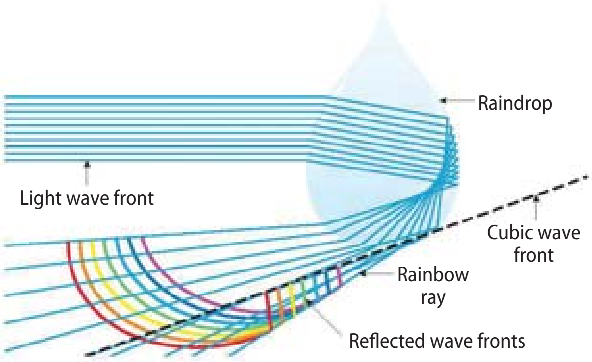
**43.** **RELATIVE ERROR** *Relative error* is the absolute error in estimating a quantity divided by its true value. The relative error of an approximation $a$ of a quantity $b$ is given by $\frac{|b-a|}{b}$. Find the relative error in approximating $e^{2.1}$ using two, three, and six terms of the exponential series.

**Approximate the value of each expression using the first four terms of the power series for sine and cosine. Then find the expected value of each.**

**44.** $\sin^2 \frac{1}{2} + \cos^2 \frac{1}{2}$

**45.** $\sec^2 1 - \tan^2 1$

**46. RAINBOWS** *Airy's equation*, which is used in physics to model the diffraction of light, can also be used to explain how a light wave front is converted into a curved wave front in forming rainbows.

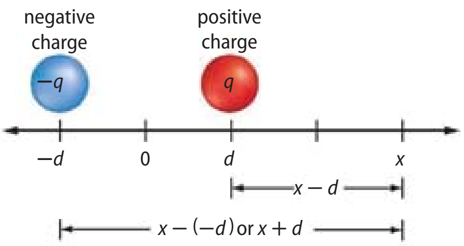



This equation can be represented by the power series shown below.

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots [(3k-1) \cdot (3k)]}$$

Use the fifth partial sum of the series to find $f(3)$. Round to the nearest hundredth.

**47. ELECTRICITY** When an electric charge is accompanied by an equal and opposite charge nearby, such an object is called an *electric dipole*. It consists of charge $q$ at the point $x = d$ and charge $-q$ at $x = -d$, as shown below.




Along the $x$-axis, the electric field strength at $x$ is the sum of the electric fields from each of the two charges. This is given by $E(x) = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2}$. Find a power series representing $E(x)$ if $k$ is a constant and $d = 1$.

**48. SOUND** The *Fourier Series* represents a periodic function of time $f(t)$ as a summation of sine waves and cosine waves with frequencies that start at 0 and increase by integer multiples. The series below represents a sound wave from the digital data fed from a CD into a CD player.

$$f(t) = 0.7 + \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n} \cos 270.6nt + \frac{1}{2n-1} \sin 270.6nt \right)$$

Graph the series for $n = 4$. Then analyze the graph.

**IDENTITIES Use power series representations from this lesson to verify each trigonometric identity.**

**49.** $\sin(-x) = -\sin x$

**50.** $\cos(-x) = \cos x$

**51. APPROXIMATIONS** The infinite series for the inverse tangent function $f(x) = \tan^{-1} x$, is given by $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. However, this series is only valid for values of $x$ on the interval $(-1, 1)$.

**a.** Write the first five terms of the infinite series representation for $f(x) = \tan^{-1} x$.

**b.** Use the first five terms of the series to approximate $\tan^{-1} 0.1$.

**c.** On the same coordinate plane, graph $f(x) = \tan^{-1} x$ and the third partial sum of the power series representing $f(x) = \tan^{-1} x$. On another coordinate plane, graph $f(x)$ and the fourth partial sum. Then graph $f(x)$ and the fifth partial sum.

**d.** Describe what happens on the interval $(-1, 1)$ and in the regions $x \geq 1$ or $x \leq -1$.

## H.O.T. Problems Use Higher-Order Thinking Skills

**52. WRITING IN MATH** Describe how using additional terms in the approximating series for $e^x$ affects the outcome.

**53. REASONING** Use the power series for sine to explain why, for $x$-values on the interval $[-0.1, 0.1]$, a close approximation of $\sin x$ is $x$.

**54. CHALLENGE** Prove that $2 \sin \theta \cos \theta = \frac{e^{2\theta i} - e^{-2\theta i}}{2i}$

**55. REASONING** For what values of $\alpha$ and $\beta$ does $e^{i\alpha} = e^{i\beta}$? Explain.

**PROOF Show that for all real numbers $x$, the following are true.**

**56.** $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

**57.** $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

**58. CHALLENGE** The hyperbolic sine and hyperbolic cosine functions are analogs of the trigonometric functions that you studied in Chapters 4 and 5. Just as the points ($\cos x$, $\sin x$) form a unit circle, the points ($\cosh t$, $\sinh t$) form the right half of an equilateral hyperbola. An equilateral hyperbola has perpendicular asymptotes. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined below. Find the power series representations for these functions.

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\qquad$ $\cosh x = \frac{e^x + e^{-x}}{2}$

## Spiral Review

**Use Pascal's triangle to expand each binomial.** (Lesson 10-5)

**59.** $\left(3m+\sqrt{2}\right)^4$ **60.** $\left(\frac{1}{2}n+2\right)^5$ **61.** $(p^2+q)^8$

**62.** Prove that $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$ for all positive integers $n$. (Lesson 10-4)

**Find each power, and express it in rectangular form.** (Lesson 9-5)

**63.** $(-2+2i)^3$ **64.** $\left(1+\sqrt{3}i\right)^4$ **65.** $\left(\sqrt{2}+\sqrt{2}i\right)^{-2}$

**66.** Given $\mathbf{t}=\langle -9,-3,c\rangle$, $\mathbf{u}=\langle 8,-4,3\rangle$, $\mathbf{v}=\langle 2,5,-6\rangle$, and that the volume of the parallelepiped having adjacent edges $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ is 93 cubic units, find $c$. (Lesson 8-5)

**Use an inverse matrix to solve each system of equations, if possible.** (Lesson 6-3)

**67.** $x-8y=-7$
$2x+5y=28$

**68.** $4x+7y=22$
$-9x+11y=4$

**69.** $w+2x+3y=18$
$4w-8x+7y=41$
$-w+9x-2y=-4$

**Determine whether $A$ and $B$ are inverse matrices.** (Lesson 6-2)

**70.** $A=\begin{bmatrix}1 & -2\\ 7 & -6\end{bmatrix}$, $B=\begin{bmatrix}-6 & 2\\ -7 & 1\end{bmatrix}$

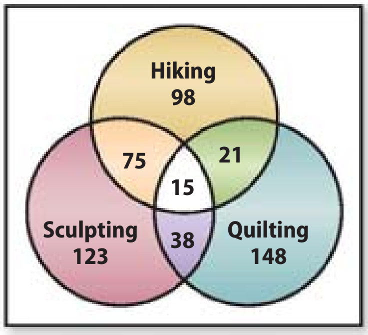
**71.** $A=\begin{bmatrix}-11 & -5\\ 9 & 4\end{bmatrix}$, $B=\begin{bmatrix}4 & 5\\ -9 & -11\end{bmatrix}$

**72.** $A=\begin{bmatrix}6 & 2\\ -2 & 8\end{bmatrix}$, $B=\begin{bmatrix}-1 & 1\\ -3 & -5\end{bmatrix}$

**73. CONFERENCE** A university sponsored a conference for 680 women. The Venn diagram shows the number of participants in three of the activities offered. Suppose women who attended the conference were randomly selected for a survey. (Lesson 0-1)

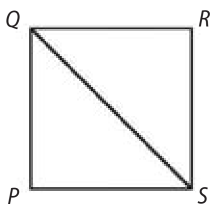
**a.** What is the probability that a woman selected participated in hiking or sculpting?

**b.** Describe a set of women such that the probability of being selected is about 0.39.




## Skills Review for Standardized Tests

**74. SAT/ACT** $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?




**A** 2

**B** $\sqrt{2}$

**C** 1

**D** $\dfrac{\sqrt{2}}{2}$

**E** $\dfrac{\sqrt{3}}{2}$

**75. REVIEW** What is the sum of the infinite geometric series $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\cdots$?

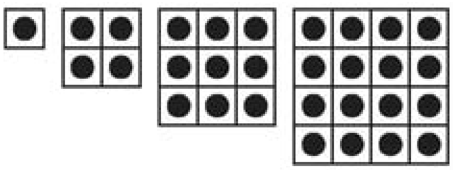
**F** $\frac{2}{3}$

**G** 1

**H** $1\frac{1}{3}$

**J** $1\frac{2}{3}$

**76. FREE RESPONSE** Consider the pattern of dots shown.



**a.** Draw the next figure in this sequence.

**b.** Write the sequence, starting with 1, that represents the number of dots that must be added to each figure in the sequence to get the number of dots in the next figure.

**c.** Find the expression for the $n$th term of the sequence found in part **b**.

**d.** Find the expression for the number of dots in the $n$th figure in the original sequence.

**e.** Prove, through mathematical induction, that the sum of the sequence found in part **b** is equal to the expression found in part **d**.

EXTEND 10-6

## Spreadsheet Lab

# Detecting Patterns in Data

## Objective

- Organize and display data using spreadsheets to detect patterns and departures from patterns.

In Chapter 10, you learned how to detect patterns in a sequence and describe them by using functions.

| Pattern in Data Sequence | Pattern in Graph of Data Sequence | Type of Sequence | Function Describing Sequence |
|---|---|---|---|
| common 1st differences | data in a linear pattern | arithmetic | linear |
| common ratio | data in an exponential pattern | geometric | exponential |

In this lab, you will use a spreadsheet to organize and display paired data in order to look for such patterns.

### Activity 1 Detect Patterns

**DOGS A certain golden retriever had a mass of 1.45 kilograms at birth. The table shows the puppy's mass in the first 70 days of its growth. Use a spreadsheet to find a pattern in the data.**

| Days after Birth | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|
| Mass (kg) | 1.91 | 2.46 | 3.17 | 4.10 | 5.22 | 6.81 | 8.63 |

**Step 1** Enter the data into the spreadsheet.

**Step 2** To determine if the sequence of masses is arithmetic, enter a formula in the next column to find the average daily rate of change in the puppy's mass.

**Step 3** To determine if the sequence is geometric, enter the formula shown in the next column to find the average ratio of change in the puppy's mass each day.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | Days after Birth | Mass (kg) | Average Rate of Change | Average Ratio of Change |
| 2 | 0 | 1.45 | | |
| 3 | 10 | 1.91 | =(B3-B2)/(A3-A2) | =(B3/B2)^(1/(A3-A2))-1 |
| 4 | 20 | 2.46 | | |
| 5 | 30 | 3.17 | | |
| 6 | 40 | 4.1 | | |
| 7 | 50 | 5.22 | | |
| 8 | 60 | 6.81 | | |
| 9 | 70 | 8.63 | | |

Sheet 1 Sheet 2 Sheet 3

## Analyze the Results

1. Explain the formulas used in the spreadsheet.
2. Describe any pattern you see in the data. What type of sequence approximates the data? Explain.
3. Use the chart tool to create a scatter plot of the data. Does this graph support your answer to Exercise 2? Explain.
4. Write an equation approximating the dog's mass $y$ after $x$ days.
5. Use your equation to predict the dog's mass 25 days after birth and 365 days after birth. Are these predictions reasonable? Explain.

You can also use a spreadsheet to detect and analyze departures from patterns.

## Activity 2 Detect Departures from Patterns

**HOMEWORK Adrian recorded the number of precalculus problems and how long he worked on them for eight nights. Look for a pattern in the data and any departures from that pattern.**

| Number of Problems | 0 | 3 | 5 | 6 | 8 | 9 | 10 | 15 |
|---|---|---|---|---|---|---|---|---|
| Time (min) | 0 | 27 | 44 | 70 | 72 | 82 | 95 | 140 |

**Step 1** Enter the data into the spreadsheet.

**Step 2** Enter formulas in the adjacent columns to detect whether the sequence of is arithmetic or geometric. Then copy these formulas into the cells below.

**Step 3** Look for patterns. Notice that all but two of the rates of change cluster around 9.

| | A | B | C |
|---|---|---|---|
| 1 | Number of Problems | Time (min) | Average Rate of Change |
| 2 | 0 | 0 | |
| 3 | 3 | 27 | 9.00 |
| 4 | 5 | 44 | 8.50 |
| 5 | 6 | 70 | 26.00 |
| 6 | 8 | 72 | 1.00 |
| 7 | 9 | 81 | 9.00 |
| 8 | 10 | 89 | 8.00 |
| 9 | 15 | 136 | 9.40 |

Sheet 1 | Sheet 2 | Sheet 3

**StudyTip**

**Series in Data** To investigate series in data, you can use the Auto Sum tool. For Activity 2, enter =B2 in cell D2 and =SUM(B2,B3) in cell D3. Copy this second formula into the remaining cells in the column to create a sequence of partial sums.

## Analyze the Results

**6.** Where does the departure in the pattern occur?

**7.** Write a spreadsheet formula that could model the data if this data value were removed.

**8.** Use the Chart Wizard to create a scatter plot that shows the actual data and the model of the data. Does this graph support your answer to Exercise 7? Explain.

**9.** Use your formula from Exercise 7 to predict how long it would take Adrian to complete 12 problems and 20 problems. Are these predictions reasonable? Explain.

## Exercises

**Use a spreadsheet to organize and identify a pattern or departure from a pattern in each set of data. Then use a calculator to write an equation to model the data.**

**10.** **INTERNET** The table shows the number of times the main page of a popular blog is read (hits) each month.

| Month | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 20 |
|---|---|---|---|---|---|---|---|---|
| Hits | 83 | 171 | 266 | 368 | 479 | 732 | 1405 | 4017 |

**11.** **COLLEGE** The table shows the composite ACT scores and grade-point averages (GPA) of 20 students after their first semester in college. (*Hint:* First use the Sort Ascending tool to organize the data.)

| ACT Score | 27 | 16 | 15 | 22 | 20 | 21 | 25 |
|---|---|---|---|---|---|---|---|
| College GPA | 3.9 | 2.9 | 2.7 | 3.6 | 3.2 | 3.4 | 3.1 |
| ACT Score | 26 | 18 | 23 | 19 | 29 | 28 | 17 |
| College GPA | 4.0 | 3.1 | 3.6 | 2.6 | 4.0 | 3.9 | 3.0 |

# Study Guide and Review

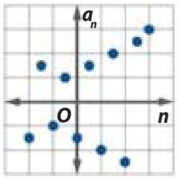
## Chapter Summary

### KeyConcepts

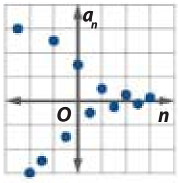
**Sequences, Series, and Sigma Notation** (Lesson 10-1)

- A *finite sequence* is a sequence with a set number of terms. An *infinite sequence* has infinitely many terms.
- A sequence with a limit is said to *converge*. A sequence with no limit is said to *diverge*.

Divergent Sequence




Convergent Series




- A *series* is the sum of all of the terms of a sequence.

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_k$$

**Arithmetic Sequences and Series** (Lesson 10-2)

- The $n$th term of an arithmetic sequence with first term $a_1$ and common difference $d$ is given by $a_n = a_1 + (n - 1)d$.
- The sum of a finite arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$.

**Geometric Sequences and Series** (Lesson 10-3)

- The $n$th term of a geometric sequence with first term $a_1$ and common ratio $r$ is given by $a_n = a_1 r^{n-1}$.
- The sum of the first $n$ terms of a geometric series is given by $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$ or $S_n = \frac{a_1 - a_n r}{1 - r}$.
- The sum of an infinite geometric series is given by $S = \frac{a_1}{1 - r}$, for $|r| < 1$.

**Mathematical Induction** (Lesson 10-4)

- If $P_n$ is a statement about a positive integer $n$, then $P_n$ is true for all positive integers $n$ if and only if $P_1$ is true, and for every positive integer $k$, if $P_k$ is true, then $P_{k+1}$ is true.

**The Binomial Theorem** (Lesson 10-5)

- The expression $(a + b)^n$ can be expanded using the $n$th row of Pascal's triangle to determine the coefficients of each term.
- The binomial coefficient of the $a^{n-r}b^r$ term in the expansion of $(a + b)^n$ is given by ${}_nC_r = \frac{n!}{(n - r)!\, r!}$.

**Functions as Infinite Series** (Lesson 10-6)

- A power series in $x$ is an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n$.
- Euler's Formula states that for any real number $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$.

### KeyVocabulary

| | |
|---|---|
| anchor step (p. 621) | geometric means (p. 611) |
| arithmetic means (p. 601) | geometric sequence (p. 608) |
| arithmetic sequence (p. 599) | geometric series (p. 611) |
| arithmetic series (p. 602) | inductive hypothesis (p. 621) |
| binomial coefficients (p. 628) | inductive step (p. 621) |
| Binomial Theorem (p. 632) | infinite sequence (p. 590) |
| common difference (p. 599) | infinite series (p. 593) |
| common ratio (p. 608) | $n$th partial sum (p. 593) |
| converge (p. 592) | power series (p. 636) |
| diverge (p. 592) | recursive formula (p. 591) |
| Euler's Formula (p. 640) | second difference (p. 601) |
| exponential series (p. 638) | sequence (p. 590) |
| Fibonacci sequence (p. 591) | series (p. 593) |
| finite sequence (p. 590) | sigma notation (p. 594) |
| finite series (p. 593) | term (p. 590) |
| first difference (p. 601) | |

### VocabularyCheck

**State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.**

1. In mathematical induction, the assumption that a conjecture works for any particular case is called the <u>inductive hypothesis</u>.
2. A sequence that has a set number of terms is called an <u>infinite sequence</u>.
3. A sequence $a_n$ defined as a function of $n$ is defined <u>recursively</u>.
4. The step in which you show that something works for the first case is called the <u>inductive step</u> in mathematical induction.
5. <u>Explicitly</u> defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms.
6. The sum of the first $n$ terms of a finite or infinite sequence is called a <u>finite series</u>.
7. The difference between the terms of an arithmetic sequence is called the <u>common ratio</u>.
8. A <u>geometric series</u> is the sum of the terms of a geometric sequence.
9. If a sequence does not have a limit, it is said to <u>converge</u>.
10. The <u>Binomial Theorem</u> is a recursive sequence that describes many patterns found in nature.

## Lesson-by-Lesson Review

### 10-1 Sequences, Series, and Sigma Notation (pp. 590–598)

Find the next four terms of each sequence.

**11.** 1, 9, 17, 25, …

**12.** −1, 2, 7, 14, …

Find the indicated sum for each sequence.

**13.** fourth partial sum of $a_n = 2n - 10$

**14.** $S_7$ of $a_n = -n^3$

Find each sum.

**15.** $\sum_{n=2}^{9} \frac{4n-6}{3}$

**16.** $\sum_{n=1}^{6} 7n - 4$

**Example 1**

**Find the sum of $\sum_{n=1}^{4} n^2 - 5$.**

Find the values for $a_n$ for $n = 1, 2, 3$, and 4.

$a_1 = (1)^2 - 5$ or $-4$ $\quad n = 1$

$a_2 = (2)^2 - 5$ or $-1$ $\quad n = 2$

$a_3 = (3)^2 - 5$ or $4$ $\quad n = 3$

$a_4 = (4)^2 - 5$ or $11$ $\quad n = 4$

Therefore, $\sum_{n=1}^{4} n^2 - 5 = -4 + (-1) + 4 + 11$ or 10.

### 10-2 Arithmetic Sequences and Series (pp. 599–607)

Write an explicit formula and a recursive formula for finding the $n$th term of each arithmetic sequence.

**17.** −6, −1, 4, …

**18.** 23, 15, 7, …

Find each sum.

**19.** 50th partial sum of 55 + 62 + 69 + … + 398

**20.** 37th partial sum of 9 + 3 + (−3) + …

Find each sum.

**21.** $\sum_{n=24}^{35} -3n - 11$

**22.** $\sum_{n=8}^{27} 4n + 14$

**Example 2**

**Find the 43rd partial sum of the arithmetic series 104 + 100 + 96 + … .**

Use the second sum of a finite arithmetic series formula.

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ $\quad$ Sum of a finite arithmetic series formula

$S_{43} = \frac{43}{2}[2(104) + (43-1)(-4)]$ $\quad$ $n = 43$, $a_1 = 104$, and $d = -4$

$= 860$ $\quad$ Simplify.

The 43rd partial sum is $S_{43} = 860$.

### 10-3 Geometric Sequences and Series (pp. 608–618)

Determine the common ratio, and find the next three terms of each geometric sequence.

**23.** 5, 7.5, 11.25, …

**24.** $3 + a, -12 - 4a, 48 + 16a, \ldots$

Write an explicit formula and a recursive formula for finding the $n$th term of each geometric sequence.

**25.** 10, −20, 40, …

**26.** 162, 54, 18, …

Find each sum.

**27.** first seven terms of $80 + 32 + \frac{64}{5} + \ldots$

**28.** $a_1 = 11$, $a_n = 360{,}448$, $r = 8$

**Example 3**

**Determine the common ratio, and find the next three terms of 27, −9, 3, … .**

First, find the common ratio.

$a_2 \div a_1 = -9 \div 27$ or $-\frac{1}{3}$ $\quad$ Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 3 \div -9$ or $-\frac{1}{3}$

The common ratio is $-\frac{1}{3}$. Multiply the third term by $-\frac{1}{3}$ to find the fourth term, and so on.

$a_4 = 3 \cdot -\frac{1}{3}$ or $-1$ $\qquad a_5 = -1 \cdot -\frac{1}{3}$ or $\frac{1}{3}$

$a_6 = \frac{1}{3} \cdot -\frac{1}{3}$ or $-\frac{1}{9}$

The next three terms are $-1, \frac{1}{3}$, and $-\frac{1}{9}$.

## 10-4 Mathematical Induction (pp. 621–627)

Use mathematical induction to prove that each conjecture is true for all positive integers $n$.

**29.** $6^n - 9$ is divisible by 3.

**30.** $7^n - 5$ is divisible by 2.

**31.** $5^n + 3$ is divisible by 4.

**32.** $2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7 + \cdots + 2n(2n + 1) = \frac{n(n + 1)(4n + 5)}{3}$

**33.** $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n + 1)(n + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)}$

Prove each inequality for the indicated values of $n$.

**34.** $4^n \geq 4n$, for all positive integers $n$

**35.** $5n < 6^n$, for all positive integers $n$

### Example 4

**Prove that $5^n - 1$ is divisible by 4 for all positive integers $n$.**

**Conjecture and Anchor Step** Let $P_n$ be the statement that $5^n - 1$ is divisible by 4. When $n = 1$, $5^n - 1 = 5^1 - 1$ or 4. Since 4 is divisible by 4, the statement $P_1$ is true.

**Inductive Hypothesis and Step** Assume that $5^k - 1$ is divisible by 4. That is, assume that $5^k - 1 = 4r$ for some integer $r$. Use this inductive hypothesis to show that $5^{k+1} - 1$ is divisible by 4.

| | |
|---|---|
| $5^k - 1 = 4r$ | Inductive hypothesis |
| $5^k = 4r + 1$ | Add 1 to each side. |
| $5 \cdot 5^k = 5(4r + 1)$ | Multiply each side by 5. |
| $5^{k+1} = 20r + 5$ | Simplify. |
| $5^{k+1} - 1 = 20r + 4$ | Subtract 1 from each side. |
| $5^{k+1} - 1 = 4(5r + 1)$ | Factor. |

Since $r$ is an integer, $5r + 1$ is an integer and $4(5r + 1)$ is divisible by 4. Therefore, $5^{k+1} - 1$ is divisible by 4.

**Conclusion** Since $P_n$ is true for $n = 1$ and $P_k$ implies $P_{k+1}$, $P_n$ is true for $n = 2$, $n = 3$, and so on. By the principle of mathematical induction, $5^n - 1$ is divisible by 4 for all positive integers $n$.

## 10-5 The Binomial Theorem (pp. 628–635)

Use Pascal's triangle to expand each binomial.

**36.** $(4x + 6)^5$

**37.** $(m - 5n)^6$

Find the coefficient of the indicated term in each expansion.

**38.** $(6x - 3y)^{10}$, $x^4y^6$ term

**39.** $(2y + 3)^{13}$, $8^{th}$ term

Use the Binomial Theorem to expand each binomial.

**40.** $(2p^2 - 7)^4$

**41.** $(4m + 3n)^7$

### Example 5

**Use the Binomial Theorem to expand $(3x + 10)^5$.**

Apply the Binomial Theorem to expand $(a + b)^5$, where $a = 3x$ and $b = 10$.

$$\begin{aligned}(3x + 10)^5 &= {}_5C_0(3x)^5(10)^0 + {}_5C_1(3x)^4(10)^1 + {}_5C_2(3x)^3(10)^2 + \\ &\quad {}_5C_3(3x)^2(10)^3 + {}_5C_4(3x)^1(10)^4 + {}_5C_5(3x)^0(10)^5 \\ &= 1(243x^5)(1) + 5(81x^4)(10) + 10(27x^3)(100) + \\ &\quad 10(9x^2)(1000) + 5(3x)(10{,}000) + 1(1)(100{,}000) \\ &= 243x^5 + 4050x^4 + 27{,}000x^3 + 90{,}000x^2 + \\ &\quad 150{,}000x + 100{,}000\end{aligned}$$

## 10-6 Functions as Infinite Series (pp. 636–644)

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the 6th partial sum of its power series.

**42.** $g(x) = \frac{1}{1-5x}$

**43.** $g(x) = \frac{3}{1-2x}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

**44.** $e^{\frac{1}{4}}$

**45.** $e^{-1.5}$

Find the value of each natural logarithm in the complex number system.

**46.** $\ln(-4)$

**47.** $\ln(-7.15)$

**Example 6**

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{4}{1-x}$. Indicate the interval on which the series converges.

A geometric series converges to $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Replace $x$ with $\frac{x+3}{4}$ since $g(x)$ is a transformation of $f(x)$ and: $g(x) = f\left(\frac{x+3}{4}\right)$. The result is

$$f\left(\frac{x+3}{4}\right) = \sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n \text{ for } \left|\frac{x+3}{4}\right| < 1.$$

Therefore, $g(x) = \frac{4}{1-x}$ can be represented by $\sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n$. This series converges for $\left|\frac{x+3}{4}\right| < 1$, which is equivalent to $-1 < \frac{x+3}{4} < 1$ or $-7 < x < 1$.

# Applications and Problem Solving

**48. RETAIL** A chain of retail coffee stores has a business plan that includes opening 6 new stores nationwide annually. If there were 480 stores on January 1, 2012, how many stores will there be at the end of the year in 2018? (Lesson 10-1)

**49. DANCE** Mara's dance team is performing at a recital. In one formation, there are three dancers in the front row and two additional dancers in each row behind the first row. (Lesson 10-2)

**a.** How many dancers are in the fourth row?

**b.** If there are eight rows, how many members does the dance team have?

**50. AGRICULTURE** The population of a herd of cows increases as shown over the course of four years. (Lesson 10-3)

| Year | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Population | 47 | 51 | 56 | 61 |

**a.** Write an explicit formula for finding the population of the herd after $n$ years. Assume that the sequence shown above is geometric.

**b.** What will the population of the herd be after 7 years?

**c.** About how many years will it take the population of the herd to exceed 85?

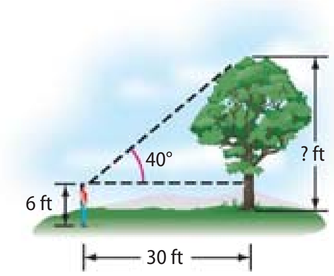
**51. NUMBER THEORY** Consider the statement $0.\overline{9} = 1$. (Lesson 10-4)

**a.** Prove that $0.9 + 0.09 + 0.009 + \cdots + \frac{9}{10^n} = \frac{10^n - 1}{10^n}$ for any positive integer $n$.

**b.** Use your understanding of limits and the statement you proved in part **a** to explain why $0.\overline{9} = 1$ is true.

**52. BASKETBALL** Julie usually makes 4 out of every 6 free throws that she attempts. What is the probability that Julie will make 5 out of 6 of the next free throws that she attempts? (Lesson 10-5)

**53. HEIGHT** Lina is estimating the height of a tree. She stands 30 feet from the base and estimates that her angle of sight to the top of the tree is 40°. If she uses the fifth partial sum of the trigonometric series for cosine and sine approximated to three decimal places to calculate the height of the tree, what is Lina's estimate? (Lesson 10-6)



# CHAPTER 10 Practice Test

Find the specified term of each sequence.

**1.** ninth term, $a_n = \dfrac{n^3}{n+3}$

**2.** sixth term, $a_1 = 156$, $a_n = \dfrac{a_{n-1} - 4}{2}$

Find the indicated sum for each series.

**3.** fifth partial sum of $a_n = 3^n + 4$

**4.** $S_8$ of $-2, 3, 8, 13, \ldots$

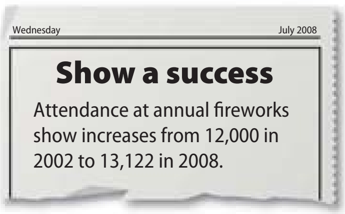
Find the indicated arithmetic means for each set of nonconsecutive terms.

**5.** 3 means; $-10$ and $-2$

**6.** 4 means; $-4$ and 56

**7.** **EXERCISE** While training for a 5-mile fun run, Kirsten ran 1 mile each workout during the first week of training and added 0.5 mile per week to each run so that she ran 5 miles each workout the week before the run. If Kirsten had training three times per week, how many miles did she run during her training?

**8.** **FIREWORKS** If the year to year increase in attendance at the fireworks show is constant, what was the attendance each year from 2003 to 2007?


Wednesday July 2008

Show a success

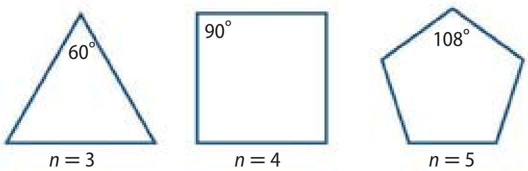
Attendance at annual fireworks show increases from 12,000 in 2002 to 13,122 in 2008.

If possible, find the sum of each infinite geometric series.

**9.** $\frac{4}{10}, \frac{4}{5}, \frac{8}{5}, \ldots$

**10.** $\sum_{n=3}^{\infty} -2(0.6)^{n-1}$

**11.** **GEOMETRY** The measure of each interior angle $a$ of a regular polygon with $n$ sides is $a_n = \dfrac{180(n-2)}{n}$, where $n \geq 3$.




**a.** Find the measure of an exterior angle of a regular triangle, square, pentagon, and hexagon.

**b.** Write a general formula for the measure of an exterior angle of a regular polygon with $n$ sides.

**c.** Determine whether the sequence is convergent or divergent. Does this make sense in the context of the situation? Explain your reasoning.

Prove each inequality for the indicated values of $n$.

**12.** $n! > 5^n$, $n \geq 12$

**13.** $4n < 7^n$, $n \geq 1$

**14.** $3^n > n + 8$, $n \geq 3$

**15.** $3n < \left(\frac{5}{2}\right)^n$, $n \geq 2$

Use Pascal's triangle to expand each binomial.

**16.** $(2x + 3y)^4$

**17.** $(x - 6)^7$

**18.** **MULTIPLE CHOICE** Ellis' basketball scoring statistics are shown below. Based on his statistics, what is the probability that he will score a 2- or 3-point field goal on 3 of his next 7 shots?

| Shots on Goal | 2-point Shots Scored | 3-point Shots Scored |
|---|---|---|
| 20 | 11 | 3 |

**A** 0.10 %

**B** 0.24 %

**C** 9.72 %

**D** 23.88 %

Use the Binomial Theorem to expand and simplify each expression.

**19.** $(x - 4y)^4$

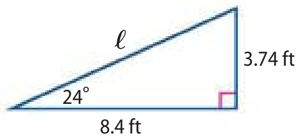
**20.** $(3a + b^3)^5$

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the 6th partial sum of its power series.

**21.** $g(x) = \dfrac{3}{1 - x}$

**22.** $g(x) = \dfrac{2}{1 - 4x}$

**23.** **SKATEBOARDING** A skateboarding ramp has an incline of 24°. Use the 5th partial sum of the trigonometric series for cosine or sine approximated to three decimal places to find the length of the ramp.




Write each complex number in exponential form.

**24.** $-\sqrt{3} - i$

**25.** $1 - \sqrt{3}i$

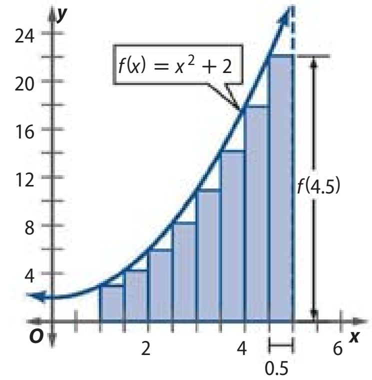
CHAPTER 10

# Connect to AP Calculus
# Riemann Sum

## Objective

- Develop notation for approximating the area of a region bound by a curve and the x-axis.

In Chapter 2, you learned to approximate the area between a curve and the *x*-axis. You divided the area into rectangles, found the area of each individual rectangle, and then calculated the sum of the areas. In calculus, this process is assigned special notation and is studied further in an effort to calculate exact areas. We will analyze the components of this process to better understand the notation.




The area *A* of the region shown above can be approximated as follows.

$A = 0.5 \cdot f(1) + 0.5 \cdot f(1.5) + 0.5 \cdot f(2.0) + 0.5 \cdot f(2.5) + 0.5 \cdot f(3.0) + 0.5 \cdot f(3.5) + 0.5 \cdot f(4.0) + 0.5 \cdot f(4.5)$

Notice that we can factor out the width.

$$A = \underbrace{0.5}_{\text{width}} \cdot \underbrace{[f(1) + f(1.5) + f(2.0) + f(2.5) + f(3.0) + f(3.5) + f(4.0) + f(4.5)]}_{\text{sum of the heights}}$$

The approximation is equal to the product of the width of the rectangles and the sum of their heights. We will examine both of these components separately.

The first component used to approximate the area of a region is the width of the rectangles. The width of the rectangles, denoted $\Delta x$, is the difference between the left endpoint and the right endpoint of a rectangle, such as 2.5 - 2 or 0.5. Generally, we are not given any of the *x*-coordinates of our rectangles. Instead, we get the *lower bound a* and the *upper bound b* of the interval [*a*, *b*] and the number of rectangles *n*.

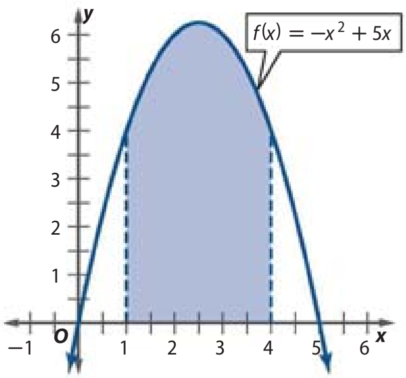
### Activity 1 Find $\Delta x$

**Find $\Delta x$ if we want to approximate the area between the graph of $f(x) = -x^2 + 5x$ and the $x$-axis on the interval [1, 4] using 6, 12, and 24 rectangles.**

**Step 1** Find the total length of the interval by calculating $b - a$.

**Step 2** Divide the answer from Step 1 by 6.

**Step 3** Repeat Step 2 for 12 and 24 rectangles.




### Analyze the Results

1. As the number of rectangles increases, what is happening to $\Delta x$? How would this affect your approximation for the area?
2. Find $\Delta x$ if we want to approximate the area between a curve and the $x$-axis on the interval [$a$, $b$] using $n$ rectangles.
3. As $n$ approaches $\infty$, what is happening to $\Delta x$?

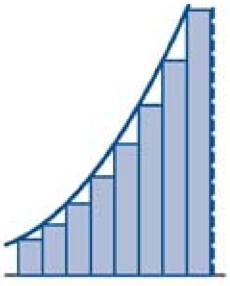


**Figure 10.CAP.1**

The second component needed to approximate the area of a region is the sum of the heights of the rectangles. The sum of the heights resembles the sum of a series. For the example presented in Activity 1, this sum is

$$f(1) + f(1.5) + f(2.0) + f(2.5) + f(3.0) + f(3.5) + f(4.0) + f(4.5).$$

Since there were 8 rectangles, $f(x)$ is evaluated for 8 values of $x$. We can write this series as

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8),$$

where $x_1, x_2, \dots, x_8$ are the $x$-coordinates used to find the heights of the rectangles, as shown in the figure. We can represent this series using sigma notation as $\sum_{i=1}^{8} f(x_i)$. For example, the sum of the heights for $f(x)$ can be written in *expanded form* as

$$\sum_{i=1}^{8} f(x_i) = \underset{x_1}{f(1)} + \underset{x_2}{f(1.5)} + \underset{x_3}{f(2.0)} + \underset{x_4}{f(2.5)} + \underset{x_5}{f(3.0)} + \underset{x_6}{f(3.5)} + \underset{x_7}{f(4.0)} + \underset{x_8}{f(4.5)}.$$

In general, the sum of the heights for $n$ rectangles can be described as $\sum_{i=1}^{n} f(x_i)$.

You now have the two components for approximating the area of a region using $n$ rectangles.

We can multiply our width by our expression for the sum of the heights to develop the notation $\sum_{i=1}^{n} f(x_i) \cdot \Delta x$. This expression is called a *Riemann sum*.

## Activity 2 Approximate Area Under a Curve

**Approximate the area between the graph of $f(x) = -x^2 + 5x$ and the $x$-axis on the interval [1, 4] using 6 rectangles. Let the left endpoint of each rectangle represent the height.**

**Step 1** Let $a = 1$, $b = 4$, and $n = 6$. Calculate $\Delta x$.

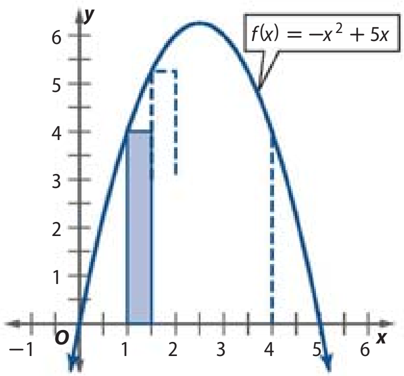
**Step 2** Write the approximation in sigma notation. Substitute the value found in Step 1 for $\Delta x$ and let $n = 6$.

$$\sum_{i=1}^{6} f(x_i) \cdot 0.5$$

**Step 3** Write the expression in expanded form.
$0.5\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$

**Step 4** Find each value for $x$. $x_1$ will start at 1. Each successive value for $x$ can be found by adding $\Delta x$ to each previous value. For example, $x_2 = 1 + 0.5$, $x_3 = x_2 + 0.5$, and so on.

**Step 5** Calculate the area of the rectangles.




**StudyTip**

**Notation** When having to evaluate the sum, it may be easier to represent the approximation for the area of a region as $\Delta x \sum_{i=1}^{n} f(x_i)$.

### Analyze the Results

4. If $n$ increased to 12, how would it change the expression in expanded form? What would happen to the $x$-values found in Step 4?
5. As $n$ approaches $\infty$, what happens to the calculation?

## Model and Apply

**Given $n$ and an interval $[a, b]$, find $\Delta x$. Then write the approximation for finding the area between the graph of $f(x) = -x^2 + 10x$ and the $x$-axis in sigma notation. Calculate the area. Let the left endpoint of each rectangle represent the height.**

6. $n = 4$; [1, 2]
7. $n = 10$; [6, 10]
8. $n = 24$; [3, 9]

CHAPTER 11

# Inferential Statistics

## Then

In Chapter 0, you found measures of center and spread and organized statistical data.

## Now

In Chapter 11, you will:

- Use the shape of a distribution to select appropriate descriptive statistics.
- Construct and use probability distributions.
- Use the Central Limit Theorem.
- Find and use confidence intervals, and perform hypothesis testing.
- Analyze and predict using bivariate data.

## Why?

**ENVIRONMENTAL ENGINEERING** Statistics are extremely important in engineering. In environmental engineering, hypothesis testing can be used to determine if a change in an emission level for a chemical has a significant impact on overall pollution. Also, confidence intervals can be used to help suggest restrictions on by-product wastes in ground water.

**PREREAD** Scan the study guide and review and use it to make two or three predictions about what you will learn in Chapter 11.



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