CHAPTER 9

Polar Coordinates and Complex Numbers

Roy Hsu/Photographer's Choice/Getty Images

Then

In **Chapter 7**, you identified and graphed rectangular equations of conic sections.

Now

In **Chapter 9**, you will:

- Graph polar coordinates and equations.
- Convert between polar and rectangular coordinates and equations.
- Identify polar equations of conic sections.
- Convert complex numbers between polar and rectangular form.

Why?

CONCERTS Polar equations can be used to model sound patterns to help determine stage arrangement, speaker and microphone placement, and volume and recording levels. Polar equations can also be used with lighting and camera angles when concerts are filmed.

PREREAD Use the Lesson Openers in Chapter 9 to make two or three predictions about what you will learn in this chapter.

Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

Graph each function using a graphing calculator. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm your answer algebraically. If odd or even, describe the symmetry of the graph of the function. (Lesson 1-2)

1. $f(x) = x^2 + 10$
2. $f(x) = -2x^3 + 5x$
3. $g(x) = \sqrt{x + 9}$
4. $h(x) = \sqrt{x^2 - 3}$
5. $g(x) = 3x^5 - 7x$
6. $h(x) = \sqrt{x^2} - 5$
7. **BALLOON** The distance in meters between a balloon and a person can be represented by $d(t) = \sqrt{t^2 + 3000}$, where t represents time in seconds. Analyze the graph to determine whether the function is *even*, *odd*, or *neither*. (Lesson 1-2)

Approximate to the nearest hundredth the relative or absolute extrema of each function. State the x-values where they occur. (Lesson 1-4)

8. $f(x) = 4x^2 - 20x + 24$
9. $g(x) = -2x^2 + 9x - 1$
10. $f(x) = -x^3 + 3x - 2$
11. $f(x) = x^3 + x^2 - 5x$
12. **ROCKET** A rocket is fired into the air. The function $h(t) = -16t^2 + 35t + 15$ represents the height h of the rocket in feet after t seconds. Find the extrema of this function. (Lesson 1-4)

Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle. (Lesson 4-2)

13. 165°
14. 205°
15. −10°
16. $\frac{\pi}{6}$
17. $\frac{4\pi}{3}$
18. $-\frac{\pi}{4}$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

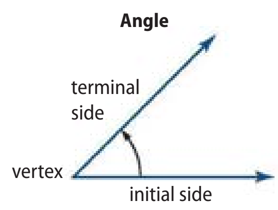
NewVocabulary

English		Español
polar coordinate system	p. 534	sistema de coordenadas polares
pole	p. 534	polo
polar axis	p. 534	eje polares
polar coordinates	p. 534	coordenadas polares
polar equation	p. 536	ecuación polar
polar graph	p. 536	gráfico polar
limaçon	p. 543	limaçon
cardioid	p. 544	cardioide
rose	p. 545	rosa
lemniscate	p. 546	lemniscate
spiral of Archimedes	p. 546	espiral de Arquímedes
complex plane	p. 569	plano complejo
real axis	p. 569	eje real
imaginary axis	p. 569	eje imaginario
Argand plane	p. 569	avión de Argand
absolute value of a complex number	p. 569	valor absoluto de un número complejo
polar form	p. 570	forma polar
trigonometric form	p. 570	forma trigonométrica
modulus	p. 570	módulo
argument	p. 570	argumento

ReviewVocabulary

initial side of an angle p. 231 lado inicial de un ángulo the starting position of the ray

terminal side of an angle p. 231 lado terminal de un ángulo the ray's position after rotation



measure of an angle p. 231 medida de un ángulo the amount and direction of rotation necessary to move from the initial side to the terminal side of the angle

LESSON 9-1

Polar Coordinates

Then

- You drew positive and negative angles given in degrees and radians in standard position. (Lesson 4-2)

Now

1. Graph points with polar coordinates.
2. Graph simple polar equations.

Why?

- To provide safe routes and travel, air traffic controllers use advanced radar systems to direct the flow of airplane traffic. This ensures that airplanes keep a sufficient distance from other aircraft and landmarks. The radar uses angle measure and directional distance to plot the positions of aircraft. Controllers can then relay this information to the pilots.

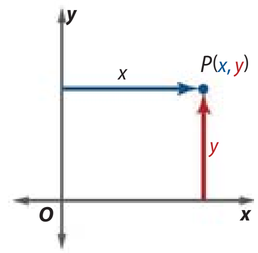
NewVocabulary
polar coordinate system
pole
polar axis
polar coordinates
polar equation
polar graph

1 Graph Polar Coordinates

To this point, you have graphed equations in a rectangular coordinate system. When air traffic controllers record the locations of airplanes using distances and angles, they are using a **polar coordinate system** or polar plane.

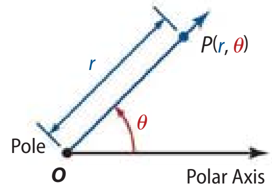
In a rectangular coordinate system, the x- and y-axes are horizontal and vertical lines, respectively, and their point of intersection O is called the origin. The location of a point P is identified by rectangular coordinates of the form (x, y), where x and y are the horizontal and vertical *directed distances*, respectively, to the point. For example, the point $(3, -4)$ is 3 units to the right of the y-axis and 4 units below the x-axis.

Rectangular Coordinate System



In a polar coordinate system, the origin is a fixed point O called the **pole**. The **polar axis** is an initial ray from the pole, usually horizontal and directed toward the right. The location of a point P in the polar coordinate system can be identified by **polar coordinates** of the form (r, θ), where r is the directed distance from the pole to the point and θ is the *directed angle* from the polar axis to $\overrightarrow{OP}$.

Polar Coordinate System



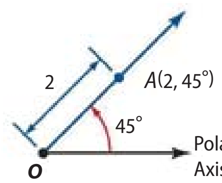
To graph a point given in polar coordinates, remember that a positive value of θ indicates a counterclockwise rotation from the polar axis, while a negative value indicates a clockwise rotation. If r is positive, then P lies on the terminal side of θ. If r is negative, P lies on the ray opposite the terminal side of θ.

Example 1 Graph Polar Coordinates

Graph each point.

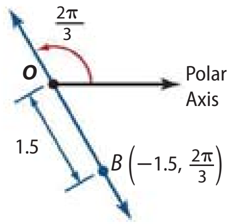
a. $A(2, 45°)$

Because $\theta = 45°$, sketch the terminal side of a 45° angle with the polar axis as its initial side. Because $r = 2$, plot a point 2 units from the pole along the terminal side of this angle.



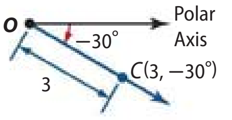
b. $B\left(-1.5, \frac{2\pi}{3}\right)$

Because $\theta = \frac{2\pi}{3}$, sketch the terminal side of a $\frac{2\pi}{3}$ angle with the polar axis as its initial side. Because r is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 1.5 units from the pole along this extended ray.



c. $C(3, -30°)$

Because $\theta = -30°$, sketch the terminal side of a $-30°$ angle with the polar axis as its initial side. Because $r = 3$, plot a point 3 units from the pole along the terminal side of this angle.



GuidedPractice

Graph each point.

1A. $D\left(-1, \frac{\pi}{2}\right)$ **1B.** $E(2.5, 240°)$ **1C.** $F\left(4, -\frac{5\pi}{6}\right)$

Just as rectangular coordinates are graphed on a rectangular grid, polar coordinates are graphed on a circular or *polar* grid representing the polar plane.

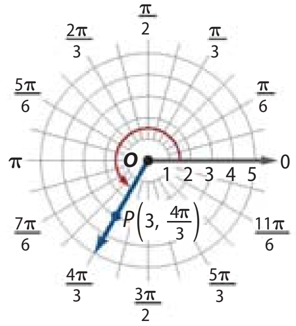
Example 2 Graph Points on a Polar Grid

Graph each point on a polar grid.

a. $P\left(3, \frac{4\pi}{3}\right)$

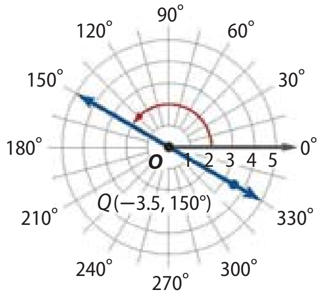
Because $\theta = \frac{4\pi}{3}$, sketch the terminal side of a $\frac{4\pi}{3}$ angle with the polar axis as its initial side.

Because $r = 3$, plot a point 3 units from the pole along the terminal side of the angle.



b. $Q(-3.5, 150°)$

Because $\theta = 150°$, sketch the terminal side of a 150° angle with the polar axis as its initial side. Because r is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 3.5 units from the pole along this extended ray.



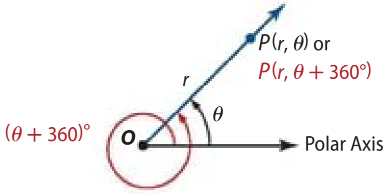
GuidedPractice

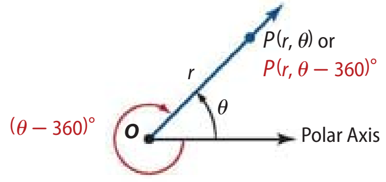
2A. $R\left(1.5, -\frac{7\pi}{6}\right)$ **2B.** $S(-2, -135°)$

StudyTip

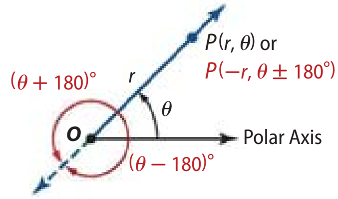
Pole The pole can be represented by $(0, \theta)$, where θ is any angle.

In a rectangular coordinate system, each point has a unique set of coordinates. This is *not* true in a polar coordinate system. In Lesson 4-2, you learned that a given angle has infinitely many coterminal angles. As a result, if a point has polar coordinates (r, θ), then it also has polar coordinates $(r, \theta \pm 360°)$ or $(r, \theta \pm 2\pi)$ as shown.





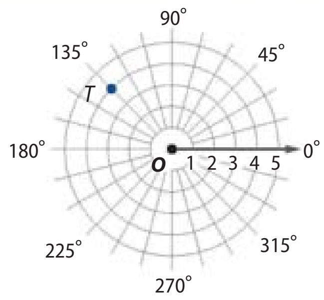
Additionally, because r is a directed distance, (r, θ) and $(-r, \theta \pm 180°)$ or $(-r, \theta \pm \pi)$ represent the same point as shown.



In general, if n is any integer, the point with polar coordinates (r, θ) can also be represented by polar coordinates of the form $(r, \theta + 360n°)$ or $(-r, \theta + (2n + 1)180°)$. Likewise, if θ is given in radians and n is any integer, the other representations of (r, θ) are of the form $(r, \theta + 2n\pi)$ or $(-r, \theta + (2n + 1)\pi)$.

Example 3 Multiple Representations of Polar Coordinates

Find four different pairs of polar coordinates that name point T if $-360° \le \theta \le 360°$.



One pair of polar coordinates that name point T is $(4, 135°)$. The other three representations are as follows.

$(4, 135°) = (4, 135° - 360°)$ Subtract 360° from θ.
$= (4, -225°)$

$(4, 135°) = (-4, 135° + 180°)$ Replace r with $-r$ and
$= (-4, 315°)$ add 180° to θ.

$(4, 135°) = (-4, 135° - 180°)$ Replace r with $-r$ and
$= (-4, -45°)$ subtract 180° from θ.

GuidedPractice

Find three additional pairs of polar coordinates that name the given point if $-360° \le \theta \le 360°$ or $-2\pi \le \theta \le \pi$.

3A. $(5, 240°)$ **3B.** $\left(2, \frac{\pi}{6}\right)$

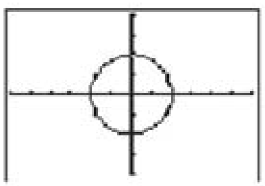
2 Graphs of Polar Equations

An equation expressed in terms of polar coordinates is called a **polar equation**. For example, $r = 2 \sin \theta$ is a polar equation. A **polar graph** is the set of all points with coordinates (r, θ) that satisfy a given polar equation.

You already know how to graph equations in the Cartesian, or *rectangular*, coordinate system. Graphs of equations involving constants like $x = 2$ and $y = -3$ are considered basic in the Cartesian coordinate system. Similarly, the graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, are considered basic in the polar coordinate system.

TechnologyTip

Graphing Polar Equations To graph the polar equation $r = 2$ on a graphing calculator, first press MODE and change the graphing setting from FUNC to POL. When you press Y=, notice that the dependent variable has changed from Y to r and the independent variable from x to θ. Graph $r = 2$.



$[0, 2\pi]$ scl: $\frac{\pi}{16}$ by $[-6, 6]$ scl: 1 by $[-4, 4]$ scl: 1

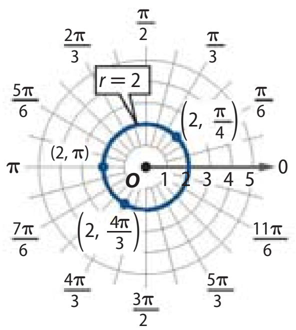
Example 4 Graph Polar Equations

Graph each polar equation.

a. $r = 2$

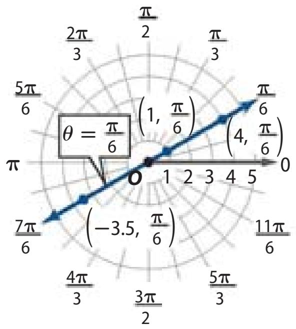
The solutions of $r = 2$ are ordered pairs of the form $(2, \theta)$, where θ is any real number.

The graph consists of all points that are 2 units from the pole, so the graph is a circle centered at the origin with radius 2.



b. $\theta = \frac{\pi}{6}$

The solutions of $\theta = \frac{\pi}{6}$ are ordered pairs of the form $\left(r, \frac{\pi}{6}\right)$, where r is any real number. The graph consists of all points on the line that makes an angle of $\frac{\pi}{6}$ with the positive polar axis.



GuidedPractice

Graph each polar equation.

4A. $r = 3$

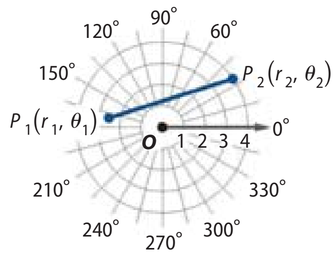
4B. $\theta = \frac{2\pi}{3}$

The distance between two points in the polar plane can be found using the following formula.

KeyConcept Polar Distance Formula

If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are two points in the polar plane, then the distance P_1P_2 is given by

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$



You will prove this formula in Exercise 63.

WatchOut!

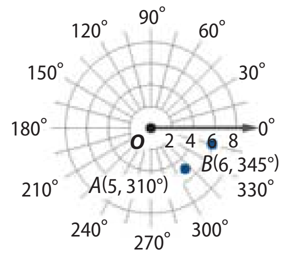
Mode When using the Polar Distance Formula, if θ is given in degrees, make sure your graphing calculator is set in degree mode.

Real-World Example 5 Find the Distance Between Polar Coordinates

AIR TRAFFIC **An air traffic controller is tracking two airplanes that are flying at the same altitude. The coordinates of the planes are $A(5, 310°)$ and $B(6, 345°)$, where the directed distance is measured in miles.**

a. Sketch a graph of this situation.

Airplane A is located 5 miles from the pole on the terminal side of the angle 310°, and airplane B is located 6 miles from the pole on the terminal side of the angle 345°, as shown.



b. If regulations prohibit airplanes from passing within three miles of each other, are these airplanes in violation? Explain.

Use the Polar Distance Formula.

$$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$ Polar Distance Formula

$$= \sqrt{5^2 + 6^2 - 2(5)(6)\cos(345° - 310°)} \text{ or about } 3.44$$ $(r_2, \theta_2) = (6, 345°)$ and $(r_1, \theta_1) = (5, 310°)$

The planes are about 3.44 miles apart, so they are not in violation of this regulation.

GuidedPractice

5. **BOATS** A naval radar is tracking two aircraft carriers. The coordinates of the two carriers are (8, 150°) and (3, 65°), with r measured in miles.

A. Sketch a graph of this situation.

B. What is the distance between the two aircraft carriers?

Keystone/Hulton Archive/Getty Images

Real-WorldLink

Germany developed a radar device in 1936 that could detect planes in an 80-mile radius. The following year, Germany was credited with supplying a battleship, the *Graf Spee*, with the first radar system.

Source: A History of the World Semiconductor Industry

Graph each point on a polar grid. (Examples 1 and 2)

1. $R(1, 120°)$

2. $T(-2.5, 330°)$

3. $F\left(-2, \frac{2\pi}{3}\right)$

4. $A\left(3, \frac{\pi}{6}\right)$

5. $Q\left(4, -\frac{5\pi}{6}\right)$

6. $B(5, -60°)$

7. $D\left(-1, -\frac{5\pi}{3}\right)$

8. $G\left(3.5, -\frac{11\pi}{6}\right)$

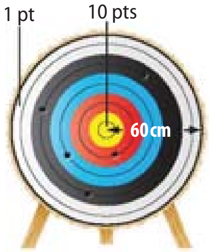
9. $C(-4, \pi)$

10. $M(0.5, 270°)$

11. $P(4.5, -210°)$

12. $W(-1.5, 150°)$

13. ARCHERY The target in competitive target archery consists of 10 evenly spaced concentric circles with score values from 1 to 10 points from the outer circle to the center. Suppose an archer using a target with a 60-centimeter radius shoots arrows at (57, 45°), (41, 315°), and (15, 240°). (Examples 1 and 2)



a. Plot the points where the archer's arrows hit the target on a polar grid.

b. How many points did the archer earn?

Find three different pairs of polar coordinates that name the given point if $-360° \leq \theta \leq 360°$ or $-2\pi \leq \theta \leq 2\pi$. (Example 3)

14. $(1, 150°)$

15. $(-2, 300°)$

16. $\left(4, -\frac{7\pi}{6}\right)$

17. $\left(-3, \frac{2\pi}{3}\right)$

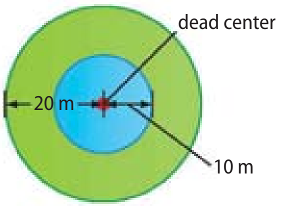
18. $\left(5, \frac{11\pi}{6}\right)$

19. $\left(-5, -\frac{4\pi}{3}\right)$

20. $(2, -30°)$

21. $(-1, -240°)$

22. SKYDIVING In competitive accuracy landing, skydivers attempt to land as near as possible to "dead center," the center of a target marked by a disk 2 meters in diameter. (Example 4)



a. Write polar equations representing the three target boundaries.

b. Graph the equations on a polar grid.

Graph each polar equation. (Example 4)

23. $r = 4$

24. $\theta = 225°$

25. $r = 1.5$

26. $\theta = -\frac{7\pi}{6}$

27. $\theta = -15°$

28. $r = -3.5$

29. DARTBOARD A certain dartboard has a radius of 225 millimeters. The bull's-eye has two sections. The 50-point section has a radius of 6.3 millimeters. The 25-point section surrounds the 50-point section for an additional 9.7 millimeters. (Example 4)

a. Write and graph polar equations representing the boundaries of the dartboard and these sections.

b. What percentage of the dartboard's area does the bull's-eye comprise?

Find the distance between each pair of points. (Example 5)

30. $(2, 30°), (5, 120°)$

31. $\left(3, \frac{\pi}{2}\right), \left(8, \frac{4\pi}{3}\right)$

32. $(6, 45°), (-3, 300°)$

33. $\left(7, -\frac{\pi}{3}\right), \left(1, \frac{2\pi}{3}\right)$

34. $\left(-5, \frac{7\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$

35. $(4, -315°), (1, 60°)$

36. $(-2, -30°), (8, 210°)$

37. $\left(-3, \frac{11\pi}{6}\right), \left(-2, \frac{5\pi}{6}\right)$

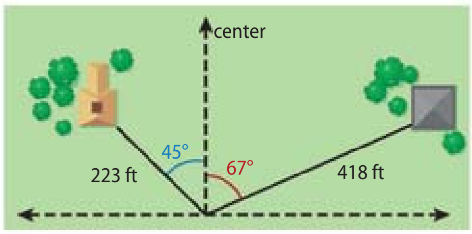
38. $\left(1, -\frac{\pi}{4}\right), \left(-5, \frac{7\pi}{6}\right)$

39. $(7, -90°), (-4, -330°)$

40. $\left(8, -\frac{2\pi}{3}\right), \left(4, -\frac{3\pi}{4}\right)$

41. $(-5, 135°), (-1, 240°)$

42. SURVEYING A surveyor mapping out the land where a new housing development will be built identifies a landmark 223 feet away and 45° left of center. A second landmark is 418 feet away and 67° right of center. Determine the distance between the two landmarks. (Example 5)



43. SURVEILLANCE A mounted surveillance camera oscillates and views part of a circular region determined by $-60° \leq \theta \leq 150°$ and $0 \leq r \leq 40$, where r is in meters.

a. Sketch a graph of the region that the security camera can view on a polar grid.

b. Find the area of the region.

Find a different pair of polar coordinates for each point such that $0 \leq \theta \leq 180°$ or $0 \leq \theta \leq \pi$.

44. $(5, 960°)$

45. $\left(-2.5, \frac{5\pi}{2}\right)$

46. $\left(4, \frac{11\pi}{4}\right)$

47. $(1.25, -920°)$

48. $\left(-1, -\frac{21\pi}{8}\right)$

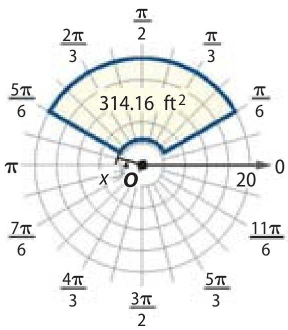
49. $(-6, -1460°)$

50. **AMPHITHEATER** Suppose a singer is performing at an amphitheater. We can model this situation with polar coordinates by assuming that the singer is standing at the pole and is facing the direction of the polar axis. The seats can then be described as occupying the area defined by $-45° \leq \theta \leq 45°$ and $30 \leq r \leq 240$, where r is measured in feet.

a. Sketch a graph of this region on a polar grid.

b. If each person needs 5 square feet of space, how many seats can fit in the amphitheater?

51. **SECURITY** A security light mounted above a house illuminates part of a circular region defined by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ and $x \leq r \leq 20$, where r is measured in feet. If the total area of the region is approximately 314.16 square feet, find x.



Find a value for the missing coordinate that satisfies the following condition.

52. $P_1 = (3, 35°)$; $P_2 = (r, 75°)$; $P_1P_2 = 4.174$

53. $P_1 = (5, 125°)$; $P_2 = (2, \theta)$; $P_1P_2 = 4$; $0 \leq \theta \leq 180°$

54. $P_1 = (3, \theta)$; $P_2 = \left(4, \frac{7\pi}{9}\right)$; $P_1P_2 = 5$; $0 \leq \theta \leq \pi$

55. $P_1 = (r, 120°)$; $P_2 = (4, 160°)$; $P_1P_2 = 3.297$

56. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between polar coordinates and rectangular coordinates.

a. **GRAPHICAL** Plot points $A\left(2, \frac{\pi}{3}\right)$ and $B\left(4, \frac{5\pi}{6}\right)$ on a polar grid. Sketch a rectangular coordinate system on top of the polar grid so that the origins coincide and the x-axis aligns with the polar axis. The y-axis should align with the line $\theta = \frac{\pi}{2}$. Form one right triangle by connecting point A to the origin and perpendicular to the x-axis. Form another right triangle by connecting point B to the origin and perpendicular to the x-axis.

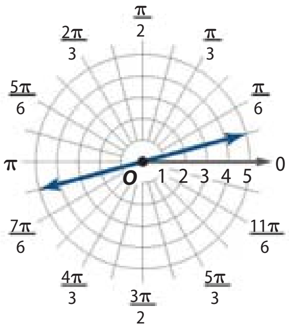
b. **NUMERICAL** Calculate the lengths of the legs of each triangle.

c. **ANALYTICAL** How do the lengths of the legs relate to rectangular coordinates for each point?

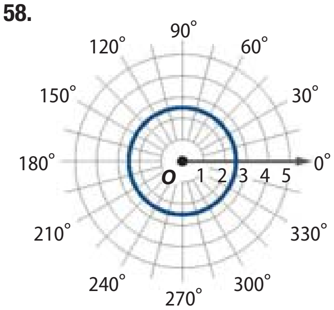
d. **ANALYTICAL** Explain the relationship between the polar coordinates (r, θ) and the rectangular coordinates (x, y).

Write an equation for each polar graph.

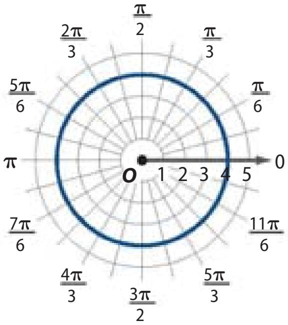
57.



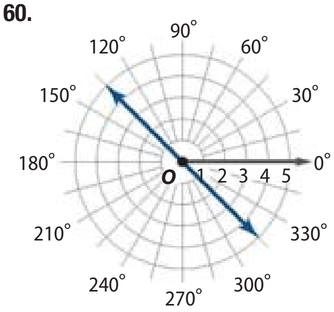
58.



59.



60.



H.O.T. Problems Use Higher-Order Thinking Skills

61. **REASONING** Explain why the order of the points used in the Polar Distance Formula is not important. That is, why can you choose either point to be P_1 and the other to be P_2?

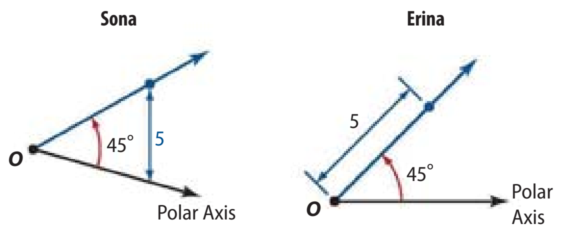
62. **CHALLENGE** Find an ordered pair of polar coordinates to represent the point with rectangular coordinates $(-3, -4)$. Round the angle measure to the nearest degree.

63. **PROOF** Prove that the distance between two points with polar coordinates $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$

64. **REASONING** Describe what happens to the Polar Distance Formula when $\theta_2 - \theta_1 = \frac{\pi}{2}$. Explain this change.

65. **ERROR ANALYSIS** Sona and Erina both graphed the polar coordinates $(5, 45°)$. Is either of them correct? Explain your reasoning.



66. **WRITING IN MATH** Make a conjecture as to why having the polar coordinates for an aircraft is not enough to determine its exact position.

Spiral Review

Find the dot product of u and v. Then determine if u and v are orthogonal. (Lesson 8-5)

67. $\mathbf{u} = \langle 4, 10, 1\rangle, \mathbf{v} = \langle -5, 1, 7\rangle$

68. $\mathbf{u} = \langle -5, 4, 2\rangle, \mathbf{v} = \langle -4, -9, 8\rangle$

69. $\mathbf{u} = \langle -8, -3, 12\rangle, \mathbf{v} = \langle 4, -6, 0\rangle$

Find each of the following for $\mathbf{a} = \langle -4, 3, -2\rangle$, $\mathbf{b} = \langle 2, 5, 1\rangle$, and $\mathbf{c} = \langle 3, -6, 5\rangle$. (Lesson 8-4)

70. $3\mathbf{a} + 2\mathbf{b} + 8\mathbf{c}$

71. $-2\mathbf{a} + 4\mathbf{b} - 5\mathbf{c}$

72. $5\mathbf{a} - 9\mathbf{b} + \mathbf{c}$

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola. (Lesson 7-1)

73. $-14(x - 2) = (y - 7)^2$

74. $(x - 7)^2 = -32(y - 12)$

75. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$

76. **STATE FAIR** If Curtis and Drew each purchased the number of game and ride tickets shown below, what was the price for each type of ticket? (Lesson 6-3)

Person	Ticket Type	Tickets	Total ($)
Curtis	game ride	6 15	93
Drew	game ride	7 12	81

Write the augmented matrix for the system of linear equations. (Lesson 6-1)

77.
$12w + 14x - 10y = 23$
$4w - 5y + 6z = 33$
$11w - 13x + 2z = -19$
$19x - 6y + 7z = -25$

78.
$-6x + 2y + 5z = 18$
$5x - 7y + 3z = -8$
$y - 12z = -22$
$8x - 3y + 2z = 9$

79.
$x + 8y - 3z = 25$
$2x - 5y + 11z = 13$
$-5x + 8z = 26$
$y - 4z = 17$

Solve each equation for all values of x. (Lesson 5-3)

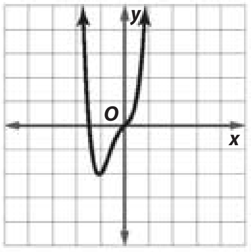
80. $2\cos^2 x + 5\sin x - 5 = 0$

81. $\tan^2 x + 2\tan x + 1 = 0$

82. $\cos^2 x + 3\cos x = -2$

Skills Review for Standardized Tests

83. **SAT/ACT** If the figure shows part of the graph of $f(x)$, then which of the following could be the range of $f(x)$?

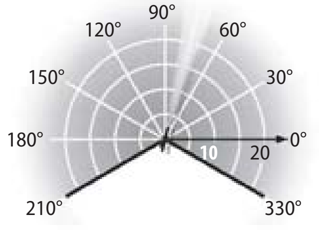


A $\{y \mid y \geq -2\}$
B $\{y \mid y \leq -2\}$
C $\{y \mid -2 < y < 1\}$
D $\{y \mid -2 \leq y \leq 1\}$
E $\{y \mid y > -2\}$

84. **REVIEW** Which of the following is the component form of $\overrightarrow{RS}$ with initial point $R(-5, 3)$ and terminal point $S(2, -7)$?

F $\langle 7, -10\rangle$
G $\langle -3, 10\rangle$
H $\langle -7, 10\rangle$
J $\langle -3, -10\rangle$

85. The lawn sprinkler shown can cover the part of a circular region determined by the polar inequalities $-30° \leq \theta \leq 210°$ and $0 \leq r \leq 20$, where r is measured in feet. What is the approximate area of this region?



A 821 square feet
B 838 square feet
C 852 square feet
D 866 square feet

86. **REVIEW** What type of conic is represented by $25y^2 = 400 + 16x^2$?

F circle
G ellipse
H hyperbola
J parabola

EXPLORE 9-2

Graphing Technology Lab

Investigate Graphs of Polar Equations

Objective

- Use a graphing calculator to explore the shape and symmetry of graphs of polar equations.

StudyTip

Square the Window To view the graphs in this activity without any distortion, square the window by selecting ZSquare under the ZOOM menu.

In Lesson 9-1, you graphed polar coordinates and simple polar equations on the polar coordinate system. Now you will explore the shape and symmetry of more complex graphs of polar equations by using a graphing calculator.

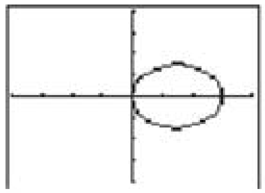
Activity Graph Polar Equations

Graph each equation. Then describe the shape and symmetry of the graph.

a. $r = 3 \cos \theta$

First, change the graph mode to polar and the angle mode to radians. Next, enter $r = 3 \cos \theta$ for r_1 in the Y= list. Use the viewing window shown.

The graph of $r = 3 \cos \theta$ is a circle with a center at (1.5, 0) and radius 1.5 units. The graph is symmetric with respect to the polar axis.

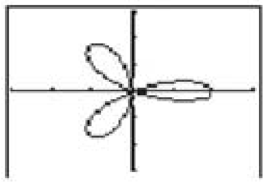


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-4, 4]$ scl: 1 by $[-4, 4]$ scl: 1

b. $r = 2 \cos 3\theta$

Clear the equation from part **a** in the Y= list and insert $r = 2 \cos 3\theta$. Use the window shown.

The graph of $r = 2 \cos 3\theta$ is a classic polar curve called a rose, which will be covered in Lesson 9-2. The graph has 3 petals and is symmetric with respect to the polar axis.

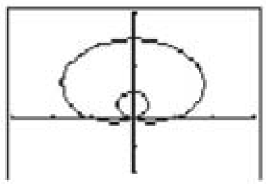


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-3, 3]$ scl: 1

c. $r = 1 + 2 \sin \theta$

Clear the equation from part **b** in the Y= list, and enter $r = 1 + 2 \sin \theta$. Adjust the window to display the entire graph.

The graph of $r = 1 + 2 \sin \theta$ is a classic polar curve called a *limaçon*, which will be covered in Lesson 9-2. The graph has a curve with an inner loop and is symmetric with respect to the line $\theta = \frac{\pi}{2}$.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-2, 4]$ scl: 1

Exercises

Graph each equation. Then describe the shape and symmetry of the graph.

1. $r = -3 \cos \theta$
2. $r = 3 \sin \theta$
3. $r = -3 \sin \theta$
4. $r = 2 \cos 4\theta$
5. $r = 2 \cos 5\theta$
6. $r = 2 \cos 6\theta$
7. $r = 2 + 4 \sin \theta$
8. $r = 1 - 3 \sin \theta$
9. $r = 1 + 2 \sin (-\theta)$

Analyze the Results

10. **ANALYTICAL** Explain how each value affects the graph of the given equation.
    - **a.** the value of n in $r = a \cos n\theta$
    - **b.** the value of $|a|$ in $r = b \pm a \sin n\theta$

11. **MAKE A CONJECTURE** Without graphing $r = 10 \cos 24\theta$, describe the shape and symmetry of the graph. Explain your reasoning.

LESSON 9-2

Graphs of Polar Equations

Then

- You graphed functions in the rectangular coordinate system. (Lesson 1-2)

Now

- 1 Graph polar equations.
- 2 Identify and graph classical curves.

Why?

- To reduce background noise, networks that broadcast sporting events use directional microphones to capture the sounds of the game. Directional microphones have the ability to pick up sound primarily from one direction or region. The sounds that these microphones can detect can be expressed as polar functions.

NewVocabulary
limaçon
cardioid
rose
lemniscate
spiral of Archimedes

1 Graphs of Polar Equations

When you graphed equations on a rectangular coordinate system, you began by using an equation to obtain a set of ordered pairs. You then plotted these coordinates as points and connected them with a smooth curve. In this lesson, you will approach the graphing of polar equations in a similar manner.

Example 1 Graph Polar Equations by Plotting Points

Graph each equation.

a. $r = \cos\theta$

Make a table of values to find the r-values corresponding to various values of θ on the interval $[0, 2\pi]$. Round each r-value to the nearest tenth.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \cos\theta$	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0	0.5	0.9	1

Graph the ordered pairs (r, θ) and connect them with a smooth curve. It appears that the graph shown in Figure 9.2.1 is a circle with center at $(0.5, 0)$ and radius 0.5 unit.

b. $r = \sin\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	0.5	0.9	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0

Graph the ordered pairs and connect them with a smooth curve. It appears that the graph shown in Figure 9.2.2 is a circle with center at $\left(0.5, \frac{\pi}{2}\right)$ and radius 0.5 unit.

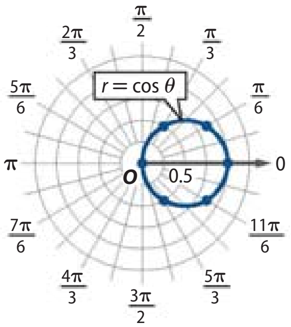


Figure 9.2.1

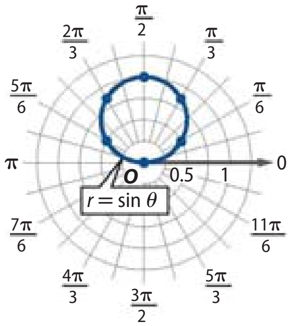


Figure 9.2.2

GuidedPractice

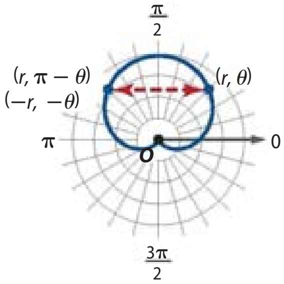
1A. $r = -\sin\theta$ **1B.** $r = 2\cos\theta$ **1C.** $r = \sec\theta$

Notice that as θ increases on $[0, 2\pi]$, each graph above is traced twice. This is because the polar coordinates obtained on $[0, \pi]$ represent the same points as those obtained on $[\pi, 2\pi]$.

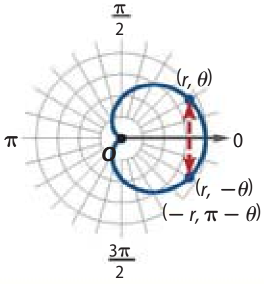
Like knowing whether a graph in the rectangular coordinate system has symmetry with respect to the x-axis, y-axis, or origin, knowing whether the graph of a polar equation is symmetric can help reduce the number of points needed to sketch its graph. Graphs of polar equations can be symmetric with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole, as shown below.

KeyConcept Symmetry of Polar Graphs

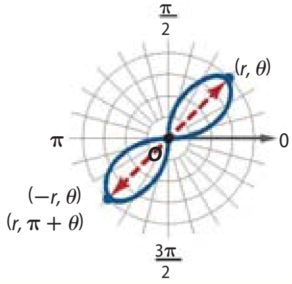
Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$



Symmetry with Respect to Polar Axis



Symmetry with Respect to the Pole



The graphical definitions above provide a way of testing a polar equation for symmetry. For example, if replacing (r, θ) in a polar equation with $(r, -\theta)$ or $(-r, \pi - \theta)$ produces an equivalent equation, then its graph is symmetric with respect to the polar axis. If an equation passes one of the symmetry tests, this is sufficient to guarantee that the equation has that type of symmetry. The converse, however, is *not* true. If a polar equation fails all of these tests, the graph may still have symmetry.

Example 2 Polar Axis Symmetry

Use symmetry to graph $r = 1 - 2 \cos \theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 1 - 2 \cos (-\theta)$. Because cosine is an even function, $\cos (-\theta) = \cos \theta$, so this equation simplifies to $r = 1 - 2 \cos \theta$. Because the replacement produced an equation equivalent to the original equation, the graph of this equation is symmetric with respect to the polar axis.

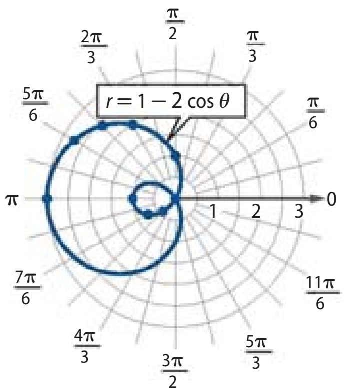
Because of this symmetry, you need only make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 1 - 2 \cos \theta$	−1	−0.7	−0.4	0	1	2	2.4	2.7	3

StudyTip
Graphing Polar Equations It is customary to graph polar functions in radians, rather than in degrees.

Plotting these points and using polar axis symmetry, you obtain the graph shown.

The type of curve is called a **limaçon** (LIM-uh-son). Some limaçons have inner loops like this one. Other limaçons come to a point, have a dimple, or just curve outward.



GuidedPractice

Use symmetry to graph each equation.

2A. $r = 1 - \cos \theta$

2B. $r = 2 + \cos \theta$

In Examples 1 and 2, notice that the graphs of $r = \cos\theta$ and $r = 1 - 2\cos\theta$ are symmetric with respect to the polar axis, while the graph of $r = \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$. These observations can be generalized as follows.

KeyConcept Quick Tests for Symmetry in Polar Graphs

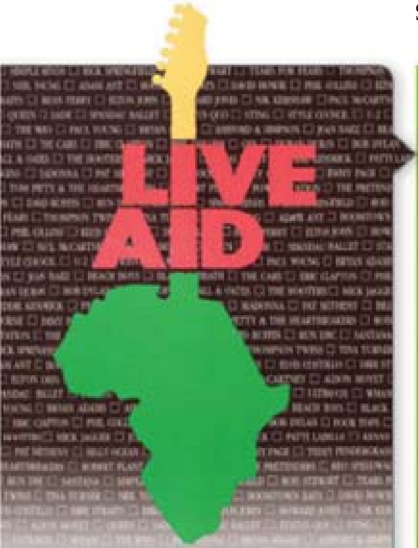
Words The graph of a polar equation is symmetric with respect to

- the polar axis if it is a function of $\cos\theta$, and
- the line $\theta = \frac{\pi}{2}$ if it is a function of $\sin\theta$.

Example The graph of $r = 3 + \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

You will justify these tests for specific cases in Exercises 65–66.

Symmetry can be used to graph polar functions that model real-world situations.



Real-WorldLink

Live Aid was a 1985 rock concert held in an effort to raise \$1 million for Ethiopian aid. Concerts in London, Philadelphia, and other cities were televised and viewed by 1.9 billion people in 150 countries. The event raised \$140 million.

Source: CNN

Real-World Example 3 Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$

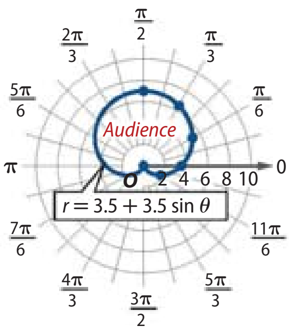
AUDIO TECHNOLOGY During a concert, a directional microphone was placed facing the audience from the center of stage to capture the crowd noise for a live recording. The area of sound the microphone captures can be represented by $r = 3.5 + 3.5\sin\theta$. Suppose the front of the stage faces due north.

a. Graph the polar pattern of the microphone.

Because this polar equation is a function of the sine function, it is symmetric with respect to the line $\theta = \frac{\pi}{2}$. Therefore, make a table and calculate the values of r on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 3.5 + 3.5\sin\theta$	0	0.5	1.0	1.8	3.5	5.25	6.0	6.5	7

Plotting these points and using symmetry with respect to the line $\theta = \frac{\pi}{2}$, you obtain the graph shown. This type of curve is called a **cardioid** (CAR-dee-oid). A cardioid is a special limaçon that has a heart shape.



b. Describe what the polar pattern tells you about the microphone.

The polar pattern indicates that the microphone will pick up sounds up to 7 units away directly in front of the microphone and up to 3.5 units away directly to the left or right of the microphone.

WatchOut!

Graphing over the Period

Usually the period of the trigonometric function used in a polar equation is sufficient to trace the entire graph, but sometimes it is not. The best way to know if you have graphed enough to discern a pattern is to plot more points.

GuidedPractice

3. **VIDEOTAPING** A high school teacher is videotaping presentations performed by her students using a stationary video camera positioned in the back of the room. The area of sound captured by the camera's microphone can be represented by $r = 5 + 2\sin\theta$. Suppose the front of the classroom is due north of the camera.

   A. Graph the polar pattern of the microphone.

   B. Describe what the polar pattern tells you about the microphone.

In Lesson 4-4, you used maximum and minimum points along with zeros to aid in graphing trigonometric functions. On the graph of a polar function, r is at its maximum for a value of θ when the distance between that point (r, θ) and the pole is maximized. To find the maximum point(s) on the graph of a polar equation, find the θ-values for which $|r|$ is maximized. Additionally, if $r = 0$ for some value of θ, you know that the graph intersects the pole.

Example 4 Symmetry, Zeros, and Maximum r-Values

Use symmetry, zeros, and maximum r-values to graph $r = 2 \cos 3\theta$.

Determine the symmetry of the graph.

This function is symmetric with respect to the polar axis, so you can find points on the interval $[0, \pi]$ and then use polar axis symmetry to complete the graph.

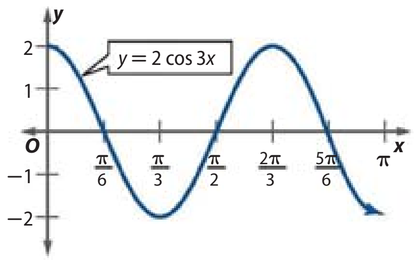
StudyTip

Alternative Method Solving the rectangular function $y = 2 \cos 3x$, we find that the function has extrema when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π. Similarly, the function has zeros when $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

Find the zeros and the maximum r-value.

Sketch the graph of the rectangular function $y = 2 \cos 3x$ on the interval $[0, \pi]$.

From the graph, you can see that $|y| = 2$ when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, and π and $y = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.



Interpreting these results in terms of the polar equation $r = 2 \cos 3\theta$, we can say that $|r|$ has a maximum value of 2 when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π and $r = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

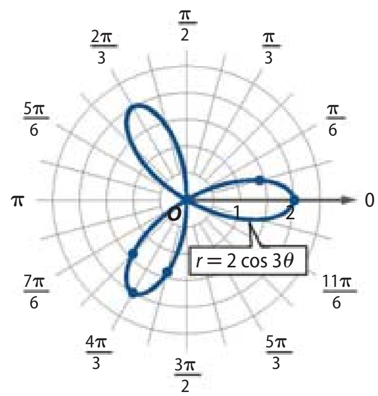
Graph the function.

Use these and a few additional points to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$r = 2 \cos 3\theta$	2	1.4	0	−1	−2	−1.4	0	1.4	2	1.4	0	−1.4	−2

Notice that polar axis symmetry can be used to complete the graph after plotting points on $\left[0, \frac{\pi}{2}\right]$.

This type of curve is called a **rose**. Roses can have three or more equal loops.



GuidedPractice

Use symmetry, zeros, and maximum r-values to graph each function.

4A. $r = 3 \sin 2\theta$

4B. $r = \cos 5\theta$

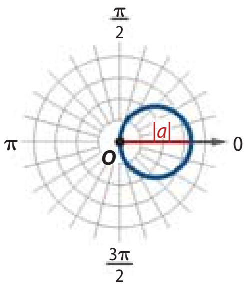
2 Classic Polar Curves

Circles, limaçons, cardioids, and roses are examples of classic curves. The forms and model graphs of these and other classic curves are summarized below.

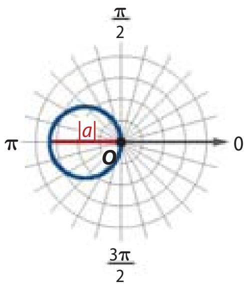
ConceptSummary Special Types of Polar Graphs

Circles

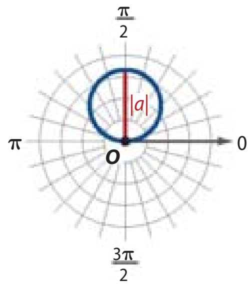
$r = a\cos\theta$ or $r = a\sin\theta$



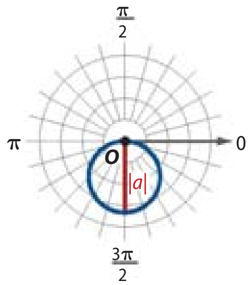
$a > 0$
$r = a\cos\theta$



$a < 0$
$r = a\cos\theta$



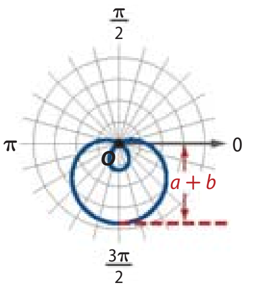
$a > 0$
$r = a\sin\theta$



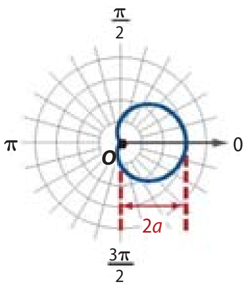
$a < 0$
$r = a\sin\theta$

Limaçons

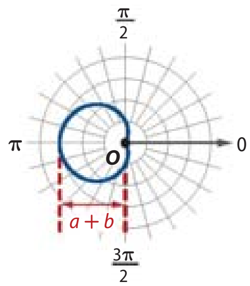
$r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where a and b are both positive



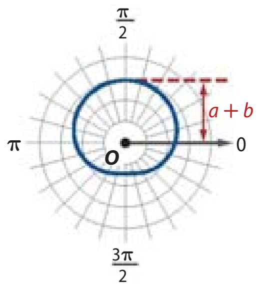
Limaçon with inner loop
$a < b$
$r = a - b\sin\theta$



Cardioid
$a = b$
$r = a + b\cos\theta$



Dimpled limaçon
$b < a < 2b$
$r = a - b\cos\theta$

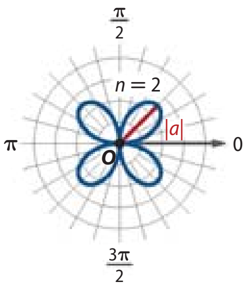


Convex limaçon
$a \geq 2b$
$r = a + b\sin\theta$

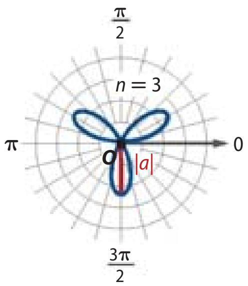
Roses

$r = a\cos n\theta$ or $r = a\sin n\theta$, where $n \geq 2$ is an integer

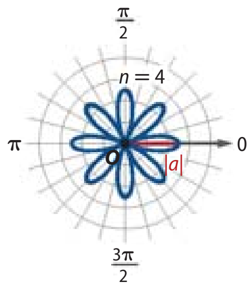
The rose has n petals if n is odd and $2n$ petals if n is even.



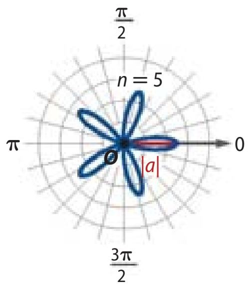
$r = a\sin n\theta$



$r = a\sin n\theta$



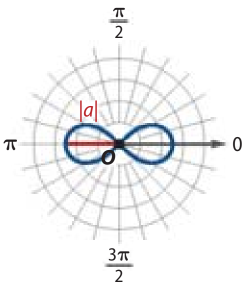
$r = a\cos n\theta$



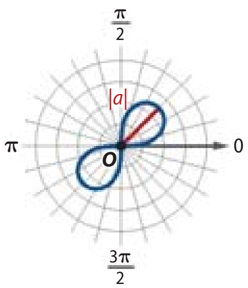
$r = a\cos n\theta$

Lemniscates (LEM-nis-keyts)

$r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$



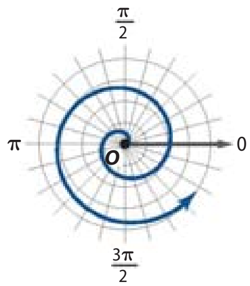
$r^2 = a^2\cos 2\theta$



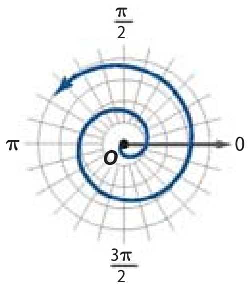
$r^2 = a^2\sin 2\theta$

Spirals of Archimedes (ahr-kuh-MEE-deez)

$r = a\theta + b$



$\theta > 0, r = a\theta + b$



$\theta < 0, r = a\theta + b$

Example 5 Identify and Graph Classic Curves

Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum *r*-values to graph the function.

a. $r^2 = 16 \sin 2\theta$

Type of Curve and Symmetry

The equation is of the form $r^2 = a^2 \sin 2\theta$, so its graph is a lemniscate. Replacing (r, θ) with $(-r, \theta)$ yields $(-r)^2 = 16 \sin 2\theta$ or $r^2 = 16 \sin 2\theta$. Therefore, the function has symmetry with respect to the pole.

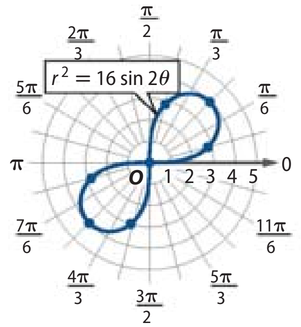
Maximum *r*-Value and Zeros

The equation $r^2 = 16 \sin 2\theta$ is equivalent to $r = \pm 4\sqrt{\sin 2\theta}$, which is undefined when $\sin 2\theta < 0$. Therefore, the domain of the function is restricted to the intervals $\left[0, \frac{\pi}{2}\right]$ or $\left[\pi, \frac{3\pi}{2}\right]$. Because you can use pole symmetry, you need only graph points in the interval $\left[0, \frac{\pi}{2}\right]$. The function attains a maximum *r*-value of $|a|$ or 4 when $\theta = \frac{\pi}{4}$ and zero *r*-value when $x = 0$ and $\frac{\pi}{2}$.

Graph

Use these points and the indicated symmetry to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	±2.8	±3.7	±4	±3.7	±2.8	0



b. $r = 3\theta$

Type of Curve and Symmetry

The equation is of the form $r = a\theta + b$, so its graph is a spiral of Archimedes. Replacing (r, θ) with $(-r, -\theta)$ yields $(-r) = 3(-\theta)$ or $r = 3\theta$. Therefore, the function has symmetry with respect to the line $\theta = \frac{\pi}{2}$.

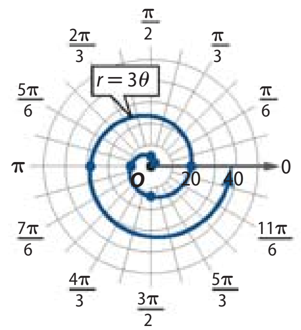
Maximum *r*-Value and Zeros

Spirals are unbounded. Therefore, the function has no maximum *r*-values and only one zero when $\theta = 0$.

Graph

Use points on the interval $[0, 4\pi]$ to sketch the graph of the function. To show symmetry, points on the interval $[-4\pi, 0]$ should also be graphed.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π
r	0	2.4	4.7	9.4	14.1	18.8	28.3	37.7



TechnologyTip

Window Settings θmin and θmax determine the values of θ that will be graphed. Normal settings for these are θmin=0 and θmax=2π, although it may be necessary to change these values to obtain a complete graph. θstep determines the interval for plotting points. The smaller this value is, the smoother the look of the graph.

GuidedPractice

5A. $r^2 = 9 \cos 2\theta$

5B. $r = 3 \sin 5\theta$

Exercises

● = Step-by-Step Solutions begin on page R29.

Graph each equation by plotting points. (Example 1)

1. $r = -\cos\theta$
2. $r = \csc\theta$
3. $r = \frac{1}{2}\cos\theta$
4. $r = 3\sin\theta$
5. $r = -\sec\theta$
6. $r = \frac{1}{3}\sin\theta$
7. $r = -4\cos\theta$
8. $r = -\csc\theta$

Use symmetry to graph each equation. (Examples 2 and 3)

9. $r = 3 + 3\cos\theta$
10. $r = 1 + 2\sin\theta$
11. $r = 4 - 3\cos\theta$
12. $r = 2 + 4\cos\theta$
13. $r = 2 - 2\sin\theta$
14. $r = 3 - 5\cos\theta$
15. $r = 5 + 4\sin\theta$
16. $r = 6 - 2\sin\theta$

Use symmetry, zeros, and maximum *r*-values to graph each function. (Example 4)

17. $r = \sin 4\theta$
18. $r = 2\cos 2\theta$
19. $r = 5\cos 3\theta$
20. $r = 3\sin 2\theta$
21. $r = \frac{1}{2}\sin 3\theta$
22. $r = 4\cos 5\theta$
23. $r = 2\sin 5\theta$
24. $r = 3\cos 4\theta$

25. **MARINE BIOLOGY** Rose curves can be observed in marine wildlife. Determine the symmetry, zeros, and maximum *r*-values of each function modeling a marine species for $0 \le \theta \le \pi$. Then use the information to graph the function. (Example 4)
   a. The pores forming the petal pattern of a sand dollar (Figure 9.2.3) can be modeled by $r = 3\cos 5\theta$.
   b. The outline of the body of a crown-of-thorns sea star (Figure 9.2.4) can be modeled by $r = 20\cos 8\theta$.

Figure 9.2.3

Figure 9.2.4

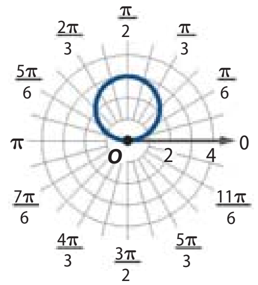
Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum *r*-values to graph the function. (Example 5)

26. $r = \frac{1}{3}\cos\theta$
27. $r = 4\theta + 1;\ \theta > 0$
28. $r = 2\sin 4\theta$
29. $r = 6 + 6\cos\theta$
30. $r^2 = 4\cos 2\theta$
31. $r = 5\theta + 2;\ \theta > 0$
32. $r = 3 - 2\sin\theta$
33. $r^2 = 9\sin 2\theta$

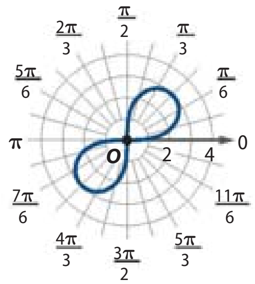
34. **FIGURE SKATING** The original focus of figure skating was to carve figures, known as *compulsory figures,* into the ice. The shape of one of these figures can be modeled by $r^2 = 25\cos 2\theta$. (Example 5)
   a. Which classic curve does the figure model?
   b. Graph the model.

Write an equation for each graph.

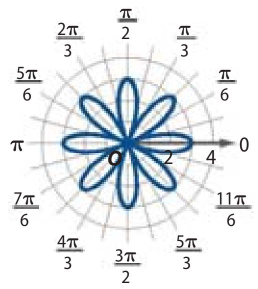
35.



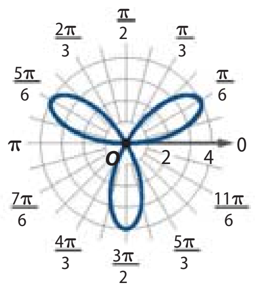
36.



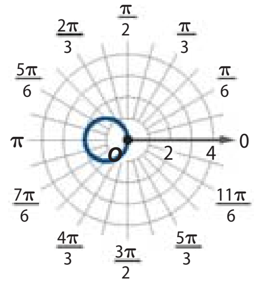
37.



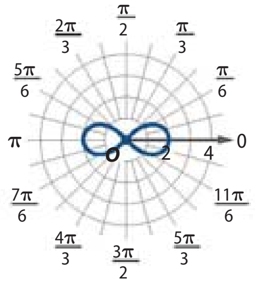
38.



39.



40.



41. **FAN** A ceiling fan has a central motor with five blades that each extend 4 units from the center. The shape of the fan can be represented by a rose curve.
   a. Write two polar equations that can be used to represent the fan.
   b. Sketch two graphs of the fan using the equations that you wrote.

Use one of the three tests to prove the specified symmetry.

42. $r = 3 + \sin\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
43. $r^2 = 4\sin 2\theta$, symmetric about the pole
44. $r = 3\sin 2\theta$, symmetric about the polar axis
45. $r = 5\cos 8\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
46. $r = 2\sin 4\theta$, symmetric about the pole

47. **FOUR-LEAF CLOVER** The shape of a certain type of clover can be represented using a rose curve. Write a polar equation for the clover if it has:
   a. 5 petals with a length of 2 units each.
   b. 4 petals with a length of 7 units each.
   c. 8 petals with a length of 6 units each.

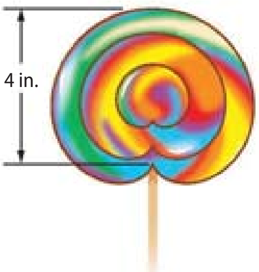
48. **CONCERT** For a concert, a circular stage is constructed and placed in the center so fans can completely surround the musicians. To record the sound of the crowd, two directional microphones are placed next to each other on the stage, one facing due east and the other facing due west. The patterns of the microphones can be represented by the polar equations $r = 2.5 + 2.5 \cos \theta$ and $r = -2.5 - 2.5 \cos \theta$.

a. Identify the type of curve given by each polar equation.

b. Sketch the graph of each microphone pattern on the same polar grid.

c. Describe what the graph tells you about the area covered by the microphones.

49. **CANDY** Write an equation that can model this lollipop in the shape of a limaçon if it is symmetric with respect to the line $\theta = \frac{\pi}{2}$ and measures 4 inches from the top of the lollipop to where the candy meets the stick.



Match each equation with its graph.

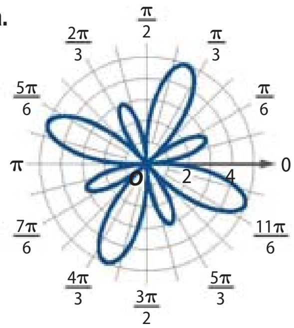
50. $r = 1 + 4 \cos 3\theta$

51. $r = 1 - 4 \sin 4\theta$

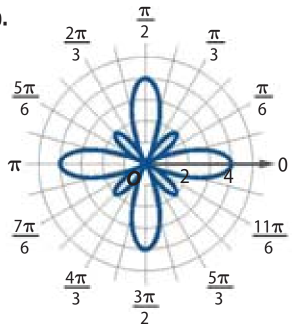
52. $r = 1 - 3 \sin 3\theta$

53. $r = 1 + 3 \cos 4\theta$

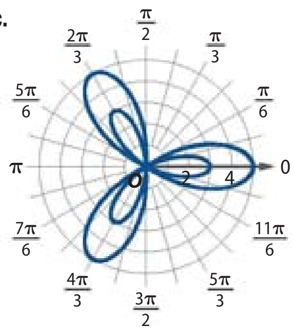
a.



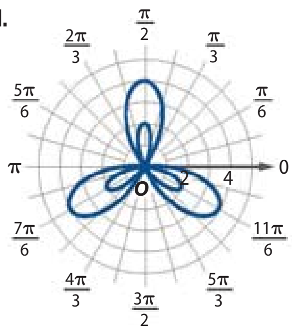
b.



c.



d.



Find x for the interval $0 \leq \theta \leq x$ so that x is a minimum and the graph is complete.

54. $r = 3 + 2 \cos \theta$

55. $r = 2 - \sin 2\theta$

56. $r = 1 + \cos \frac{\theta}{3}$

Match each equation with an equation that produces an equivalent graph.

57. $r = 5 + 4 \cos \theta$

58. $r = -5 + 4 \sin \theta$

59. $r = 5 - 4 \sin \theta$

60. $r = -5 - 4 \cos \theta$

a. $r = 5 + 4 \sin \theta$

b. $r = -5 + 4 \cos \theta$

c. $r = 5 - 4 \cos \theta$

d. $r = -5 - 4 \sin \theta$

61. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a spiral of Archimedes.

a. GRAPHICAL Sketch separate graphs of $r = \theta$ for the intervals $0 \leq \theta \leq 3\pi$, $-3\pi \leq \theta \leq 0$, and $-3\pi \leq \theta \leq 3\pi$.

b. VERBAL Make a conjecture as to the symmetry of $r = \theta$. Explain your reasoning.

c. ANALYTICAL Prove your conjecture from part **b** by using one of the symmetry tests discussed in this lesson.

d. VERBAL How does changing the interval for θ affect the other classic curves? How does this differ from how the interval affects a spiral of Archimedes? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

62. **ERROR ANALYSIS** Haley and Ella are graphing polar equations. Ella says that $r = 7 \sin 2\theta$ is not a function because it does not pass the vertical line test. Haley says the vertical line test does not apply in a polar grid. Is either of them correct? Explain your reasoning.

63. **REASONING** Sketch the graphs of $r_1 = \cos \theta$, $r_2 = \cos\left(\theta - \frac{\pi}{2}\right)$, and $r_3 = \cos(\theta - \pi)$ on the same polar grid. Describe the relationship between the three graphs. Make a conjecture as to the change in a graph when a value d is subtracted from θ.

64. **CHALLENGE** Solve the following system of polar equations algebraically on $[0, 2\pi]$. Graph the system and compare the points of intersection with the solutions that you found. Explain any discrepancies.

$r = 1 + 2 \sin \theta$

$r = 4 \sin \theta$

65. **PROOF** Prove that the graph of $r = a + b \cos 2\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

66. **PROOF** Prove that the graph of $r = a \sin 2\theta$ is symmetric with respect to the polar axis.

67. **WRITING IN MATH** Describe the effect of a in the graph of $r = a \cos \theta$.

68. **OPEN ENDED** Sketch the graph of a rose with 8 petals. Then write the equation for your graph.

Spiral Review

Graph each polar equation. (Lesson 9-1)

69. $r = 3.5$

70. $\theta = -\frac{\pi}{3}$

71. $\theta = 225°$

Find the angle θ between vectors u and v to the nearest tenth of a degree. (Lesson 8-5)

72. $\mathbf{u} = \langle 4, -3, 5\rangle$, $\mathbf{v} = \langle 2, 6, -8\rangle$

73. $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$

74. $\mathbf{u} = \langle -1, 1, 5\rangle$, $\mathbf{v} = \langle 7, -6, 9\rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j. (Lesson 8-2)

75. $D\left(-5, \frac{2}{3}\right), E\left(-\frac{4}{5}, 0\right)$

76. $D\left(-\frac{1}{2}, \frac{4}{7}\right), E\left(-\frac{3}{4}, \frac{5}{7}\right)$

77. $D(9.7, -2.4), E(-6.1, -8.5)$

78. **YARDWORK** Kyle is pushing a wheelbarrow full of leaves with a force of 525 newtons at a 48° angle with the ground. (Lesson 8-1)

a. Draw a diagram that shows the resolution of the force that Kyle is exerting into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components of the force.

Graph the hyperbola given by each equation. (Lesson 7-3)

79. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

80. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

81. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

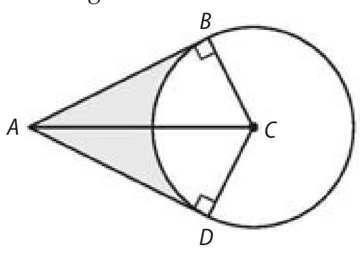
Write an equation for and graph each parabola with focus F and the given characteristics. (Lesson 7-1)

82. $F(-5, 8)$; opens right; contains $(-5, 12)$

83. $F(-1, -5)$; opens left; contains $(-1, 5)$

Skills Review for Standardized Tests

84. **SAT/ACT** In the figure, C is the center of the circle, $AC = 12$, and $m\angle BAD = 60°$. What is the perimeter of the shaded region?



A $12 + 3\pi$

B $6\sqrt{3} + 4\pi$

C $6\sqrt{3} + 3\pi$

D $12\sqrt{3} + 3\pi$

E $12\sqrt{3} + 4\pi$

85. **REVIEW** While mapping a level site, a surveyor identifies a landmark 450 feet away and 30° left of center and another landmark 600 feet away and 50° right of center. What is the approximate distance between the two landmarks?

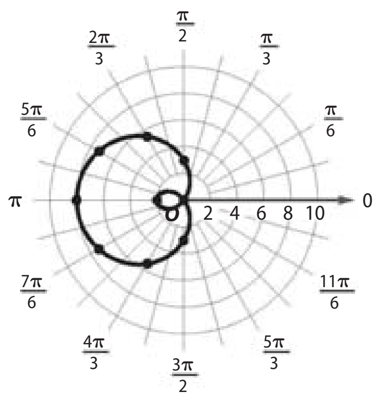
F 672 feet

G 685 feet

H 691 feet

J 703 feet

86. Which type of curve does the figure represent?



A lemniscate

B limaçon

C rose

D cardioid

87. **REVIEW** An air traffic controller is tracking two jets at the same altitude. The coordinates of the jets are (5, 310°) and (6, 345°), with r measured in miles. What is the approximate distance between the jets?

F 2.97 miles

G 3.25 miles

H 3.44 miles

J 3.71 miles

LESSON 9-3

Polar and Rectangular Forms of Equations

Then

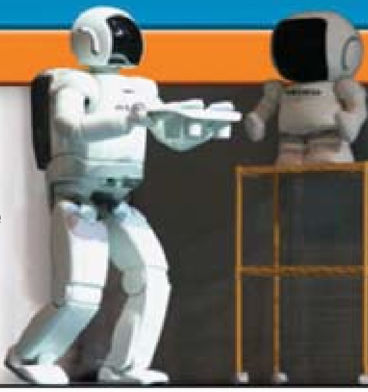
- You used a polar coordinate system to graph points and equations. (Lessons 9-1 and 9-2)

Now

1. Convert between polar and rectangular coordinates.
2. Convert between polar and rectangular equations.

Why?

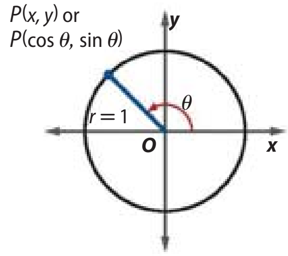
- An ultrasonic sensor attached to a robot emits an outward beam that rotates through a full circle. The sensor receives a return signal when the beam intercepts an object, and it calculates the position of the object in terms of its distance r and the angle measure θ relative to the front of the robot. The sensor relays these polar coordinates to the robot, which converts them to rectangular coordinates so it can plot the object on an internal map.



1 Polar and Rectangular Coordinates

Recall from Chapter 4 that the coordinates of a point $P(x, y)$ corresponding to an angle θ on a unit circle with radius 1 can be written in terms of θ as $P(\cos\theta, \sin\theta)$ because

$$\cos\theta = \frac{x}{r} = \frac{x}{1} \text{ or } x \qquad \text{and} \qquad \sin\theta = \frac{y}{r} = \frac{y}{1} \text{ or } y.$$



If we let r take on any real value, we can write a point $P(x, y)$ in terms of both r and θ.

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

$$r\cos\theta = x \qquad\qquad r\sin\theta = y \qquad \text{Multiply each side by } r.$$

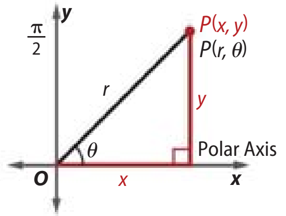
If we let the polar axis and pole in the polar coordinate system coincide with the positive x-axis and origin in the rectangular coordinate system, respectively, we now have a means of converting polar coordinates to rectangular coordinates.

KeyConcept Convert Polar to Rectangular Coordinates

If a point P has polar coordinates (r, θ), then the rectangular coordinates (x, y) of P are given by

$x = r\cos\theta$ and $y = r\sin\theta$.

That is, $(x, y) = (r\cos\theta, r\sin\theta)$.



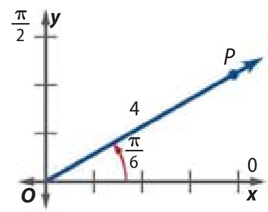
Example 1 Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates for each point with the given polar coordinates.

a. $P\left(4, \frac{\pi}{6}\right)$

For $P\left(4, \frac{\pi}{6}\right)$, $r = 4$ and $\theta = \frac{\pi}{6}$.

$x = r\cos\theta$	Conversion formula	$y = r\sin\theta$
$= 4\cos\frac{\pi}{6}$	$r = 4$ and $\theta = \frac{\pi}{6}$	$= 4\sin\frac{\pi}{6}$
$= 4\left(\frac{\sqrt{3}}{2}\right)$	Simplify.	$= 4\left(\frac{1}{2}\right)$
$= 2\sqrt{3}$		$= 2$

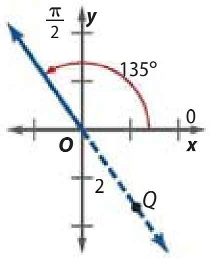


The rectangular coordinates of P are $(2\sqrt{3}, 2)$ or approximately $(3.46, 2)$ as shown.

b. $Q(-2, 135°)$

For $Q(-2, 135°)$, $r = -2$ and $\theta = 135°$.

x		y
$x = r\cos\theta$	Conversion formula	$y = r\sin\theta$
$= -2\cos 135°$	$r = -2$ and $\theta = 135°$	$= -2\sin 135°$
$= -2\left(-\frac{\sqrt{2}}{2}\right)$	Simplify.	$= -2\left(\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2}$		$= -\sqrt{2}$

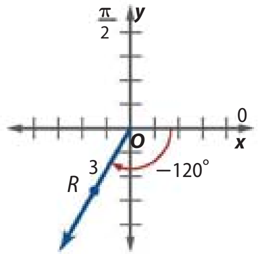


The rectangular coordinates of Q are $(\sqrt{2}, -\sqrt{2})$ or approximately (1.41, −1.41) as shown.

c. $V(3, -120°)$

For $V(3, -120°)$, $r = 3$ and $\theta = -120°$.

x		y
$x = r\cos\theta$	Conversion formula	$y = r\sin\theta$
$= 3\cos -120°$	$r = 3$ and $\theta = -120°$	$= 3\sin -120°$
$= 3\left(-\frac{1}{2}\right)$	Simplify.	$= 3\left(-\frac{\sqrt{3}}{2}\right)$
$= -\frac{3}{2}$		$= -\frac{3\sqrt{3}}{2}$



The rectangular coordinates of V are $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ or approximately (−1.5, −2.6) as shown.

Guided Practice

1A. $R(-6, -120°)$ **1B.** $S\left(5, \frac{\pi}{3}\right)$ **1C.** $T(-3, 45°)$

Study Tip

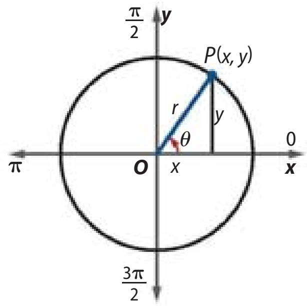
Coordinate Conversions The process for converting rectangular coordinates to polar coordinates is the same as the process used to determine the magnitude and direction of vectors in Chapter 8.

To write a pair of rectangular coordinates in polar form, you need to find the distance r a point (x, y) is from the origin or pole and the angle measure θ that point is from the x- or polar axis.

To find the distance r from the point (x, y) to the origin, use the Pythagorean Theorem.

$r^2 = x^2 + y^2$ Pythagorean Theorem

$r = \sqrt{x^2 + y^2}$ Take the positive square root of each side.

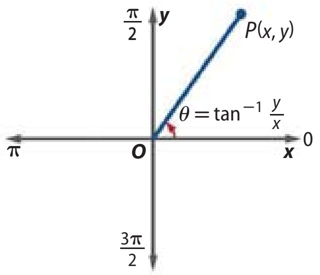


The angle θ is related to x and y by the tangent function.

$\tan\theta = \frac{y}{x}$ Tangent Ratio

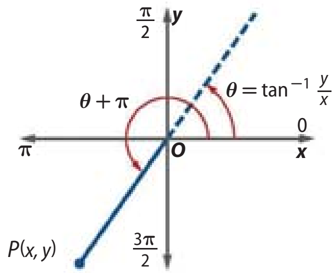
$\theta = \tan^{-1}\frac{y}{x}$ Definition of inverse tangent function

Recall that the inverse tangent function is only defined on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or [−90°, 90°]. In the rectangular coordinate system, this refers to θ-values in Quadrants I and IV or when $x > 0$, as shown in Figure 9.3.1. If a point is located in Quadrant II or III, which is when $x < 0$, you must add π or 180° to the angle measure given by the inverse tangent function, as shown in Figure 9.3.2.



When $x > 0$, $\theta = \tan^{-1}\frac{y}{x}$.

Figure 9.3.1



When $x < 0$, $\theta = \tan^{-1}\frac{y}{x} + \pi$ or $\theta = \tan^{-1}\frac{y}{x} + 180°$.

Figure 9.3.2

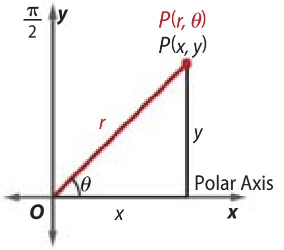
KeyConcept Convert Rectangular to Polar Coordinates

If a point P has rectangular coordinates (x, y) then the polar coordinates (r, θ) of P are given by

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, when $x > 0$

$\theta = \tan^{-1}\frac{y}{x} + \pi$ or

$\theta = \tan^{-1}\frac{y}{x} + 180°$, when $x < 0$.



Recall that polar coordinates are not unique. The conversion from rectangular coordinates to polar coordinates results in just *one* representation of the polar coordinates. There are, however, infinitely many polar representations for a point given in rectangular form.

TechnologyTip

Coordinate Conversions To convert rectangular coordinates to polar coordinates using a calculator, press [2nd] [APPS] to view the ANGLE menu. Select R▸Pr(and enter the coordinates. This will calculate the value of r. To calculate θ, repeat this process but select R▸Pθ).

Example 2 Rectangular Coordinates to Polar Coordinates

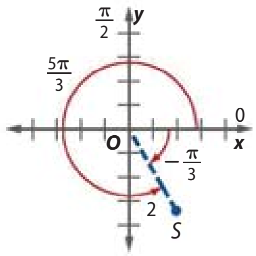
Find two pairs of polar coordinates for each point with the given rectangular coordinates.

a. $S(1, -\sqrt{3})$

For $S(x, y) = (1, -\sqrt{3})$, $x = 1$ and $y = -\sqrt{3}$. Because $x > 0$, use $\tan^{-1}\frac{y}{x}$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x}$
$= \sqrt{1^2 + (-\sqrt{3})^2}$	$x = 1$ and $y = -\sqrt{3}$	$= \tan^{-1}\frac{-\sqrt{3}}{1}$
$= \sqrt{4}$ or 2	Simplify.	$= -\frac{\pi}{3}, \frac{5\pi}{3}$

One set of polar coordinate for S is $\left(2, -\frac{\pi}{3}\right)$. Another representation that uses a positive θ-value is $\left(2, -\frac{\pi}{3} + 2\pi\right)$ or $\left(2, \frac{5\pi}{3}\right)$, as shown.



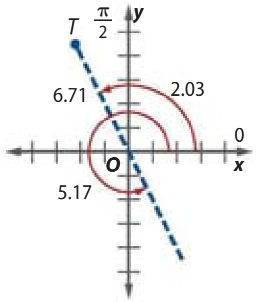
b. $T(-3, 6)$

For $T(x, y) = (-3, 6)$, $x = -3$ and $y = 6$.

Because $x < 0$, use $\tan^{-1}\frac{y}{x} + \pi$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x} + \pi$
$= \sqrt{(-3)^2 + 6^2}$	$x = -3$ and $y = 6$	$= \tan^{-1}\left(-\frac{6}{3}\right) + \pi$
$= \sqrt{45}$ or about 6.71	Simplify.	$= \tan^{-1}(-2) + \pi$ or about 2.03

One set of polar coordinates for T is approximately (6.71, 2.03). Another representation that uses a negative r-value is $(-6.71, 2.03 + \pi)$ or $(-6.71, 5.17)$, as shown.



GuidedPractice

Find two pairs of polar coordinates for each point with the given rectangular coordinates. Round to the nearest hundredth, if necessary.

2A. $V(8, 10)$

2B. $W(-9, -4)$

For some real-world phenomena, it is useful to be able to convert between polar coordinates and rectangular coordinates.

Real-WorldLink

NASA's Special Purpose Dexterous Manipulator, or Dextre, is a 3400-pound robot that stands 12 feet tall with an arm span of 11 feet. Dextre is responsible for performing jobs in space that previously required astronauts.

Source: *The New York Times*

Real-World Example 3 Conversion of Coordinates

ROBOTICS **Refer to the beginning of the lesson. Suppose the robot is facing due east and its sensor detects an object at (5, 295°).**

a. What are the rectangular coordinates that the robot will need to calculate?

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= 5 \cos 295°$	$r = 5$ and $\theta = 295°$	$= 5 \sin 295°$
≈ 2.11	Simplify.	≈ -4.53

The object is located at the rectangular coordinates (2.11, −4.53).

b. If a previously detected object has rectangular coordinates of (3, 7), what are the distance and angle measure of the object relative to the front of the robot?

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1} \frac{y}{x}$
$= \sqrt{3^2 + 7^2}$	$x = 3$ and $y = 7$	$= \tan^{-1} \frac{7}{3}$
≈ 7.62	Simplify.	$\approx 66.8°$

The object is located at the polar coordinates (7.62, 66.8°).

GuidedPractice

3. **FISHING** A fish finder is a type of radar that is used to locate fish under water. Suppose a boat is facing due east, and a fish finder gives the polar coordinates of a school of fish as (6, 125°).

A. What are the rectangular coordinates for the school of fish?

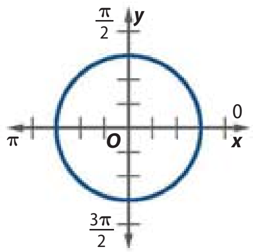
B. If a previously detected school of fish had rectangular coordinates of (−2, 6), what are the distance and angle measure of the school relative to the front of the boat?

2 Polar and Rectangular Equations

In calculus, you will sometimes need to convert from the rectangular form of an equation to its polar form and vice versa to facilitate some calculations. Some complicated rectangular equations have much simpler polar equations. Consider the rectangular and polar equations of the circle graphed below.

Rectangular Equation

$x^2 + y^2 = 9$



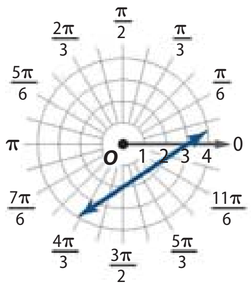
Polar Equation

$r = 3$

Likewise, some polar equations have much simpler rectangular equations, such as the line graphed below.

Polar Equation

$r = \frac{6}{2 \cos \theta - 3 \sin \theta}$



Rectangular Equation

$2x - 3y = 6$

The conversion of a rectangular equation to a polar equation is fairly straightforward. Replace x with $r \cos \theta$ and y with $r \sin \theta$, and then simplify the resulting equation using algebraic manipulations and trigonometric identities.

Example 4 Rectangular Equations to Polar Equations

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

a. $(x - 4)^2 + y^2 = 16$

The graph of $(x - 4)^2 + y^2 = 16$ is a circle with radius 4 centered at (4, 0). To find the polar form of this equation, replace x with $r \cos \theta$ and y with $r \sin \theta$. Then simplify.

$(x - 4)^2 + y^2 = 16$	Original equation
$(r \cos \theta - 4)^2 + (r \sin \theta)^2 = 16$	$x = r \cos \theta$ and $y = r \sin \theta$
$r^2 \cos^2 \theta - 8r \cos \theta + 16 + r^2 \sin^2 \theta = 16$	Multiply.
$r^2 \cos^2 \theta - 8r \cos \theta + r^2 \sin^2 \theta = 0$	Subtract 16 from each side.
$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 8r \cos \theta$	Isolate the squared terms.
$r^2(\cos^2 \theta + \sin^2 \theta) = 8r \cos \theta$	Factor.
$r^2(1) = 8r \cos \theta$	Pythagorean Identity
$r = 8 \cos \theta$	Divide each side by r.

The graph of this polar equation (Figure 9.3.3) is a circle with radius 4 centered at (4, 0).

StudyTip

Trigonometric Identities You will find it helpful to review the trigonometric identities you studied in Chapter 5 to help you simplify the polar forms of rectangular equations. A summary of these identities is found inside the back cover of this text.

b. $y = x^2$

The graph of $y = x^2$ is a parabola with vertex at the origin that opens up.

$y = x^2$	Original equation
$r \sin \theta = (r \cos \theta)^2$	$x = r \cos \theta$ and $y = r \sin \theta$
$r \sin \theta = r^2 \cos^2 \theta$	Multiply.
$\dfrac{\sin \theta}{\cos^2 \theta} = r$	Divide each side by $r \cos^2 \theta$.
$\dfrac{\sin \theta}{\cos \theta} \cdot \dfrac{1}{\cos \theta} = r$	Rewrite.
$\tan \theta \sec \theta = r$	Quotient and Reciprocal Identities

The graph of the polar equation $r = \tan \theta \sec \theta$ (Figure 9.3.4) is a parabola with vertex at the pole that opens up.

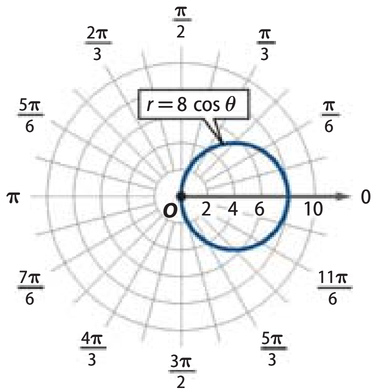


Figure 9.3.3

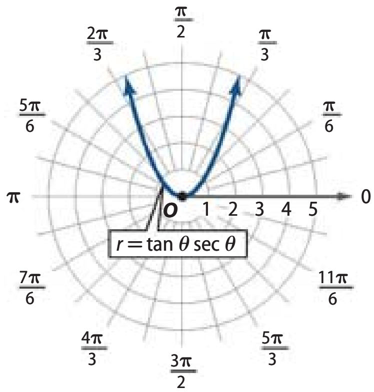


Figure 9.3.4

GuidedPractice

4A. $x^2 + (y - 3)^2 = 9$

4B. $x^2 - y^2 = 1$

To write a polar equation in rectangular form, you also make use of the relationships $r^2 = x^2 + y^2$, $x = r\cos\theta$, and $y = r\sin\theta$, as well as the relationship $\tan\theta = \frac{y}{x}$. The process, however, is not as straightforward as converting from rectangular to polar form.

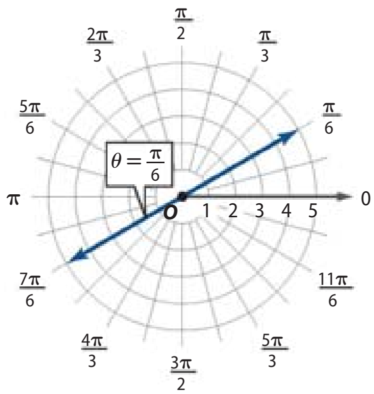
Example 5 Polar Equations to Rectangular Equations

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

a. $\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6}$	Original equation
$\tan\theta = \frac{\sqrt{3}}{3}$	Find the tangent of each side.
$\frac{y}{x} = \frac{\sqrt{3}}{3}$	$\tan\theta = \frac{y}{x}$
$y = \frac{\sqrt{3}}{3}x$	Multiply each side by x.

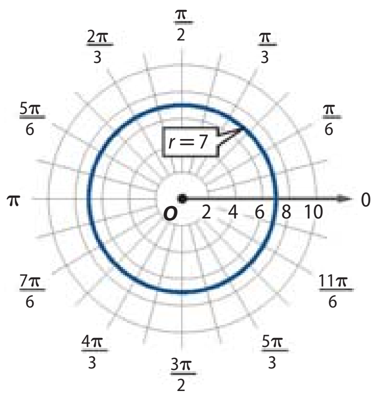
The graph of this equation is a line through the origin with slope $\frac{\sqrt{3}}{3}$ or about $\frac{2}{3}$, as supported by the graph of $\theta = \frac{\pi}{6}$ shown.



b. $r = 7$

$r = 7$	Original equation
$r^2 = 49$	Square each side.
$x^2 + y^2 = 49$	$r^2 = x^2 + y^2$

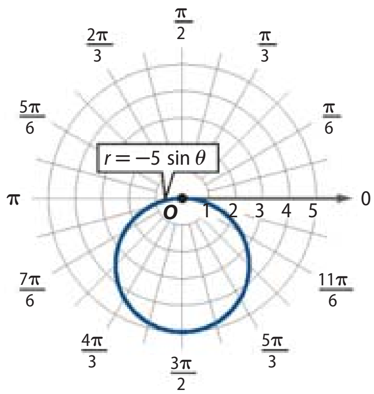
The graph of this rectangular equation is a circle with center at the origin and radius 7, supported by the graph of $r = 7$ shown.



c. $r = -5\sin\theta$

$r = -5\sin\theta$	Original equation
$r^2 = -5r\sin\theta$	Multiply each side by r.
$x^2 + y^2 = -5y$	$r^2 = x^2 + y^2$ and $y = r\sin\theta$
$x^2 + y^2 + 5y = 0$	Add $5y$ to each side.

Because in standard form, $x^2 + (y + 2.5)^2 = 6.25$, you can identify the graph of this equation as a circle centered at $(0, -2.5)$ with radius 2.5, as supported by the graph of $r = -5\sin\theta$.



Guided Practice

5A. $r = -3$ **5B.** $\theta = \frac{\pi}{3}$ **5C.** $r = 3\cos\theta$

Study Tip

Alternative Method Two points on the line $\theta = \frac{\pi}{6}$ are $\left(2, \frac{\pi}{6}\right)$ and $\left(4, \frac{\pi}{6}\right)$. In rectangular form, these points are $(\sqrt{3}, 1)$ and $(2\sqrt{3}, 2)$. The equation of the line through these points is $y = \frac{\sqrt{3}}{3}x$.

Study Tip

Converting to Rectangular Form Other useful substitutions are variations of the equations $x = r\cos\theta$ and $y = r\sin\theta$, such as $r = \frac{x}{\cos\theta}$ and $r = \frac{y}{\sin\theta}$.

Exercises

= Step-by-Step Solutions begin on page R29.

Find the rectangular coordinates for each point with the given polar coordinates. Round to the nearest hundredth, if necessary. (Example 1)

1. $\left(2, \frac{\pi}{4}\right)$ **2.** $\left(\frac{1}{4}, \frac{\pi}{2}\right)$

3. $(5, 240°)$ **4.** $(2.5, 250°)$

5. $\left(-2, \frac{4\pi}{3}\right)$ **6.** $(-13, -70°)$

7. $\left(3, \frac{\pi}{2}\right)$ **8.** $\left(\frac{1}{2}, \frac{3\pi}{4}\right)$

9. $(-2, 270°)$ **10.** $(4, 210°)$

11. $\left(-1, -\frac{\pi}{6}\right)$ **12.** $\left(5, \frac{\pi}{3}\right)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \le \theta \le 2\pi$. Round to the nearest hundredth, if necessary. (Example 2)

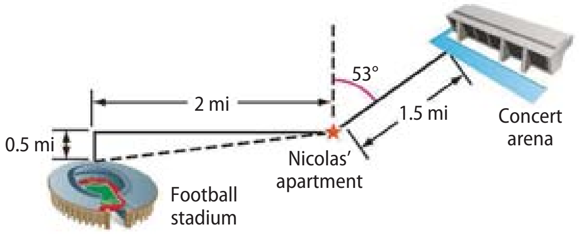
13. $(7, 10)$ **14.** $(-13, 4)$ **15.** $(-6, -12)$

16. $(4, -12)$ **17.** $(2, -3)$ **18.** $(0, -173)$

19. $(a, 3a), a > 0$ **20.** $(-14, 14)$ **21.** $(52, -31)$

22. $(3b, -4b), b > 0$ **23.** $(1, -1)$ **24.** $(2, \sqrt{2})$

25. DISTANCE Standing on top of his apartment building, Nicolas determines that a concert arena is 53° east of north. Suppose the arena is exactly 1.5 miles from Nicolas' apartment. (Example 3)



a. How many miles north and east will Nicolas have to travel to reach the arena?

b. If a football stadium is 2 miles west and 0.5 mile south of Nicolas' apartment, what are the polar coordinates of the stadium if Nicolas' apartment is at the pole?

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Example 4)

26. $x = -2$ **27.** $(x + 5)^2 + y^2 = 25$

28. $y = -3$ **29.** $x = y^2$

30. $(x - 2)^2 + y^2 = 4$ **31.** $(x - 1)^2 - y^2 = 1$

32. $x^2 + (y + 3)^2 = 9$ **33.** $y = \sqrt{3}x$

34. $x^2 + (y + 1)^2 = 1$ **35.** $x^2 + (y - 8)^2 = 64$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation. (Example 5)

36. $r = 3 \sin \theta$ **37.** $\theta = -\frac{\pi}{3}$

38. $r = 10$ **39.** $r = 4 \cos \theta$

40. $\tan \theta = 4$ **41.** $r = 8 \csc \theta$

42. $r = -4$ **43.** $\cot \theta = -7$

44. $\theta = \frac{3\pi}{4}$ **45.** $r = \sec \theta$

46. EARTHQUAKE An equation to model the seismic waves of an earthquake is $r = 12.6 \sin \theta$, where r is measured in miles. (Example 5)

a. Graph the polar pattern of the earthquake.

b. Write an equation in rectangular form to model the seismic waves.

c. Find the rectangular coordinates of the epicenter of the earthquake, and describe the area that is affected by the earthquake.

47. MICROPHONE The polar pattern for a directional microphone at a football game is given by $r = 2 + 2 \cos \theta$. (Example 5)

a. Graph the polar pattern.

b. Will the microphone detect a sound that originates from the point with rectangular coordinates $(-2, 0)$? Explain.

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

48. $r = \frac{1}{\cos \theta + \sin \theta}$ **49.** $r = 10 \csc \left(\theta + \frac{7\pi}{4}\right)$

50. $r = 3 \csc \left(\theta - \frac{\pi}{2}\right)$ **51.** $r = -2 \sec \left(\theta - \frac{11\pi}{6}\right)$

52. $r = 4 \sec \left(\theta - \frac{4\pi}{3}\right)$ **53.** $r = \frac{5 \cos \theta + 5 \sin \theta}{\cos^2 \theta - \sin^2 \theta}$

54. $r = 2 \sin \left(\theta + \frac{\pi}{3}\right)$ **55.** $r = 4 \cos \left(\theta + \frac{\pi}{2}\right)$

56. ASTRONOMY Polar equations are used to model the paths of satellites or other orbiting bodies in space. Suppose the path of a satellite is modeled by $r = \frac{4}{4 + 3 \sin \theta}$, where r is measured in tens of thousands of miles, with Earth at the pole.

a. Sketch a graph of the path of the satellite.

b. Determine the minimum and maximum distances the satellite is from Earth at any time.

c. Suppose a second satellite passes through a point with rectangular coordinates $(1.5, -3)$. Are the two satellites at risk of ever colliding at this point? Explain.

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

57. $6x - 3y = 4$

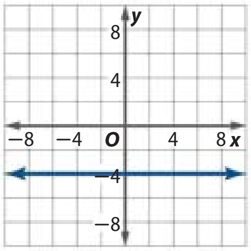
58. $2x + 5y = 12$

59. $(x - 6)^2 + (y - 8)^2 = 100$

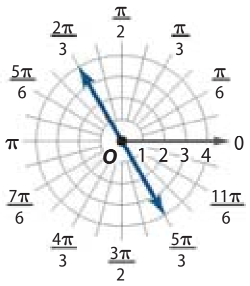
60. $(x + 3)^2 + (y - 2)^2 = 13$

Write rectangular and polar equations for each graph.

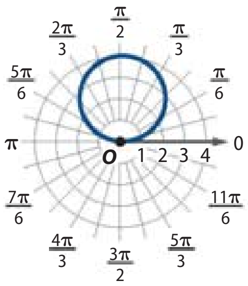
61.



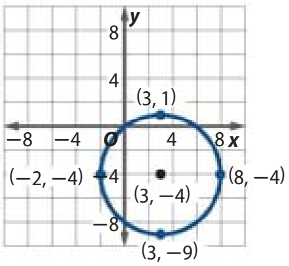
62.



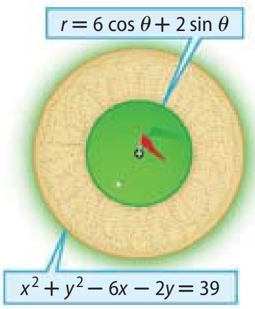
63.



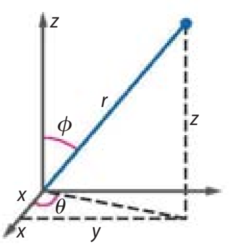
64.



65. GOLF On the 18th hole at Hilly Pines Golf Course, the circular green is surrounded by a ring of sand as shown in the figure. Find the area of the region covered by sand assuming the hole acts as the pole for both equations and units are given in yards.



66. CONSTRUCTION Boom cranes operate on three-dimensional counterparts of polar coordinates called *spherical coordinates*. A point in space has spherical coordinates (r, θ, ϕ), where r represents the distance from the pole, θ represents the angle of rotation about the vertical axis, and ϕ represents the polar angle from the positive vertical axis. Given a point in spherical coordinates (r, θ, ϕ) find the rectangular coordinates (x, y, z) in terms of r, θ, and ϕ.



67. MULTIPLE REPRESENTATIONS In this problem, you will investigate the relationship between complex numbers and polar coordinates.

a. GRAPHICAL The complex number $a + bi$ can be plotted on a complex plane using the ordered pair (a, b), where the x-axis is the real axis R and the y-axis is the imaginary axis $\boldsymbol{i}$. Graph the complex number $6 + 8i$.

b. NUMERICAL Find polar coordinates for the complex number using the rectangular coordinates plotted in part **a** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

c. GRAPHICAL Graph the complex number $-3 + 3i$ on a rectangular coordinate system.

d. GRAPHICAL Find polar coordinates for the complex number using the rectangular coordinates plotted in part **c** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

e. ANALYTICAL For a complex number $a + bi$, find an expression for converting to polar coordinates.

H.O.T. Problems Use Higher-Order Thinking Skills

68. ERROR ANALYSIS Becky and Terrell are writing the polar equation $r = \sin\theta$ in rectangular form. Terrell believes that the answer is $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. Becky believes that the answer is simply $y = \sin x$. Is either of them correct? Explain your reasoning.

69. CHALLENGE The equation for a circle is $r = 2a\cos\theta$. Write this equation in rectangular form. Find the center and radius of the circle.

70. REASONING Given a set of rectangular coordinates (x, y) and a value for r, write expressions for finding θ in terms of sine and in terms of cosine. (*Hint:* You may have to write multiple expressions for each function, similar to the expressions given in this lesson using tangent.)

71. WRITING IN MATH Make a conjecture about when graphing an equation is made easier by representing the equation in polar form rather than rectangular form and vice versa.

72. PROOF Use $x = r\cos\theta$ and $y = r\sin\theta$ to prove that $r = x\sec\theta$ and $r = y\csc\theta$.

73. CHALLENGE Write $r^2(4\cos^2\theta + 3\sin^2\theta) + r(-8a\cos\theta + 6b\sin\theta) = 12 - 4a^2 - 3b^2$ in rectangular form. (*Hint:* Distribute before substituting values for r^2 and r. The rectangular equation should be a conic.)

74. WRITING IN MATH Use the definition of a polar axis given in Lesson 9-1 to explain why it was necessary to state that the robot in Example 3 was facing due east. How can the use of quadrant bearings help to eliminate this?

Spiral Review

Use symmetry to graph each equation. (Lesson 9-2)

75. $r = 1 - 2 \sin \theta$

76. $r = -2 - 2 \sin \theta$

77. $r = 2 \sin 3\theta$

Find three different pairs of polar coordinates that name the given point if $-360° < \theta \leq 360°$ or $-2\pi < \theta \leq 2\pi$. (Lesson 9-1)

78. $T(1.5, 180°)$

79. $U\left(-1, \frac{\pi}{3}\right)$

80. $V(4, 315°)$

Find the angle θ between u and v to the nearest tenth of a degree. (Lesson 8-3)

81. $\mathbf{u} = \langle 6, -4 \rangle, \mathbf{v} = \langle -5, -7 \rangle$

82. $\mathbf{u} = \langle 2, 3 \rangle, \mathbf{v} = \langle -9, 6 \rangle$

83. $\mathbf{u} = \langle 1, 10 \rangle, \mathbf{v} = \langle 8, -2 \rangle$

Write each pair of parametric equations in rectangular form. Then graph and state any restrictions on the domain. (Lesson 7-5)

84. $y = t + 6$ and $x = \sqrt{t}$

85. $y = \frac{t}{2} + 1$ and $x = \frac{t^2}{4}$

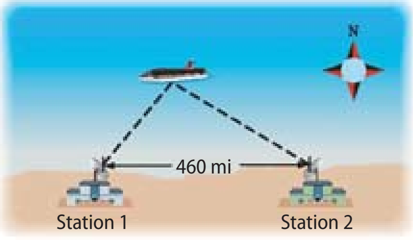
86. $y = -3 \sin t$ and $x = 3 \cos t$

87. NAVIGATION Two LORAN broadcasting stations are located 460 miles apart. A ship receives signals from both stations and determines that it is 108 miles farther from Station 2 than Station 1. (Lesson 7-3)

a. Determine the equation of the hyperbola centered at the origin on which the ship is located.

b. Graph the equation, indicating on which branch of the hyperbola the ship is located.

c. Find the coordinates of the location of the ship on the coordinate grid if it is 110 miles from the x-axis.



88. BICYCLES Woodland Bicycles makes two models of off-road bicycles: the Adventure, which sells for \$250, and the Grande Venture, which sells for \$350. Both models use the same frame. The painting and assembly time required for the Adventure is 2 hours, while the time is 3 hours for the Grande Venture. If there are 175 frames and 450 hours of labor available for production, how many of each model should be produced to maximize revenue? What is the maximum revenue? (Lesson 6-5)

Solve each system of equations using Gauss-Jordan elimination. (Lesson 6-1)

89. $3x + 9y + 6z = 21$
$4x - 10y + 3z = 15$
$-5x + 12y - 2z = -6$

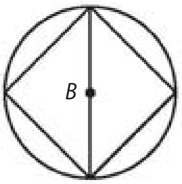
90. $x + 5y - 3z = -14$
$2x - 4y + 5z = 18$
$-7x - 6y - 2z = 1$

91. $2x - 4y + z = 20$
$5x + 2y - 2z = -4$
$6x + 3y + 5z = 23$

Skills Review for Standardized Tests

92. SAT/ACT A square is inscribed in circle B. If the circumference of the circle is 50π, what is the length of the diagonal of the square?

A $10\sqrt{2}$
B 25
C $25\sqrt{2}$
D 50
E $50\sqrt{2}$



93. REVIEW Which of the following could be an equation for a rose with three petals?

F $r = 3 \sin \theta$
G $r = \sin 3\theta$
H $r = 6 \sin \theta$
J $r = \sin 6\theta$

94. What is the polar form of $x^2 + (y - 2)^2 = 4$?

A $r = \sin \theta$
B $r = 2 \sin \theta$
C $r = 4 \sin \theta$
D $r = 8 \sin \theta$

95. REVIEW Which of the following could be an equation for a spiral of Archimedes that passes through $A\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

F $r = \frac{\sqrt{2}\pi}{2} \cos \theta$
G $r = \theta$
H $r = \frac{3}{4}$
J $r = \frac{\theta}{2}$

CHAPTER 9

Mid-Chapter Quiz

Lessons 9-1 through 9-3

Graph each point on a polar grid. (Lesson 9-1)

1. $A(-2, 45°)$

2. $D(1, 315°)$

3. $C\left(-1.5, -\frac{4\pi}{3}\right)$

4. $B\left(3, -\frac{5\pi}{6}\right)$

Graph each polar equation. (Lesson 9-1)

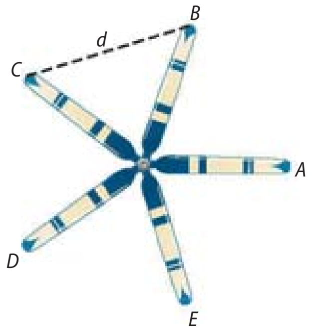
5. $r = 3$

6. $\theta = -\frac{3\pi}{4}$

7. $\theta = 60°$

8. $r = -1.5$

9. HELICOPTERS A helicopter rotor consists of five equally spaced blades. Each blade is 11.5 feet long. (Lesson 9-1)



a. If the angle blade *A* makes with the polar axis is 3°, write an ordered pair to represent the tip of each blade on a polar grid. Assume that the rotor is centered at the pole.

b. What is the distance *d* between the tips of the helicopter blades to the nearest tenth of a foot?

Graph each equation. (Lesson 9-2)

10. $r = \frac{1}{4}\sec\theta$

11. $r = \frac{1}{3}\cos\theta$

12. $r = 3\csc\theta$

13. $r = 4\sin\theta$

14. STAINED GLASS A rose window is a circular window seen in gothic architecture. The pattern of the window radiates from the center. The window shown can be approximated by the equation $r = 3\sin 6\theta$. Use symmetry, zeros, and maximum *r*-values of the function to graph the function. (Lesson 9-2)

Identify and graph each classic curve. (Lesson 9-2)

15. $r = \frac{1}{2}\sin\theta$

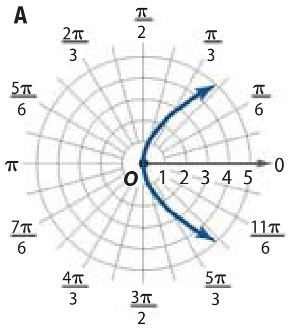
16. $r = \frac{1}{3}\theta + 3, \theta \geq 0$

17. $r = 1 + 2\cos\theta$

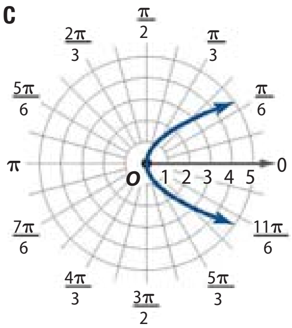
18. $r = 5\sin 3\theta$

19. MULTIPLE CHOICE Identify the polar graph of $y^2 = \frac{1}{2}x$. (Lesson 9-3)

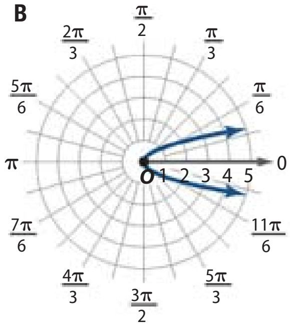
A



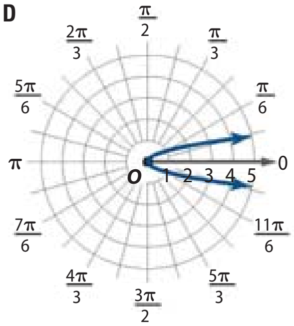
C



B



D



Find the rectangular coordinates for each point with the given polar coordinates. (Lesson 9-3)

20. $\left(4, \frac{2\pi}{3}\right)$

21. $\left(-2, -\frac{\pi}{4}\right)$

22. $(-1, 210°)$

23. $(3, 30°)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth. (Lesson 9-3)

24. $(-3, 5)$

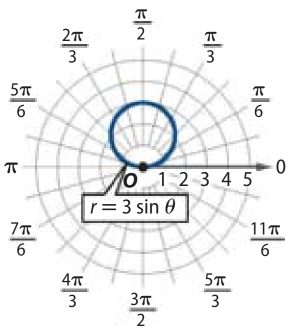
25. $(8, 1)$

26. $(7, -6)$

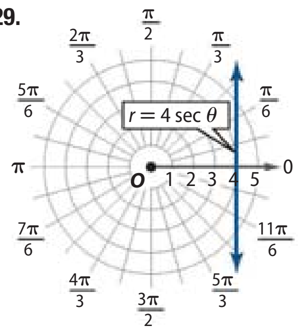
27. $(-4, -10)$

Write a rectangular equation for each graph. (Lesson 9-3)

28.



29.



LESSON 9-4 Polar Forms of Conic Sections

Then

- You defined conic sections. (Lessons 7-1 through 7-3)

Now

1. Identify polar equations of conics.
2. Write and graph the polar equation of a conic given its eccentricity and the equation of its directrix.

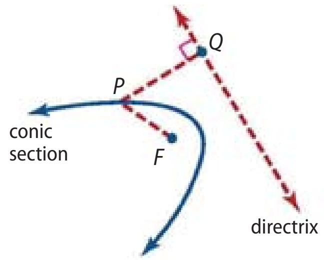
Why?

- Polar equations of conic sections can be used to model orbital motion, such as the orbit of a planet around the Sun or the orbit of a satellite around a planet.

1 Use Polar Equations of Conics

In Chapter 7, you defined conic sections in terms of the distance between a focus and directrix (parabola) or between two foci (ellipse and hyperbola). Alternatively, we can define all of these curves using the focus-directrix definition of a parabola.

In general, a conic section can be defined as the locus of points such that the distance from a point P to the focus and the distance from the point to a fixed line not containing P (the directrix) is a constant ratio. This constant ratio $\frac{PF}{PQ}$ represents the eccentricity of a conic and is denoted e.

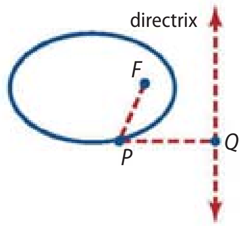
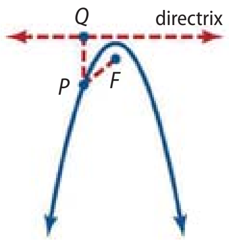
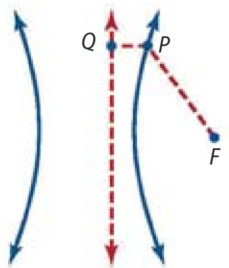


e as Constant Ratio $\qquad$ **e as Constant Multiplier**

$$e = \frac{PF}{PQ} \qquad \text{or} \qquad PF = e \cdot PQ$$

Recall that for a parabola, $PF = PQ$. Therefore, a parabola has eccentricity $\frac{PQ}{PQ}$ or 1. Other values of e give us other conics. These eccentricities are summarized below.

ConceptSummary Eccentricities of Conics

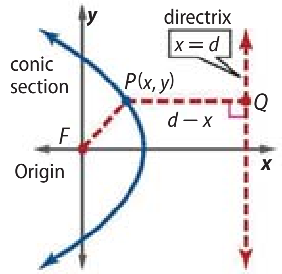
Ellipse	Parabola	Hyperbola
$0 < e < 1$	$e = 1$	$e > 1$
		
$0 < \frac{PF}{PQ} < 1$	$\frac{PF}{PQ} = 1$	$\frac{PF}{PQ} > 1$

Recall too that when the center of a conic section lies at the origin, the rectangular equations of conics take on a simpler form.

Ellipses	Parabolas	Hyperbolas
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	$x^2 = 4py$ or $y^2 = 4px$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Using the focus-directrix definition, the equation of a conic in polar form is simplified if a *focus* of the conic lies at the origin.

Consider a conic with its focus located at the origin and its directrix to the right at $x = d$. For any point $P(x, y)$ on the curve, the distance PF is given by $\sqrt{x^2 + y^2}$, and the distance PQ is given by $d - x$. We can substitute these expressions in the definition of a conic section.



StudyTip

Other Conics When defining conics in terms of their eccentricity, *e* is a strictly positive constant. There are no circles, lines, or other degenerate conics.

$PF = e \cdot PQ$ — Definition of a conic section

$\sqrt{x^2 + y^2} = e(d - x)$ — $PF = \sqrt{x^2 + y^2}$ and $PQ = d - x$

The expression $\sqrt{x^2 + y^2}$ should make you think of polar coordinates. In fact, the equation above has a simpler form in the polar coordinate system.

$\sqrt{x^2 + y^2} = e(d - x)$ — Rectangular form of conic defined in terms of its eccentricity e

$r = e(d - r\cos\theta)$ — $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$

$r = ed - er\cos\theta$ — Distributive Property

$r + er\cos\theta = ed$ — Isolate r-terms.

$r(1 + e\cos\theta) = ed$ — Factor.

$r = \dfrac{ed}{1 + e\cos\theta}$ — Solve for r.

This last equation is the polar form of an equation for the conic sections with focus at the pole and vertical directrix and center or vertex to the right of the pole. Different orientations of the focus and directrix can produce different forms of this polar equation as summarized below.

ReadingMath

Eccentricity In each of these polar equations, the letter *e* is a variable that represents the eccentricity of the conic. It should *not* be confused with the transcendental number *e*, which is a constant.

KeyConcept Polar Equations of Conics

The conic section with eccentricity $e > 0$, $d > 0$, and focus at the pole has the polar equation:

- $r = \dfrac{ed}{1 + e\cos\theta}$ if the directrix is the vertical line $x = d$ (Figure 9.4.1),
- $r = \dfrac{ed}{1 - e\cos\theta}$ if the directrix is the vertical line $x = -d$ (Figure 9.4.2),
- $r = \dfrac{ed}{1 + e\sin\theta}$ if the directrix is the horizontal line $y = d$ (Figure 9.4.3), and
- $r = \dfrac{ed}{1 - e\sin\theta}$ if the directrix is the horizontal line $y = -d$ (Figure 9.4.4).

In each of the examples below, $e = 1$, so the conic takes the form of a parabola.

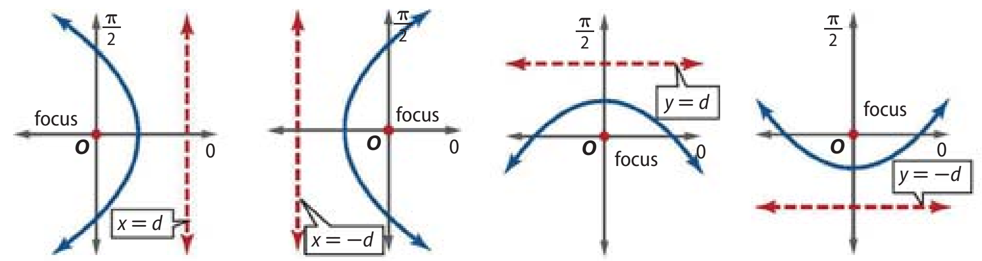


Figure 9.4.1 Figure 9.4.2 Figure 9.4.3 Figure 9.4.4

You will derive the last three of these equations in Exercises 50–52.

Notice that for $r = \dfrac{ed}{1 - e\cos\theta}$, the directrix of the conic is to the left of the pole. For $r = \dfrac{ed}{1 + e\sin\theta}$, the directrix is above the pole. For $r = \dfrac{ed}{1 - e\sin\theta}$, the directrix is below the pole.

To analyze the polar equation of a conic, begin by writing the equation in standard form, $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$. In this form, determine the eccentricity and use this value to identify the type of conic the equation represents. Then determine the equation of the directrix, and use it to describe the orientation of the conic.

Example 1 Identify Conics from Polar Equations

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

a. $r = \frac{9}{3 + 2.25 \cos \theta}$

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

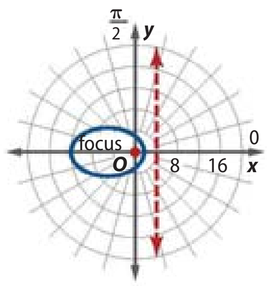
$r = \frac{9}{3 + 2.25 \cos \theta}$ — Original equation

$r = \frac{3(3)}{3(1 + 0.75 \cos \theta)}$ — Factor the numerator and denominator.

$r = \frac{3}{1 + 0.75 \cos \theta}$ — Divide the numerator and denominator by 3.

In this form, you can see from the denominator that $e = 0.75$. Therefore, the conic is an ellipse. For polar equations of this form, the equation of the directrix is $x = d$. From the numerator, we know that $ed = 3$, so $d = 3 \div 0.75$ or 4. Therefore, the equation of the directrix is $x = 4$.

CHECK Sketch the graph of $r = \frac{9}{3 + 2.25 \cos \theta}$ and its directrix $x = 4$ using either the techniques shown in Lesson 9-2 or a graphing calculator. The graph is an ellipse with its directrix to the right of the pole. ✔



> **StudyTip**
> **Focus-Directrix Pairs** While a parabola has one focus and one directrix, ellipses and hyperbolas have two foci-directrix pairs. Either focus-directrix pair can be used to generate the conic.

b. $r = \frac{-16}{4 \sin \theta - 2}$

Write the equation in standard form.

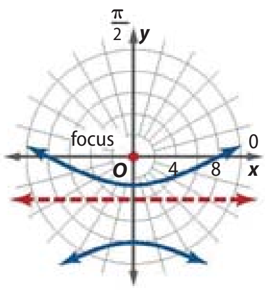
$r = \frac{-16}{4 \sin \theta - 2}$ — Original equation

$r = \frac{-2(8)}{-2(1 - 2 \sin \theta)}$ — Factor the numerator and denominator.

$r = \frac{8}{1 - 2 \sin \theta}$ — Divide the numerator and denominator by −2.

The equation is of the form $r = \frac{ed}{1 - e \sin \theta}$, so $e = 2$. Therefore, the conic is a hyperbola. For polar equations of this form, the equation of the directrix is $y = -d$. Because $ed = 8$, $d = 8 \div 2$ or 4. Therefore, the equation of the directrix is $y = -4$.

CHECK Sketch the graph of $r = \frac{-16}{4 \sin \theta - 2}$ and its directrix $y = -4$. The graph is a hyperbola with one focus at the origin, above the directrix. ✔



GuidedPractice

1A. $r = \frac{-6}{3 \cos \theta - 1}$

1B. $r = \frac{9}{3 + 3 \sin \theta}$

1C. $r = \frac{1}{6 + 1.2 \cos \theta}$

2 Write Polar Equations of Conics

You can write the polar equation of a conic given its eccentricity and the equation of the directrix or its eccentricity and some other characteristics.

Example 2 Write Polar Equations of Conics

Write and graph a polar equation and directrix for the conic with the given characteristics.

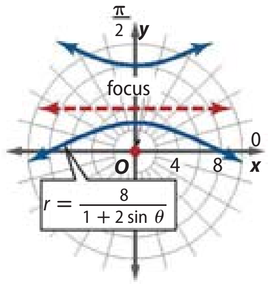
a. **$e = 2$; directrix: $y = 4$**

Because $e = 2$, the conic is a hyperbola. The directrix $y = 4$ is above the pole, so the equation is of the form $r = \frac{ed}{1 + e \sin \theta}$. Use the values for e and d to write the equation.

$r = \frac{ed}{1 + e \sin \theta}$ — Polar form of conic with directrix $y = d$

$r = \frac{2(4)}{1 + 2 \sin \theta}$ or $\frac{8}{1 + 2 \sin \theta}$ — $e = 2$ and $d = 4$

Sketch the graph of this polar equation and its directrix. The graph is a hyperbola with its directrix above the pole.



b. **$e = 0.5$; vertices at $(-4, 0)$ and $(12, 0)$**

Because $e = 0.5$, the conic is an ellipse. The center of the ellipse is at $(4, 0)$, the midpoint of the segment between the given vertices. This point is to the right of the pole. Therefore, the directrix will be to the left of the pole at $x = -d$. The polar equation of a conic with this directrix is $r = \frac{ed}{1 - e \cos \theta}$.

Use the value of e and the polar form of a point on the conic to find the value of d. The vertex point $(12, 0)$ has polar coordinates $(r, \theta) = \left(\sqrt{12^2 + 0^2}, \tan^{-1} \frac{0}{12}\right)$ or $(12, 0)$.

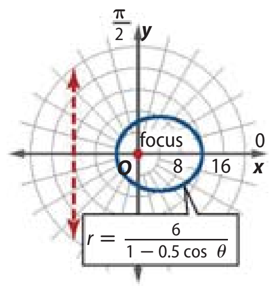
$r = \frac{ed}{1 - e \cos \theta}$ — Polar form of conic with directrix $x = -d$

$12 = \frac{0.5d}{1 - 0.5 \cos 0}$ — $e = 0.5$, $r = 12$, and $\theta = 0$

$12 = \frac{0.5d}{0.5}$ — $\cos 0 = 1$

$12 = d$ — Simplify.

Therefore, the equation for the ellipse is $r = \frac{0.5 \cdot 12}{1 - 0.5 \cos \theta}$ or $r = \frac{6}{1 - 0.5 \cos \theta}$. Because $d = 12$, the equation of the directrix is $x = -12$. The graph is an ellipse with vertices at $(-4, 0)$ and $(12, 0)$.



StudyTip

Effects of Various Eccentricities You will investigate the effects of various eccentricities for a fixed directrix and various directrixes for a fixed eccentricity in Exercise 49.

GuidedPractice

2A. $e = 1$; directrix: $x = 2$

2B. $e = 2.5$; vertices at $(0, -3)$ and $(0, -7)$

In Lessons 7-1 through 7-3, you analyzed the rectangular equations of conics in standard form to describe the geometric properties of parabolas, ellipses, and hyperbolas. You can use the geometric analysis of the graph of a conic given in polar form to write the equation in rectangular form.

Example 3 Write the Polar Form of Conics in Rectangular Form

Write each polar equation in rectangular form.

a. $r = \frac{4}{1 - \sin\theta}$

Step 1 Analyze the polar equation.

For this equation, $e = 1$ and $d = 4$. The eccentricity and form of the equation determine that this is a parabola that opens vertically with focus at the pole and a directrix $y = -4$. The general equation of such a parabola in rectangular form is $(x - h)^2 = 4p(y - k)$.

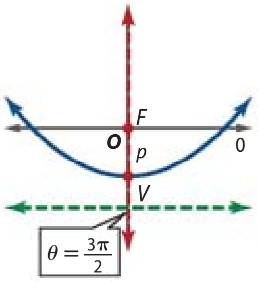


Figure 9.4.5

Step 2 Determine values for h, k, and p.

The vertex lies between the focus F and directrix of the parabola, occurring when $\theta = \frac{3\pi}{2}$, as shown in Figure 9.4.5. Evaluating the function at this value, we find that the vertex lies at polar coordinates $\left(2, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates $(0, -2)$. So, $(h, k) = (0, -2)$. The distance p from the vertex at $(0, -2)$ to the focus at $(0, 0)$ is 2.

Step 3 Substitute the values for h, k, and p into the standard form of an equation for a parabola.

$(x - h)^2 = 4p(y - k)$ — Standard form of a parabola

$(x - 0)^2 = 4(2)[y - (-2)]$ — $h = 0$, $k = -2$, and $p = 2$

$x^2 = 8y + 16$ — Simplify.

b. $r = \frac{3.2}{1 - 0.6\cos\theta}$

Step 1 Analyze the polar equation.

For this equation, $e = 0.6$ and $d \approx 5.3$. The eccentricity and form of the equation determine that this is an ellipse with directrix $x = -5.3$. Therefore, the major axis of the ellipse lies along the polar or x-axis. The general equation of such an ellipse in rectangular form is $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.

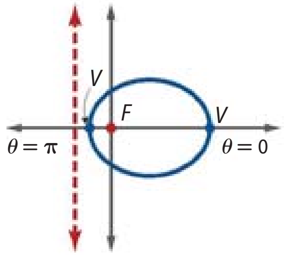


Figure 9.4.6

Step 2 Determine values for h, k, a, and b.

The vertices are the endpoints of the major axis and occur when $\theta = 0$ and π as shown in Figure 9.4.6. Evaluating the function at these values, we find that the vertices have polar coordinates $(8, 0)$ and $(2, \pi)$, which correspond to rectangular coordinates $(8, 0)$ and $(-2, 0)$. The ellipse's center is the midpoint of the segment between the vertices, so $(h, k) = (3, 0)$.

The distance a between the center and each vertex is 5. The distance c from the center to the focus at $(0, 0)$ is 3. By the Pythagorean relation $b = \sqrt{a^2 - c^2}$, $b = \sqrt{5^2 - 3^2}$ or 4.

Step 3 Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ — Standard form of an ellipse

$\frac{(x - 3)^2}{5^2} + \frac{(y - 0)^2}{4^2} = 1$ — $h = 3$, $k = 0$, $a = 5$, and $b = 4$

$\frac{(x - 3)^2}{25} + \frac{y^2}{16} = 1$ — Simplify.

GuidedPractice

3A. $r = \frac{2.5}{1 - 1.5\cos\theta}$

3B. $r = \frac{5}{1 + \sin\theta}$

Exercises

= Step-by-Step Solutions begin on page R29.

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. (Example 1)

1. $r = \dfrac{20}{4 + 4 \sin \theta}$

2. $r = \dfrac{18}{2 - 6 \cos \theta}$

3. $r = \dfrac{21}{3 \cos \theta + 1}$

4. $r = \dfrac{24}{4 \sin \theta + 8}$

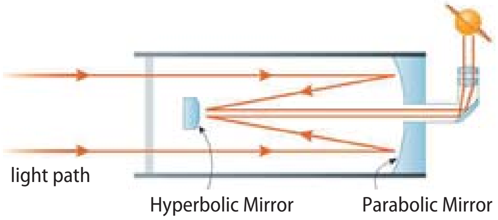
5. $r = \dfrac{-12}{6 \cos \theta - 6}$

6. $r = \dfrac{9}{4 - 3 \sin \theta}$

7. $r = \dfrac{-8}{\sin \theta - 0.25}$

8. $r = \dfrac{10}{2.5 + 2.5 \cos \theta}$

9. **TELESCOPES** The Cassegrain Telescope, invented in 1692, produces an image by reflecting light off of parabolic and hyperbolic mirrors. Determine the eccentricity, type of conic, and the equation of the directrix for each equation modeling a mirror in the telescope. (Example 1)



a. $r = \dfrac{7}{2 \sin \theta + 2}$

b. $r = \dfrac{28}{12.5 \cos \theta + 5}$

Write and graph a polar equation and directrix for the conic with the given characteristics. (Example 2)

10. $e = 1$: directrix: $y = 6$

11. $e = 0.75$; directrix: $x = -8$

12. $e = 5$; directrix: $x = 2$

13. $e = 0.1$; directrix: $y = 8$

14. $e = 6$; directrix: $y = -7$

15. $e = 1$; directrix: $x = -1.5$

16. $e = 0.8$; vertices at $(-36, 0)$ and $(4, 0)$

17. $e = 1.5$; vertices at $(-3, 0)$ and $(-15, 0)$

Write each polar equation in rectangular form. (Example 3)

18. $r = \dfrac{4.8}{1 + \sin \theta}$

19. $r = \dfrac{30}{4 + \cos \theta}$

20. $r = \dfrac{5}{1 - 1.5 \cos \theta}$

21. $r = \dfrac{5.1}{1 + 0.7 \sin \theta}$

22. $r = \dfrac{12}{1 - \cos \theta}$

23. $r = \dfrac{6}{0.25 - 0.75 \sin \theta}$

24. $r = \dfrac{4.5}{1 + 1.25 \sin \theta}$

25. $r = \dfrac{8.4}{1 - 0.4 \cos \theta}$

GRAPHING CALCULATOR **Determine the type of conic for each polar equation. Then graph each equation.**

26. $r = \dfrac{2}{2 + \sin\left(\theta + \frac{\pi}{3}\right)}$

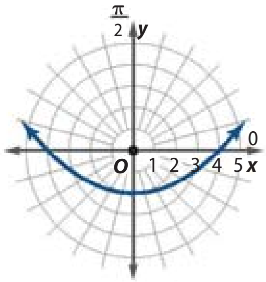
27. $r = \dfrac{3}{1 + \cos\left(\theta - \frac{\pi}{4}\right)}$

28. $r = \dfrac{2}{1 - \cos\left(\theta + \frac{\pi}{6}\right)}$

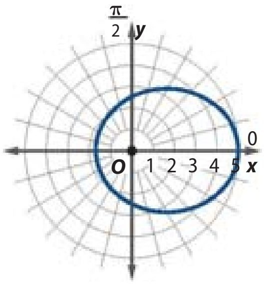
29. $r = \dfrac{4}{1 + 2 \sin\left(\theta + \frac{3\pi}{4}\right)}$

Match each polar equation with its graph.

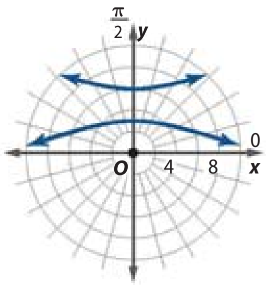
a.



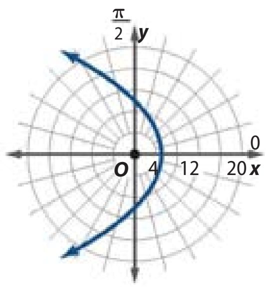
b.



c.



d.



30. $r = \dfrac{10}{1 + \cos \theta}$

31. $r = \dfrac{4}{1 - \sin \theta}$

32. $r = \dfrac{5}{2 - \cos \theta}$

33. $r = \dfrac{12}{1 + 3 \sin \theta}$

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. Then sketch the graph of the equation, and label the directrix.

34. $r = \dfrac{12}{2 - 0.75 \cos \theta}$

35. $r = \dfrac{1}{0.2 - 0.2 \sin \theta}$

36. $r = \dfrac{6}{1.2 \sin \theta + 0.3}$

37. $r = \dfrac{8}{\cos \theta + 5}$

38. **ASTRONOMY** The comet Borrelly travels in an elliptical orbit around the Sun with eccentricity $e = 0.624$. The point in a comet's orbit nearest to the Sun is defined as the *perihelion,* while the farthest point from the Sun is defined as the *aphelion.* The aphelion occurs at a distance of 5.83 AU (astronomical units, based on the distance between Earth and the Sun) from the Sun and the perihelion occurs at a distance of 1.35 AU. The diameter of the Sun is about 0.0093 AU.

a. Write a polar equation for the elliptical orbit of the comet Borrelly, and graph the equation.

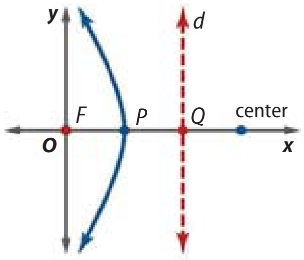
b. Determine the distance in miles between the comet Borrelly and the Sun at the aphelion and perihelion if 1 AU ≈ 93 million miles.

PROOF **Prove each of the following.**

39. $b = a\sqrt{1 - e^2}$ for an ellipse

40. $b = a\sqrt{e^2 - 1}$ for a hyperbola

41. PROOF Use the definition for the eccentricity of a conic, $PF = ePQ$, and the drawing of the hyperbola shown below, to verify that $d = \frac{a(e^2 - 1)}{e}$ for any hyperbola.



Write each rectangular equation in polar form.

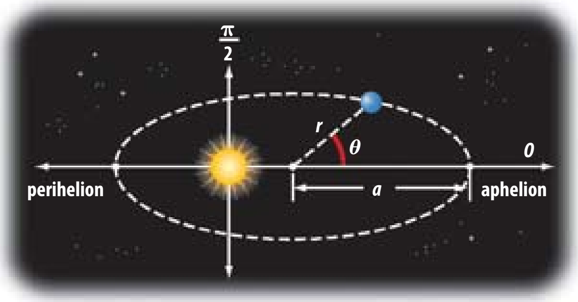
42. $x^2 = 4y + 4$

43. $-10y + 25 = x^2$

44. $\frac{(x - 2)^2}{16} + \frac{y^2}{12} = 1$

45. $\frac{(x + 4)^2}{64} + \frac{y^2}{48} = 1$

46. ASTRONOMY The planets travel around the Sun in approximately elliptical orbits with the Sun at one focus, as shown below.



a. Show that the polar equation of the planets' orbit can be written as $r = \frac{a(1 - e^2)}{(1 - e \cos \theta)}$.

b. Prove that the perihelion distance of any planet is $a(1 - e)$, and the aphelion distance is $a(1 + e)$.

c. Use the formulas from part **a** to find the perihelion and aphelion distances for each of the planets.

Planet	a	e	Planet	a	e
Earth	1.000	0.017	Neptune	30.06	0.009
Jupiter	5.203	0.048	Saturn	9.539	0.056
Mars	1.524	0.093	Uranus	19.18	0.047
Mercury	0.387	0.206	Venus	0.723	0.007

d. For which planet is the distance between the perihelion and aphelion the smallest? the greatest?

Write each equation in polar form. (*Hint*: Translate each conic so that a focus lies on the pole.)

47. $\frac{(x - 2)^2}{64} - \frac{y^2}{36} = 1$

48. $3(x + 5)^2 + 4y^2 = 192$

49. MULTIPLE REPRESENTATIONS In this problem, you will investigate the effects of varying the eccentricity and the directrix on graphs of conic sections.

a. NUMERICAL Write an equation for a conic section with focus (0, 0) and directrix $x = 3$ for $e = 0.4, 0.6, 1, 1.6$, and 2. Then identify the type of conic that each equation represents.

b. GRAPHICAL Graph and label the eccentricity for each of the equations that you found in part **a** on the same coordinate plane.

c. VERBAL Describe the changes in the graphs from part **b** as e approaches 2.

d. NUMERICAL Write an equation for a conic section with focus (0, 0) and eccentricity $e = 0.5$ for $d = 0.25, 1$, and 4.

e. GRAPHICAL Graph each of the equations on the same coordinate plane.

f. VERBAL Describe the relationship between the value of d and the distances between the vertices and the foci of the graphs from part **e**.

Derive each of the following polar equations of conics as shown on page 562 for the equation $r = \frac{ed}{1 + e \cos \theta}$. Include a diagram with each derivation.

50. $r = \frac{ed}{1 - e \cos \theta}$

51. $r = \frac{ed}{1 + e \sin \theta}$

52. $r = \frac{ed}{1 - e \sin \theta}$

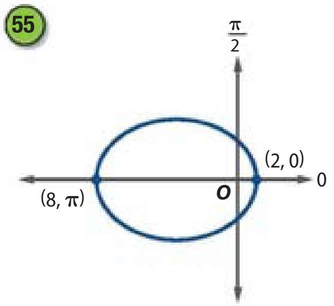
H.O.T. Problems Use Higher-Order Thinking Skills

53. WRITING IN MATH Describe two definitions that can be used to define a conic section.

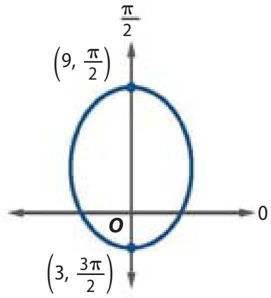
54. REASONING Explain why $r = \frac{ed}{1 + e \sin \theta}$ does not produce a true circle for any value of e.

CHALLENGE Determine a polar equation for the ellipse with the given vertices if one focus is at the pole.

55.



56.



57. WRITING IN MATH Explain how you can find the distance from the focus at (0, 0) to any point on the conic when the rectangular coordinates, polar coordinates, or θ is provided.

Spiral Review

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \le \theta \le 2\pi$. If necessary, round to the nearest hundredth. (Lesson 9-3)

58. $(-\sqrt{2}, \sqrt{2})$

59. $(-2, -5)$

60. $(8, -12)$

Identify and graph each classic curve. (Lesson 9-2)

61. $r = 3 + 3\cos\theta$

62. $r = -2\sin 3\theta$

63. $r = \frac{5}{2}\theta, \theta \ge 0$

Determine an equation of an ellipse with each set of characteristics. (Lesson 7-2)

64. co-vertices (5, 8), (5, 0); foci (8, 4), (2, 4)

65. major axis (−2, 4) to (8, 4); minor axis (3, 1) to (3, 7)

66. foci (1, −1), (9, −1); length of minor axis equals 6

67. **OLYMPICS** In the Olympic Games, team standings are determined according to each team's total points. Each type of Olympic medal earns a team a given number of points. Use the information to determine the Olympics in which the United States earned the most points. (Lesson 6-2)

Olympics	Gold	Silver	Bronze
1996	44	32	25
2000	37	24	31
2004	35	39	29
2008	36	38	36

Medal	Points
gold	3
silver	2
bronze	1

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval. (Lesson 5-5)

68. $\sin\theta = \frac{2}{3}, (0°, 90°)$

69. $\tan\theta = -\frac{24}{7}, \left(\frac{\pi}{2}, \pi\right)$

70. $\sin\theta = -\frac{4}{5}, \left(\pi, \frac{3\pi}{2}\right)$

Locate the vertical asymptotes, and sketch the graph of each function. (Lesson 4-5)

71. $y = \sec\left(x + \frac{\pi}{3}\right)$

72. $y = 4\cot\frac{x}{2}$

73. $y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$

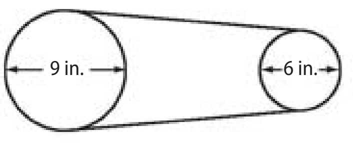
Find the exact values of the five remaining trigonometric functions of θ. (Lesson 4-3)

74. $\sec\theta = 2$, where $\sin\theta > 0$ and $\cos\theta > 0$

75. $\csc\theta = \sqrt{5}$, where $\sin\theta > 0$ and $\cos\theta > 0$

Skills Review for Standardized Tests

76. **SAT/ACT** A pulley with a 9-inch diameter is belted to a pulley with a 6-inch diameter, as shown in the figure. If the larger pulley runs at 120 rpm (revolutions per minute), how fast does the smaller pulley run?



A 80 rpm
B 120 rpm
C 160 rpm
D 180 rpm
E 200 rpm

77. What type of conic is given by $r = \frac{3}{2 - 0.5\cos\theta}$?

F circle
G ellipse
H parabola
J hyperbola

78. **REVIEW** Which of the following includes the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, 4, -2)$ and terminal point $B(-5, 2, 1)$?

A $\langle -8, -2, 3\rangle, \sqrt{77}$
B $\langle 8, -2, 3\rangle, \sqrt{77}$
C $\langle -8, -2, 3\rangle, \sqrt{109}$
D $\langle 8, -2, 3\rangle, \sqrt{109}$

79. **REVIEW** What is the eccentricity of the ellipse described by $\frac{y^2}{47} + \frac{(x - 12)^2}{34} = 1$?

F 0.38
G 0.41
H 0.53
J 0.62

LESSON 9-5 Complex Numbers and DeMoivre's Theorem

:: Then

- You performed operations with complex numbers written in rectangular form. (Lesson 0-6)

:: Now

1. Convert complex numbers from rectangular to polar form and vice versa.
2. Find products, quotients, powers, and roots of complex numbers in polar form.

:: Why?

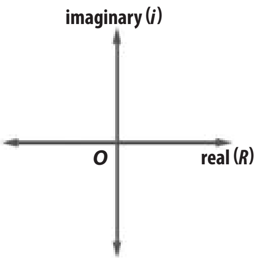
- Electrical engineers use complex numbers to describe certain relationships of electricity. Voltage E, impedance Z, and current I are the three quantities related by the equation $E = I \cdot Z$ used to describe alternating current. Each variable can be written as a complex number in the form $a + bj$, where j is an imaginary number (engineers use j to not be confused with current I). For impedance, the real part a represents the opposition to current flow due to resistors, and the imaginary part b is related to the opposition due to inductors and capacitors.

NewVocabulary

complex plane
real axis
imaginary axis
Argand plane
absolute value of a complex number
polar form
trigonometric form
modulus
argument
pth roots of unity

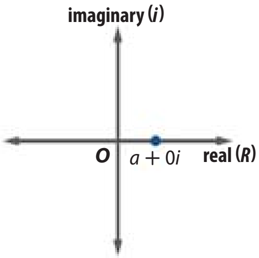
1 Polar Forms of Complex Numbers

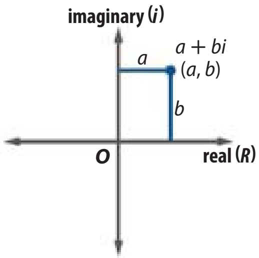
A complex number given in rectangular form, $a + bi$, has a real component a and an imaginary component bi. You can graph a complex number on the **complex plane** by representing it with the point (a, b). Similar to a coordinate plane, we need two axes to graph a complex number. The real component is plotted on the horizontal axis called the **real axis**, and the imaginary component is plotted on the vertical axis called the **imaginary axis**. The complex plane may also be referred to as the **Argand Plane** (ar GON).



Complex Plane

Consider a complex number where $b = 0$, $a + 0i$. The result is a real number a that can be graphed using just a real number line or the real axis. When $b \neq 0$, the imaginary axis is needed to represent the imaginary component.



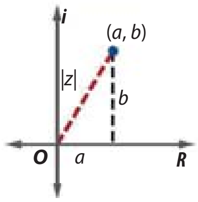


Recall that the absolute value of a real number is its distance from zero on the number line. Similarly, the **absolute value of a complex number** is its distance from zero in the complex plane. When $a + bi$ is graphed in the complex plane, the distance from zero can be calculated using the Pythagorean Theorem.

KeyConcept Absolute Value of a Complex Number

The absolute value of the complex number $z = a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

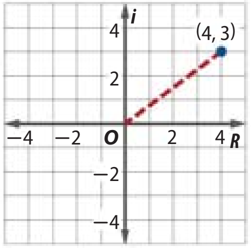


Example 1 Graphs and Absolute Values of Complex Numbers

Graph each number in the complex plane, and find its absolute value.

a. $z = 4 + 3i$

$(a, b) = (4, 3)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

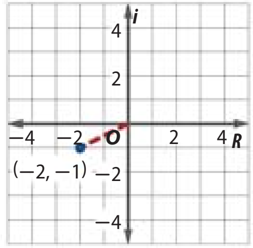
$= \sqrt{4^2 + 3^2}$ $a = 4$ and $b = 3$

$= \sqrt{25}$ or 5 Simplify.

The absolute value of $4 + 3i = 5$.

b. $z = -2 - i$

$(a, b) = (-2, -1)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

$= \sqrt{(-2)^2 + (-1)^2}$ $a = -2$ and $b = -1$

$= \sqrt{5}$ or 2.24 Simplify.

The absolute value of $-2 - i \approx 2.24$.

GuidedPractice

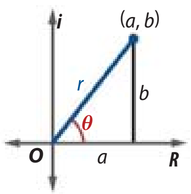
1A. $5 + 2i$

1B. $-3 + 4i$

WatchOut!

Polar Form The *polar form of a complex number* should not be confused with *polar coordinates of a complex number*. The polar form of a complex number is another way to represent a complex number. Polar coordinates of a complex number will be discussed later in this lesson.

Just as rectangular coordinates (x, y) can be written in polar form, so can the coordinates that represent the graph of a complex number in the complex plane. The same trigonometric ratios that were used to find values of x and y can be applied to represent values for a and b.



$\cos \theta = \frac{a}{r}$ and $\sin \theta = \frac{b}{r}$

$r \cos \theta = a$ $r \sin \theta = b$ Multiply each side by r.

Substituting the polar representations for a and b, we can calculate the **polar form** or **trigonometric form** of a complex number.

$z = a + bi$ Original complex number

$= r \cos \theta + (r \sin \theta)i$ $a = r \cos \theta$ and $b = r \sin \theta$

$= r(\cos \theta + i \sin \theta)$ Factor.

In the case of a complex number, r represents the absolute value, or **modulus**, of the complex number and can be found using the same process you used when finding the absolute value, $r = |z| = \sqrt{a^2 + b^2}$. The angle θ is called the **argument** of the complex number. Similar to finding θ with rectangular coordinates (x, y), when using a complex number, $\theta = \tan^{-1} \frac{b}{a}$ or $\theta = \tan^{-1} \frac{b}{a} + \pi$ if $a < 0$.

StudyTip

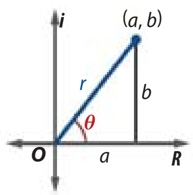
Argument The argument of a complex number is also called the *amplitude*. Just as in polar coordinates, θ is not unique, although it is normally given in the interval $-2\pi < \theta < 2\pi$.

KeyConcept Polar Form of a Complex Number

The polar or trigonometric form of the complex number $z = a + bi$ is

$$z = r(\cos \theta + i \sin \theta), \text{ where}$$

$r = |z| = \sqrt{a^2 + b^2}$, $a = r \cos \theta$, $b = r \sin \theta$, and $\theta = \tan^{-1} \frac{b}{a}$ for $a > 0$ or $\theta = \tan^{-1} \frac{b}{a} + \pi$ for $a < 0$.



Example 2 Complex Numbers in Polar Form

Express each complex number in polar form.

a. $-6 + 8i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ — Conversion formula

$= \sqrt{(-6)^2 + 8^2}$ or 10 — $a = -6$ and $b = 8$

$\theta = \tan^{-1}\frac{b}{a} + \pi$

$= \tan^{-1} -\frac{8}{6} + \pi$ or about 2.21

The polar form of $-6 + 8i$ is about $10(\cos 2.21 + i \sin 2.21)$.

b. $4 + \sqrt{3}i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ — Conversion formula

$= \sqrt{4^2 + (\sqrt{3})^2}$ — $a = 4$ and $b = \sqrt{3}$

$= \sqrt{19}$ or about 4.36 — Simplify.

$\theta = \tan^{-1}\frac{b}{a}$

$= \tan^{-1}\frac{\sqrt{3}}{4}$

≈ 0.41

The polar form of $4 + \sqrt{3}i$ is about $4.36(\cos 0.41 + i \sin 0.41)$.

GuidedPractice

2A. $9 + 7i$

2B. $-2 - 2i$

> **ReadingMath**
>
> **Polar Form** $r(\cos\theta + i\sin\theta)$ is often abbreviated as $r \text{ cis } \theta$. In Example 2a, $-6 + 8i$ can also be expressed as 10 cis 2.21, where $10 = \sqrt{(-6)^2 + 8^2}$ and $2.21 = \tan^{-1} -\frac{8}{6}$.

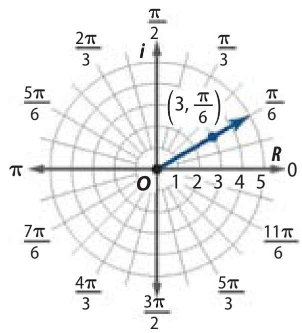
You can use the polar form of a complex number to graph the number on a polar grid by using the r and θ values as your polar coordinates (r, θ). You can also take a complex number written in polar form and convert it to rectangular form by evaluating.

Example 3 Graph and Convert the Polar Form of a Complex Number

Graph $z = 3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ on a polar grid. Then express it in rectangular form.

The value of r is 3, and the value of θ is $\frac{\pi}{6}$.

Plot the polar coordinates $\left(3, \frac{\pi}{6}\right)$.



To express the number in rectangular form, evaluate the trigonometric values and simplify.

$3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ — Polar form

$= 3\left[\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$ — Evaluate for cosine and sine.

$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$ — Distributive Property

The rectangular form of $z = 3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ is $z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

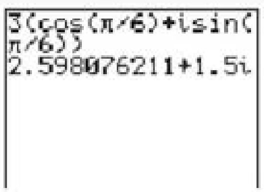
GuidedPractice

Graph each complex number on a polar grid. Then express it in rectangular form.

3A. $5\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

3B. $4\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right)$

> **TechnologyTip**
>
> **Complex Number Conversions** You can convert a complex number in polar form to rectangular form by entering the expression in polar form, then selecting ENTER. To be in polar mode, select MODE then a + b*i*.



2 Products, Quotients, Powers, and Roots of Complex Numbers

The polar form of complex numbers, along with the sum and difference formulas for cosine and sine, greatly aid in the multiplication and division of complex numbers. The formula for the product of two complex numbers in polar form can be derived by performing the multiplication.

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$	Original equation
$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$	FOIL
$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)]$	$i^2 = -1$ and group imaginary terms.
$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$	Factor out i.
$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$	Sum identities for cosine and sine

KeyConcept Product and Quotient of Complex Numbers in Polar Form

Given the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

Product Formula $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Quotient Formula $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$, where z_2 and $r_2 \neq 0$

You will prove the Quotient Formula in Exercise 77.

ReadingMath

Plural Forms *Moduli* is the plural of *modulus.*

Notice that when multiplying complex numbers, you multiply the moduli and add the arguments. When dividing, you divide the moduli and subtract the arguments.

Example 4 Product of Complex Numbers in Polar Form

Find $2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$	Original expression
$= 2(4)\left[\cos\left(\frac{5\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{5\pi}{3}+\frac{\pi}{6}\right)\right]$	Product Formula
$= 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$	Simplify.

Now find the rectangular form of the product.

$8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$	Polar form
$= 8\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$	Evaluate.
$= 4\sqrt{3} - 4i$	Distributive Property

The polar form of the product is $8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$. The rectangular form of the product is $4\sqrt{3} - 4i$.

GuidedPractice

Find each product in polar form. Then express the product in rectangular form.

4A. $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

4B. $-6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\cdot 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

As mentioned at the beginning of this lesson, quotients of complex numbers can be used to show relationships in electricity.

Real-WorldCareer

Electrical Engineers Electrical engineers design and create new technology used to manufacture global positioning systems, giant generators that power entire cities, turbine engines used in jet aircrafts, and radar and navigation systems. They also work on improving various products such as cell phones, cars, and robots.

Real-World Example 5 Quotient of Complex Numbers in Polar Form

ELECTRICITY **If a circuit has a voltage E of 150 volts and an impedance Z of $6 - 3j$ ohms, find the current I amps in the circuit in rectangular form. Use $E = I \cdot Z$.**

Express each number in polar form.

$150 = 150(\cos 0 + j \sin 0)$ — $r = \sqrt{150^2 + 0^2}$ or 150; $\theta = \tan^{-1}\frac{0}{150}$ or 0

$6 - 3j = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$ — $r = \sqrt{6^2 + (-3)^2}$ or $3\sqrt{5}$; $\theta = \tan^{-1} -\frac{3}{6}$ or about -0.46

Solve for the current I in $I \cdot Z = E$.

$I \cdot Z = E$	Original equation
$I = \frac{E}{Z}$	Divide each side by Z.
$I = \frac{150(\cos 0 + j \sin 0)}{3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]}$	$E = 150(\cos 0 + j \sin 0)$ and $Z = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$
$I = \frac{150}{3\sqrt{5}}\{\cos[0 - (-0.46)] + j \sin[0 - (-0.46)]\}$	Quotient Formula
$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$	Simplify.

Now convert the current to rectangular form.

$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$	Original equation
$= 10\sqrt{5}(0.90 + 0.44j)$	Evaluate.
$= 20.12 + 9.84j$	Distributive Property

The current is about $20.12 + 9.84j$ amps.

GuidedPractice

5. **ELECTRICITY** If a circuit has a voltage of 120 volts and a current of $8 + 6j$ amps, find the impedance of the circuit in rectangular form.

Before calculating the powers and roots of complex numbers, it may be helpful to express the complex numbers in polar form. Abraham DeMoivre (duh MWAHV ruh) is credited with discovering a useful pattern for evaluating powers of complex numbers.

We can use the formula for the product of complex numbers to help visualize the pattern that DeMoivre discovered.

First, find z^2 by taking the product of $z \cdot z$.

$z \cdot z = r(\cos\theta + i \sin\theta) \cdot r(\cos\theta + i \sin\theta)$	Multiply.
$z^2 = r^2[\cos(\theta + \theta) + i \sin(\theta + \theta)]$	Product Formula
$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$	Simplify.

Now find z^3 by calculating $z^2 \cdot z$.

$z^2 \cdot z = r^2(\cos 2\theta + i \sin 2\theta) \cdot r(\cos\theta + i \sin\theta)$	Multiply.
$z^3 = r^3[\cos(2\theta + \theta) + i \sin(2\theta + \theta)]$	Product Formula
$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$	Simplify.

Notice that when calculating these powers of a complex number, you take the nth power of the modulus and multiply the argument by n.

Math HistoryLink

Abraham DeMoivre
(1667–1754)
A French mathematician, DeMoivre is known for the theorem named for him and his book on probability theory, *The Doctrine of Chances*. He is credited with being one of the pioneers of analytic geometry and probability.

This pattern is summarized below.

KeyConcept DeMoivre's Theorem

If the polar form of a complex number is $z = r(\cos\theta + \boldsymbol{i}\sin\theta)$, then for positive integers n

$$z^n = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^n \text{ or } r^n(\cos n\theta + \boldsymbol{i}\sin n\theta).$$

You will prove DeMoivre's Theorem in Lesson 10-4.

Example 6 DeMoivre's Theorem

Find $(4 + 4\sqrt{3}\boldsymbol{i})^6$, and express it in rectangular form.

First, write $4 + 4\sqrt{3}\boldsymbol{i}$ in polar form.

$r = \sqrt{a^2 + b^2}$	Conversion formula	$\theta = \tan^{-1}\frac{b}{a}$
$= \sqrt{4^2 + (4\sqrt{3})^2}$	$a = 4$ and $b = 4\sqrt{3}$	$= \tan^{-1}\frac{4\sqrt{3}}{4}$
$= \sqrt{16 + 48}$	Simplify.	$= \tan^{-1}\sqrt{3}$
$= 8$	Simplify.	$= \frac{\pi}{3}$

The polar form of $4 + 4\sqrt{3}\boldsymbol{i}$ is $8\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$.

Now use DeMoivre's Theorem to find the sixth power.

$(4 + 4\sqrt{3}\boldsymbol{i})^6 = \left[8\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)\right]^6$	Original equation
$= 8^6\left[\cos 6\left(\frac{\pi}{3}\right) + \boldsymbol{i}\sin 6\left(\frac{\pi}{3}\right)\right]$	DeMoivre's Theorem
$= 262{,}144(\cos 2\pi + \boldsymbol{i}\sin 2\pi)$	Simplify.
$= 262{,}144(1 + 0\boldsymbol{i})$	Evaluate.
$= 262{,}144$	Simplify.

Therefore, $(4 + 4\sqrt{3}\boldsymbol{i})^6 = 262{,}144$.

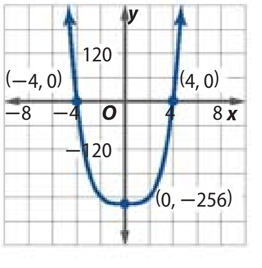
GuidedPractice

Find each power, and express it in rectangular form.

6A. $(1 + \sqrt{3}\boldsymbol{i})^4$

6B. $(2\sqrt{3} - 2\boldsymbol{i})^8$

In the real number system, $x^4 = 256$ has two solutions, 4 and -4. The graph of $y = x^4 - 256$ shows that there are two real zeros at $x = 4$ and -4. In the complex number system, however, there are two real solutions and two complex solutions.



In Lesson 2-4, you learned through the Fundamental Theorem of Algebra that polynomials of degree n have exactly n zeros in the complex number system. Therefore, the equation $x^4 = 256$, rewritten as $x^4 - 256 = 0$, has exactly four solutions, or roots: 4, -4, $4\boldsymbol{i}$, and $-4\boldsymbol{i}$. In general, all nonzero complex numbers have p distinct pth roots. That is, they each have two square roots, three cube roots, four fourth roots, and so on.

Review Vocabulary

Fundamental Theorem of Algebra A polynomial function of degree n, where $n > 0$, has at least one zero (real or imaginary) in the complex number system. (Lesson 2-4)

To find all of the roots of a polynomial, we can use DeMoivre's Theorem to arrive at the following expression.

Key Concept Distinct Roots

For a positive integer p, the complex number $r(\cos\theta + i\sin\theta)$ has p distinct pth roots. They are found by

$$r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + i\sin\frac{\theta + 2n\pi}{p}\right),$$

where $n = 0, 1, 2, \ldots, p - 1$.

We can use this formula for the different values of n, but we can stop when $n = p - 1$. When n equals or exceeds p, the roots repeat as the following shows.

$$\frac{\theta + 2\pi p}{p} = \frac{\theta}{p} + 2\pi \qquad \text{Coterminal with } \frac{\theta}{p}\text{, when } n = 0$$

Example 7 *p*th Roots of a Complex Number

Find the fourth roots of $-4 - 4i$.

First, write $-4 - 4i$ in polar form.

$$-4 - 4i = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \qquad r = \sqrt{(-4)^2 + (-4)^2} \text{ or } 4\sqrt{2};\ \theta = \tan^{-1}\frac{-4}{-4} + \pi \text{ or } \frac{5\pi}{4}$$

Now write an expression for the fourth roots.

$$(4\sqrt{2})^{\frac{1}{4}}\left(\cos\frac{\frac{5\pi}{4} + 2n\pi}{4} + i\sin\frac{\frac{5\pi}{4} + 2n\pi}{4}\right) \qquad \theta = \frac{5\pi}{4},\ p = 4, \text{ and } r^{\frac{1}{p}} = (4\sqrt{2})^{\frac{1}{4}}$$

$$= \sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right)\right] \qquad \text{Simplify.}$$

Let $n = 0, 1, 2,$ and 3 successively to find the fourth roots.

Let $n = 0$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right)\right]$ Distinct Roots

$= \sqrt[8]{32}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)$ or $0.86 + 1.28i$ First fourth root

Let $n = 1$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{13\pi}{16} + i\sin\frac{13\pi}{16}\right)$ or $-1.28 + 0.86i$ Second fourth root

Let $n = 2$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{21\pi}{16} + i\sin\frac{21\pi}{16}\right)$ or $-0.86 - 1.28i$ Third fourth root

Let $n = 3$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{29\pi}{16} + i\sin\frac{29\pi}{16}\right)$ or $1.28 - 0.86i$ Fourth fourth root

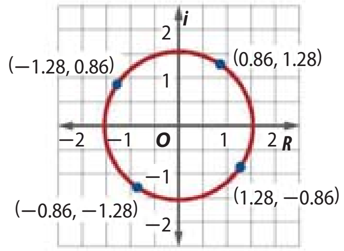
The fourth roots of $-4 - 4i$ are approximately $0.86 + 1.28i$, $-1.28 + 0.86i$, $-0.86 - 1.28i$, and $1.28 - 0.86i$.

Guided Practice

7A. Find the cube roots of $2 + 2i$.

7B. Find the fifth roots of $4\sqrt{3} - 4i$.

We can make observations about the distinct roots of a number by graphing the roots on a coordinate plane. As shown at the right, the four fourth roots found in Example 7 lie on a circle. If we look at the polar form of each complex number, each has the same modulus of $\sqrt[8]{32}$, which acts as the radius of the circle. The roots are also equally spaced around the circle as a result of the arguments differing by $\frac{\pi}{2}$.



A special case of finding roots occurs when finding the pth roots of 1. When 1 is written in polar form, $r = 1$. As mentioned in the previous paragraph, the modulus of our roots is the radius of the circle that is formed from plotting the roots on a coordinate plane. Thus, the pth roots of 1 lie on the unit circle. Finding the pth roots of 1 is referred to as finding the **pth roots of unity**.

StudyTip

The pth Roots of a Complex Number Each root will have the same modulus of $r^{\frac{1}{p}}$. The argument of the first root is $\frac{\theta}{p}$, and each successive root is found by repeatedly adding $\frac{2\pi}{p}$ to the argument.

Example 8 The pth Roots of Unity

Find the eighth roots of unity.

First, write 1 in polar form.

$1 = 1 \cdot (\cos 0 + i \sin 0)$ — $r = \sqrt{1^2 + 0^2}$ or 1 and $\theta = \tan^{-1} \frac{0}{1}$ or 0

Now write an expression for the eighth roots.

$1\left(\cos \frac{0 + 2n\pi}{8} + i \sin \frac{0 + 2n\pi}{8}\right)$ — $\theta = 0$, $p = 8$, and $r^{\frac{1}{p}} = 1^{\frac{1}{8}}$ or 1

$= \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4}$ — Simplify.

Let $n = 0$ to find the first root of 1.

$n = 0 \qquad \cos \frac{(0)\pi}{4} + i \sin \frac{(0)\pi}{4}$ — Distinct Roots

$= \cos 0 + i \sin 0$ or 1 — First root

Notice that the modulus of each complex number is 1. The arguments are found by $\frac{n\pi}{4}$, resulting in θ increasing by $\frac{\pi}{4}$ for each successive root. Therefore, we can calculate the remaining roots by adding $\frac{\pi}{4}$ to each previous θ.

$\cos 0 + i \sin 0$ or 1 — 1st root

$\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$ or $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ — 2nd root

$\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ or i — 3rd root

$\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$ or $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ — 4th root

$\cos \pi + i \sin \pi$ or -1 — 5th root

$\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}$ or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ — 6th root

$\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$ or $-i$ — 7th root

$\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}$ or $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ — 8th root

The eighth roots of 1 are 1, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, i, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, -1, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $-i$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ as shown in Figure 9.5.1.

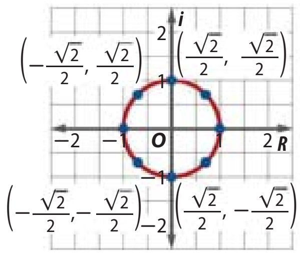


Figure 9.5.1

GuidedPractice

8A. Find the cube roots of unity.

8B. Find the seventh roots of unity.

Exercises

= Step-by-Step Solutions begin on page R29.

Graph each number in the complex plane, and find its absolute value. (Example 1)

1. $z = 4 + 4i$

2. $z = -3 + i$

3. $z = -4 - 6i$

4. $z = 2 - 5i$

5. $z = 3 + 4i$

6. $z = -7 + 5i$

7. $z = -3 - 7i$

8. $z = 8 - 2i$

9. **VECTORS** The force on an object is given by $z = 10 + 15i$, where the components are measured in newtons (N). (Example 1)

a. Represent z as a vector in the complex plane.

b. Find the magnitude and direction angle of the vector.

Express each complex number in polar form. (Example 2)

10. $4 + 4i$

11. $-2 + i$

12. $4 - \sqrt{2}i$

13. $2 - 2i$

14. $4 + 5i$

15. $-2 + 4i$

16. $-1 - \sqrt{3}i$

17. $3 + 3i$

Graph each complex number on a polar grid. Then express it in rectangular form. (Example 3)

18. $10(\cos 6 + i \sin 6)$

19. $2(\cos 3 + i \sin 3)$

20. $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

21. $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

22. $\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$

23. $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

24. $-3(\cos 180° + i \sin 180°)$

25. $\frac{3}{2}(\cos 360° + i \sin 360°)$

Find each product or quotient, and express it in rectangular form. (Examples 4 and 5)

26. $6\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \cdot 4\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

27. $5(\cos 135° + i \sin 135°) \cdot 2\,(\cos 45° + i \sin 45°)$

28. $3\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div \frac{1}{2}(\cos \pi + i \sin \pi)$

29. $2(\cos 90° + i \sin 90°) \cdot 2(\cos 270° + i \sin 270°)$

30. $3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \div 4\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

31. $4\left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}\right) \div 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$

32. $\frac{1}{2}(\cos 60° + i \sin 60°) \cdot 6(\cos 150° + i \sin 150°)$

33. $6\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div 2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

34. $5(\cos 180° + i \sin 180°) \cdot 2(\cos 135° + i \sin 135°)$

35. $\frac{1}{2}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \div 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

Find each power, and express it in rectangular form. (Example 6)

36. $(2 + 2\sqrt{3}i)^6$

37. $(12i - 5)^3$

38. $\left[4\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\right]^4$

39. $(\sqrt{3} - i)^3$

40. $(3 - 5i)^4$

41. $(2 + 4i)^4$

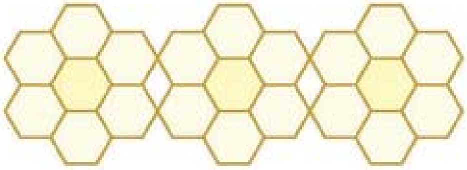
42. $(3 - 6i)^4$

43. $(2 + 3i)^2$

44. $\left[3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right]^3$

45. $\left[2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]^4$

46. **DESIGN** Stella works for an advertising agency. She wants to incorporate a design comprised of regular hexagons as the artwork for one of her proposals. Stella can locate the vertices of one of the central regular hexagons by graphing the solutions to $x^6 - 1 = 0$ in the complex plane. Find the vertices of this hexagon. (Example 7)



Find all of the distinct pth roots of the complex number. (Examples 7 and 8)

47. sixth roots of i

48. fifth roots of $-i$

49. fourth roots of $4\sqrt{3} - 4i$

50. cube roots of $-117 + 44i$

51. fifth roots of $-1 + 11\sqrt{2}i$

52. square root of $-3 - 4i$

53. find the square roots of unity

54. find the fourth roots of unity

55 **ELECTRICITY** The impedance in one part of a series circuit is $5(\cos 0.9 + j \sin 0.9)$ ohms. In the second part of the circuit, it is $8(\cos 0.4 + j \sin 0.4)$ ohms.

a. Convert each expression to rectangular form.

b. Add your answers from part **a** to find the total impedance in the circuit.

c. Convert the total impedance back to polar form.

Find each product. Then repeat the process by multiplying the polar forms of each pair of complex numbers using the Product Formula.

56. $(1 - i)(4 + 4i)$

57. $(3 + i)(3 - i)$

58. $(4 + i)(3 - i)$

59. $(-6 + 5i)(2 - 3i)$

60. $(\sqrt{2} + 2i)(1 + i)$

61. $(3 - 2i)(1 + \sqrt{3}i)$

62. **FRACTALS** A *fractal* is a geometric figure that is made up of a pattern that is repeated indefinitely on successively smaller scales, as shown below.

In this problem, you will generate a fractal through iterations of $f(z) = z^2$. Consider $z_0 = 0.8 + 0.5\boldsymbol{i}$.

a. Calculate $z_1, z_2, z_3, z_4, z_5, z_6$, and z_7 where $z_1 = f(z_0)$, $z_2 = f(z_1)$, and so on.

b. Graph each of the numbers on the complex plane.

c. Predict the location of z_{100}. Explain.

63. **TRANSFORMATIONS** There are certain operations with complex numbers that correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (3 - 4\boldsymbol{i})$

b. w is the complex conjugate of z.

c. $w = \boldsymbol{i} \cdot z$

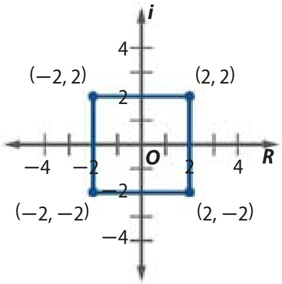
d. $w = 0.25z$

Find z and the pth roots of z given each of the following.

64. $p = 3$, one cube root is $\frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$

65. $p = 4$, one fourth root is $-1 - \boldsymbol{i}$

66. **GRAPHICS** By representing each vertex by a complex number in polar form, a programmer dilates and then rotates the square below 45° counterclockwise so that the new vertices lie at the midpoints of the sides of the original square.



a. By what complex number should the programmer multiply each number to produce this transformation?

b. What happens if the numbers representing the original vertices are multiplied by the square of your answer to part **a**?

Use the Distinct Roots Formula to find all of the solutions of each equation. Express the solutions in rectangular form.

67. $x^3 = \boldsymbol{i}$

68. $x^3 + 3 = 128$

69. $x^4 = 81\boldsymbol{i}$

70. $x^5 - 1 = 1023$

71. $x^3 + 1 = \boldsymbol{i}$

72. $x^4 - 2 + \boldsymbol{i} = -1$

H.O.T. Problems Use Higher-Order Thinking Skills

73. **ERROR ANALYSIS** Alma and Blake are evaluating $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}\right)^5$. Alma uses DeMoivre's Theorem and gets an answer of $\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}$. Blake tells her that she has only completed part of the problem. Is either of them correct? Explain your reasoning.

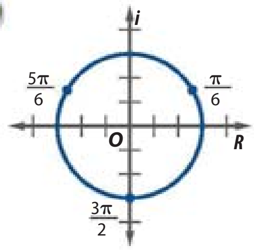
74. **REASONING** Suppose $z = a + b\boldsymbol{i}$ is one of the 29th roots of 1.

a. What is the maximum value of a?

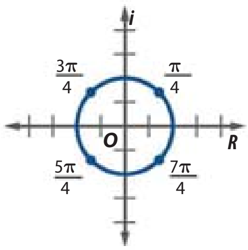
b. What is the maximum value of b?

CHALLENGE **Find the roots shown on each graph and write them in polar form. Then identify the complex number with the given roots.**

75.



76.



77. **PROOF** Given $z_1 = r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)$, where $r_2 \neq 0$, prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)]$.

REASONING **Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.**

78. The pth roots of a complex number z are equally spaced around the circle centered at the origin with radius $r^{\frac{1}{p}}$.

79. The complex conjugate of $z = a + b\boldsymbol{i}$ is $\bar{z} = a - b\boldsymbol{i}$. For any z, $z + \bar{z}$ and $z \cdot \bar{z}$ are real numbers.

80. **OPEN ENDED** Find two complex numbers $a + b\boldsymbol{i}$ in which $a \neq 0$ and $b \neq 0$ with an absolute value of $\sqrt{17}$.

81. **WRITING IN MATH** Explain why the sum of the imaginary parts of the p distinct pth roots of any positive real number must be zero. (*Hint*: The roots are the vertices of a regular polygon.)

Spiral Review

Write each polar equation in rectangular form. (Lesson 9-4)

82. $r = \frac{15}{1 + 4\cos\theta}$

83. $r = \frac{14}{2\cos\theta + 2}$

84. $r = \frac{-6}{\sin\theta - 2}$

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Lesson 9-3)

85. $(x - 3)^2 + y^2 = 9$

86. $x^2 - y^2 = 1$

87. $x^2 + y^2 = 2y$

Graph the conic given by each equation. (Lesson 7-4)

88. $y = x^2 + 3x + 1$

89. $y^2 - 2x^2 - 16 = 0$

90. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

Find the center, foci, and vertices of each ellipse. (Lesson 7-2)

91. $\frac{(x + 8)^2}{9} + \frac{(y - 7)^2}{81} = 1$

92. $25x^2 + 4y^2 + 150x + 24y = -161$

93. $4x^2 + 9y^2 - 56x + 108y = -484$

Solve each system of equations using Gauss-Jordan elimination. (Lesson 6-1)

94. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

95. $9g + 7h = -30$
$8h + 5j = 11$
$-3g + 10j = 73$

96. $2k - n = 2$
$3p = 21$
$4k + p = 19$

97. POPULATION In the beginning of 2008, the world's population was about 6.7 billion. If the world's population grows continuously at a rate of 2%, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2008. (Lesson 3-4)

a. According to this model, what will be the world's population in 2018?

b. Some experts have estimated that the world's food supply can support a population of at most 18 billion people. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

Skills Review for Standardized Tests

98. SAT/ACT The graph on the xy-plane of the quadratic function g is a parabola with vertex at $(3, -2)$. If $g(0) = 0$, then which of the following must also equal 0?

A $g(2)$
B $g(3)$
C $g(4)$
D $g(6)$
E $g(7)$

99. Which of the following expresses the complex number $20 - 21i$ in polar form?

F $29(\cos 5.47 + i \sin 5.47)$
G $29(\cos 5.52 + i \sin 5.52)$
H $32(\cos 5.47 + i \sin 5.47)$
J $32(\cos 5.52 + i \sin 5.52)$

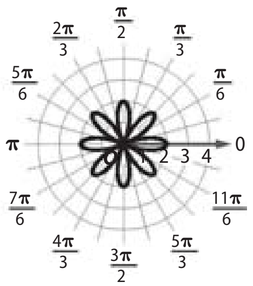
100. FREE RESPONSE Consider the graph at the right.

a. Give a possible equation for the function.

b. Describe the symmetries of the graph.

c. Give the zeroes of the function over the domain $0 \le \theta \le 2\pi$.

d. What is the minimum value of r over the domain $0 \le \theta \le 2\pi$?



CHAPTER 9

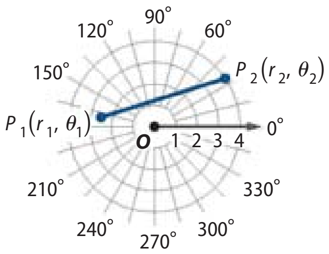
Study Guide and Review

Chapter Summary

KeyConcepts

Polar Coordinates (Lesson 9-1)

- In the polar coordinate system, a point (r, θ) is located using its directed distance r and directed angle θ.
- The distance between $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in the polar plane is $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$.



Graphs of Polar Equations (Lesson 9-2)

- Circle: $r = a \cos \theta$ or $r = a \sin \theta$
- Limaçon: $r = a \pm b \cos \theta$ or $r = a \pm b \sin \theta$, $a > 0$, $b > 0$
- Rose: $r = a \cos n\theta$ or $r = a \sin n\theta$, $n \geq 2$, $n \in \mathbb{Z}$
- Lemniscate: $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$
- Spirals of Archimedes: $r = a\theta + b$, $\theta \geq 0$

Polar and Rectangular Forms of Equations (Lesson 9-3)

- A point $P(r, \theta)$ has rectangular coordinates $(r \cos \theta, r \sin \theta)$.
- To convert a point $P(x, y)$ from rectangular to polar coordinates, use the equations $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$, when $x > 0$ or $\theta = \tan^{-1} \frac{y}{x} + \pi$, when $x < 0$.

Polar Forms of Conic Sections (Lesson 9-4)

- The polar equation of a conic section is of the form $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$, depending on the location and orientation of the directrix.

Complex Numbers and DeMoivre's Theorem (Lesson 9-5)

- The polar or trigonometric form of the complex number $a + bi$ is $r(\cos \theta + \boldsymbol{i} \sin \theta)$.
- The product formula for two complex numbers z_1 and z_2 is $z_1z_2 = r_1r_2 [\cos(\theta_1 + \theta_2) + \boldsymbol{i} \sin(\theta_1 + \theta_2)]$.
- The quotient formula for two complex numbers z_1 and z_2 is $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)]$, where z_2 and $r_2 \neq 0$.
- DeMoivre's Theorem states that if the polar form of a complex number is $z = r(\cos \theta + \boldsymbol{i} \sin \theta)$, then $z^n = r^n(\cos n\theta + \boldsymbol{i} \sin n\theta)$ for positive integers n.

KeyVocabulary

absolute value of a complex number (p. 569)
Argand plane (p. 569)
argument (p. 570)
cardioid (p.544)
complex plane (p. 569)
imaginary axis (p. 569)
lemniscate (p. 546)
limaçon (p. 543)
modulus (p. 570)
polar axis (p. 534)
polar coordinate system (p. 534)
polar coordinates (p. 534)
polar equation (p. 536)
polar form (p. 570)
polar graph (p. 536)
pole (p. 534)
*p*th roots of unity (p. 576)
real axis (p. 569)
rose (p. 545)
spiral of Archimedes (p. 546)
trigonometric form (p. 570)

VocabularyCheck

Choose the correct term from the list above to complete each sentence.

1. A(n) ________ is the set of all points with coordinates (r, θ) that satisfy a given polar equation.
2. A plane that has an axis for the real component and an axis for the imaginary component is a(n) ________.
3. The location of a point in the ________ is identified using the directed distance from a fixed point and the angle from a fixed axis.
4. A special type of limaçon with equation of the form $r = a + b \cos \theta$ where $a = b$ is called a(n) ________.
5. The ________ is the angle θ of a complex number written in the form $r(\cos \theta + \boldsymbol{i} \sin \theta)$.
6. The origin of a polar coordinate system is called the ________.
7. The absolute value of a complex number is also called the ________.
8. The ________ is another name for the complex plane.
9. The graph of a polar equation of the form $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$ is called a(n) ________.
10. The ________ is an initial ray from the pole, usually horizontal and directed toward the right.

Lesson-by-Lesson Review

9-1 Polar Coordinates (pp. 534–540)

Graph each point on a polar grid.

11. $W(-0.5, 210°)$

12. $X\left(1.5, \frac{7\pi}{4}\right)$

13. $Y(4, -120°)$

14. $Z\left(-3, \frac{5\pi}{6}\right)$

Graph each polar equation.

15. $\theta = -60°$

16. $r = \frac{9}{2}$

17. $r = 7$

18. $\theta = \frac{11\pi}{6}$

Find the distance between each pair of points.

19. $\left(5, \frac{\pi}{2}\right), \left(2, -\frac{7\pi}{6}\right)$

20. $(-3, 60°), (4, 240°)$

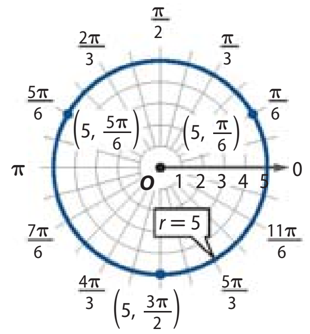
21. $(-1, -45°), (6, 270°)$

22. $\left(7, \frac{5\pi}{6}\right), \left(2, \frac{4\pi}{3}\right)$

Example 1

Graph $r = 5$.

The solutions of $r = 5$ are ordered pairs of the form $(5, \theta)$ where θ is any real number. The graph consists of all points that are 5 units from the pole, so the graph is a circle centered at the pole with radius 5.



9-2 Graphs of Polar Equations (pp. 542–550)

Use symmetry, zeros, and maximum r-values to graph each function.

23. $r = \sin 3\theta$

24. $r = 2 \cos \theta$

25. $r = 5 \cos 2\theta$

26. $r = 4 \sin 4\theta$

27. $r = 2 + 2 \cos \theta$

28. $r = 1.5\theta, \theta \geq 0$

Use symmetry to graph each equation.

29. $r = 2 - \sin \theta$

30. $r = 1 + 5 \cos \theta$

31. $r = 3 - 2 \cos \theta$

32. $r = 4 + 4 \sin \theta$

33. $r = -3 \sin \theta$

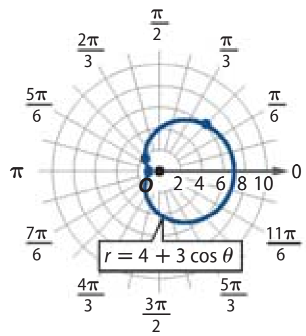
34. $r = -5 + 3 \cos \theta$

Example 2

Use symmetry to graph $r = 4 + 3 \cos \theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 4 + 3 \cos(-\theta)$, which simplifies to $r = 4 + 3 \cos \theta$ because cosine is even. The equations are equivalent, so the graph of this equation is symmetric with respect to the polar axis. Therefore, you can make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

θ	r
0	7
$\frac{\pi}{4}$	$\frac{8 + 3\sqrt{2}}{2}$
$\frac{\pi}{2}$	4
$\frac{3\pi}{4}$	$\frac{8 - 3\sqrt{2}}{2}$
π	1



By plotting these points and using polar axis symmetry, you obtain the graph shown.

Study Guide and Review *Continued*

9-3 Polar and Rectangular Forms (pp. 551–559)

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth.

35. $(-1, 5)$

36. $(3, 7)$

37. $(2a, 0), a > 0$

38. $(4b, -6b), b > 0$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

39. $r = 5$

40. $r = -4 \sin \theta$

41. $r = 6 \sec \theta$

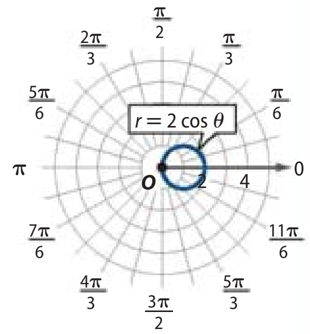
42. $r = \frac{1}{3} \csc \theta$

Example 3

Write $r = 2 \cos \theta$ in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

$r = 2 \cos \theta$	Original equation
$r^2 = 2r \cos \theta$	Multiply each side by r.
$x^2 + y^2 = 2x$	$r^2 = x^2 + y^2$ and $x = r \cos \theta$
$x^2 + y^2 - 2x = 0$	Subtract $2x$ from each side.

In standard form, $(x - 1)^2 + y^2 = 1$, you can identify the graph of this equation as a circle centered at $(1, 0)$ with radius 1, as supported by the graph of $r = 2 \cos \theta$.



9-4 Polar Forms of Conic Sections (pp. 561–568)

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

43. $r = \frac{3.5}{1 + \sin \theta}$

44. $r = \frac{1.2}{1 + 0.3 \cos \theta}$

45. $r = \frac{14}{1 - 2 \sin \theta}$

46. $r = \frac{6}{1 - \cos \theta}$

Write and graph a polar equation and directrix for the conic with the given characteristics.

47. $e = 0.5$; vertices at $(0, -2)$ and $(0, 6)$

48. $e = 1.5$; directrix: $x = 5$

Write each polar equation in rectangular form.

49. $r = \frac{1.6}{1 - 0.2 \sin \theta}$

50. $r = \frac{5}{1 + \cos \theta}$

Example 4

Determine the eccentricity, type of conic, and equation of the directrix for $r = \frac{7}{3.5 - 3.5 \cos \theta}$.

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

$r = \frac{7}{3.5 - 3.5 \cos \theta}$	Original equation
$r = \frac{3.5(2)}{3.5(1 - \cos \theta)}$	Factor the numerator and denominator.
$r = \frac{2}{1 - \cos \theta}$	Divide the numerator and denominator by 3.5.

In this form, you can see from the denominator that $e = 1$; therefore, the conic is a parabola. For polar equations of this form, the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 2$, so $d = 2 \div 1$ or 2. Therefore, the equation of the directrix is $x = -2$.

9-5 Complex Numbers and DeMoivre's Theorem (pp. 569–579)

Graph each number in the complex plane, and find its absolute value.

51. $z = 3 - i$

52. $z = 4i$

53. $z = -4 + 2i$

54. $z = 6 - 3i$

Express each complex number in polar form.

55. $3 + \sqrt{2}i$

56. $-5 + 8i$

57. $-4 - \sqrt{3}i$

58. $\sqrt{2} + \sqrt{2}i$

Graph each complex number on a polar grid. Then express it in rectangular form.

59. $z = 3\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

60. $z = 5\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

61. $z = -2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

62. $z = 4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

Find each product or quotient, and express it in rectangular form.

63. $-2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) \cdot -4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

64. $8(\cos 225° + i \sin 225°) \cdot \frac{1}{2}(\cos 120° + i \sin 120°)$

65. $5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \div \frac{1}{3}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

66. $6(\cos 210° + i \sin 210°) \div 3(\cos 150° + i \sin 150°)$

Find each power, and express it in rectangular form.

67. $(4 - i)^5$

68. $(\sqrt{2} + 3i)^4$

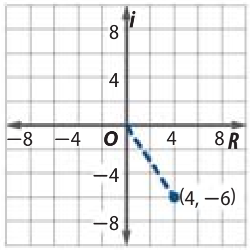
Find all of the distinct pth roots of the complex number.

69. cube roots of $6 - 4i$

70. fourth roots of $1 + i$

Example 5

Graph $4 - 6i$ in the complex plane and express in polar form.



Find the modulus.

$r = \sqrt{a^2 + b^2}$ — Conversion formula

$= \sqrt{4^2 + (-6)^2}$ or $2\sqrt{13}$ — $a = 4$ and $b = -6$

Find the argument.

$\theta = \tan^{-1}\frac{b}{a}$ — Conversion formula

$= \tan^{-1}\left(-\frac{6}{4}\right)$ — $a = 4$ and $b = -6$

$= -0.98$ — Simplify.

The polar form of $4 - 6i$ is approximately $2\sqrt{13}\,[\cos(-0.98) + i \sin(-0.98)]$.

Example 6

Find $-3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \cdot 5\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$-3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \cdot 5\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$ — Original expression

$= (-3 \cdot 5)\left[\cos\left(\frac{\pi}{4} + \frac{7\pi}{6}\right) + i \sin\left(\frac{\pi}{4} + \frac{7\pi}{6}\right)\right]$ — Product Formula

$= -15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$ — Simplify.

Now find the rectangular form of the product.

$-15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$ — Polar form

$= -15[-0.26 + i(-0.97)]$ — Evaluate.

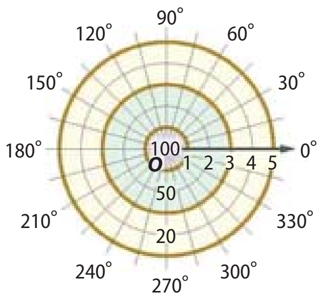
$= 3.9 + 14.5i$ — Distributive Property

The polar form of the product is $-15\left[\cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)\right]$

The rectangular form of the product is $3.9 + 14.5i$.

Applications and Problem Solving

71. GAMES An arcade game consists of rolling a ball up an incline at a target. The region in which the ball lands determines the number of points earned. The model shows the point value for each region. (Lesson 9-1)



a. If, on a turn, a player rolls the ball to the point (3.5, 165°), how many points does he get?

b. Give two possible locations that a player will receive 50 points.

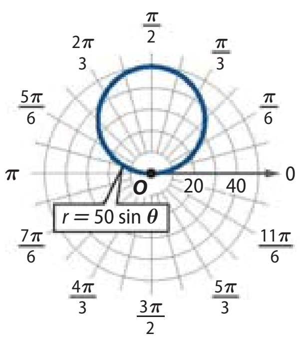
72. LANDSCAPING A landscaping company uses an adjustable lawn sprinkler that can rotate 360° and can cover a circular region with radius of 20 feet. (Lesson 9-1)

a. Graph the dimensions of the region that the sprinkler can cover on a polar grid if it is set to rotate 360°.

b. Find the area of the region that the sprinkler covers if the rotation is adjusted to $-30° \leq \theta \leq 210°$.

73. BIOLOGY The pattern on the shell of a snail can be modeled using $r = \frac{1}{3}\theta + \frac{1}{2}$, $\theta \geq 0$. Identify and graph the classic curve that models this pattern. (Lesson 9-2)

74. RIDES The path of a Ferris wheel can be modeled by $r = 50 \sin \theta$, where r is given in feet. (Lesson 9-3)



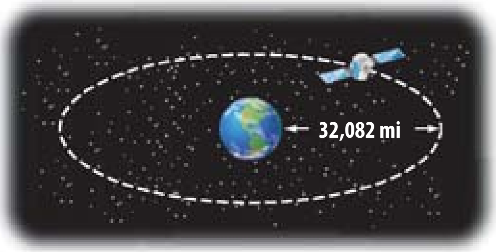
a. What are the polar coordinates of a rider located at $\theta = \frac{\pi}{12}$? Round to the nearest tenth, if necessary.

b. What are the rectangular coordinates of the rider's location? Round to the nearest tenth, if necessary.

c. What is the rider's distance above the ground if the polar axis represents the ground?

75. ORIENTEERING Orienteering requires participants to make their way through an area using a topographic map. One orienteer starts at Checkpoint A and walks 5000 feet at an angle of 35° measured clockwise from due east. A second orienteer starts at Checkpoint A and walks 3000 feet due west and then 2000 feet due north. How far, to the nearest foot, are the two orienteers from each other? (Lesson 9-3)

76. SATELLITE The orbit of a satellite around Earth has eccentricity of 0.05, and the distance from a vertex of the path to the center of Earth is 32,082 miles. Write a polar equation that can be used to model the path of the satellite if Earth is located at the focus closest to the given vertex. (Lesson 9-4)

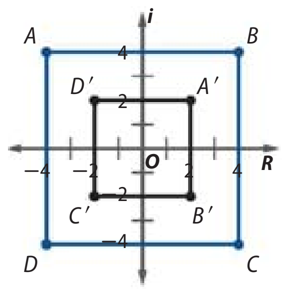


77. ELECTRICITY Most circuits in Europe are designed to accommodate 220 volts. For parts **a** and **b**, use $E = I \cdot Z$, where voltage E is measured in volts, impedance Z is measured in ohms, and current I is measured in amps. Round to the nearest tenth. (Lesson 9-5)

a. If the circuit has a current of $2 + 5j$ amps, what is the impedance?

b. If a circuit has an impedance of $1 - 3j$ ohms, what is the current?

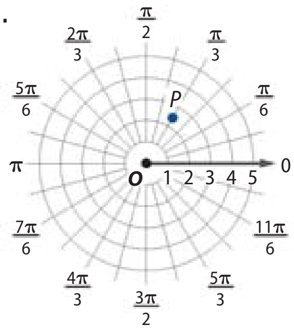
78. COMPUTER GRAPHICS Geometric transformation of figures can be performed using complex numbers. If a programmer starts with square $ABCD$, as shown below, each of the vertices can be represented by a complex number in polar form. Multiplication can then be used to rotate and dilate the square, producing the square $A'B'C'D'$. By what complex number should the programmer multiply each number to produce this transformation? (Lesson 9-5)



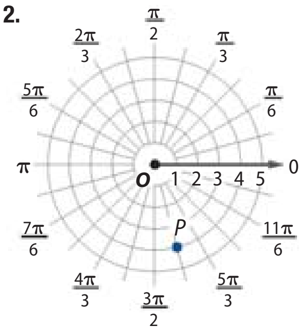
CHAPTER 9 Practice Test

Find four different pairs of polar coordinates that name point P if $-2\pi \le \theta \le 2\pi$.

1.



2.



Graph each polar equation.

3. $\theta = 30°$

4. $r = 1$

5. $r = 2.5$

6. $\theta = \frac{5\pi}{3}$

7. $r = \frac{2}{3}\sin\theta$

8. $r = -\frac{1}{2}\sec\theta$

9. $r = -4\csc\theta$

10. $r = 2\cos\theta$

Identify and graph each classic curve.

11. $r = 1.5 + 1.5\cos\theta$

12. $r^2 = 6.25\sin 2\theta$

13. RADAR An air traffic controller is tracking an airplane with a current location of (66, 115°). The value of r is given in miles.

a. What are the rectangular coordinates of the airplane? Round to the nearest tenth mile.

b. If a second plane is located at the point (50, −75), what are the polar coordinates of the plane if $r > 0$ and $0 \le \theta \le 360°$? Round to the nearest mile and the nearest tenth of a degree, if necessary.

c. What is the distance between the two planes? Round to the nearest mile.

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

14. $(x - 7)^2 + y^2 = 49$

15. $y = 3x^2$

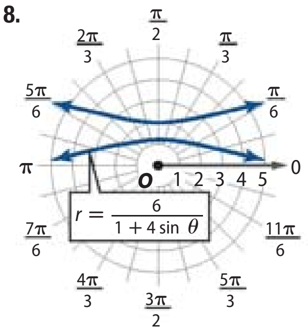
Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

16. $r = \frac{2}{1 - 0.4\sin\theta}$

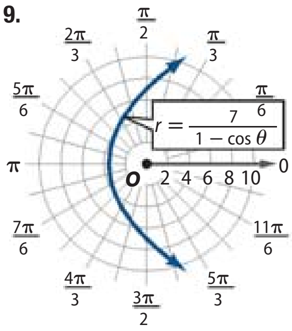
17. $r = \frac{6}{2\cos\theta + 1}$

Write the equation for each polar graph in rectangular form.

18.



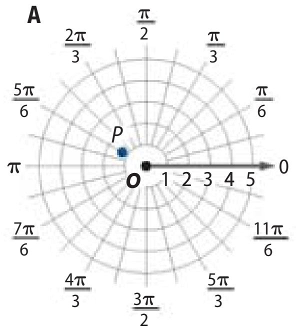
19.



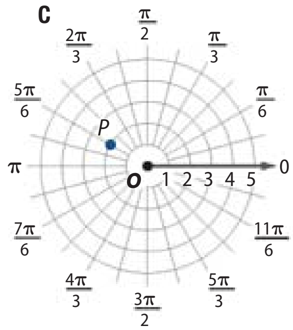
20. ELECTRICITY If a circuit has a voltage E of 135 volts and a current I of $3 - 4j$ amps, find the impedance Z of the circuit in ohms in rectangular form. Use the equation $E = I \cdot Z$.

21. MULTIPLE CHOICE Identify the graph of point P with complex coordinates $(-\sqrt{3}, -1)$ on the polar coordinate plane.

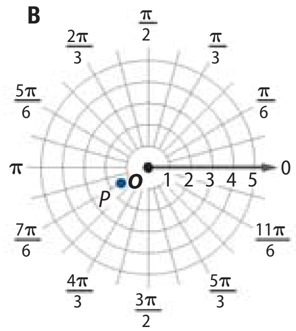
A



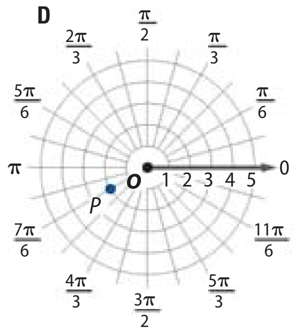
C



B



D



Find each power, and express it in rectangular form. Round to the nearest tenth.

22. $(-1 + 4i)^3$

23. $(-7 - 3i)^5$

24. $(6 + i)^4$

25. $(2 - 5i)^6$

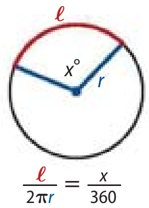
CHAPTER 9

Connect to AP Calculus
Arc Length

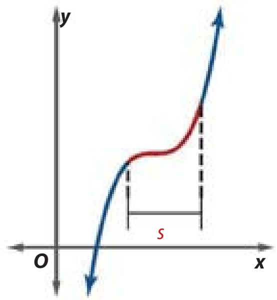
Objective

- Approximate the arc length of a curve.

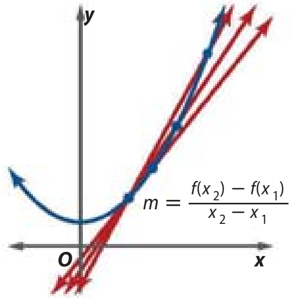
You can find the length of a line segment by using the Distance Formula. You can find the length of an arc by using proportions. In calculus, you will need to calculate many lengths that are not represented by line segments or sections of a circle.



As mentioned in Chapter 2, *integral calculus* focuses on areas, volumes, and lengths. It can be used to find the length of a curve for which we do not have a standard equation, such as a curve defined by a quadratic, cubic, or polar function. *Riemann sums* and *definite integrals*, two concepts that you will learn more about in the following chapters, are needed to calculate the exact length of a curve, or *arc length*, denoted *s*.



In this lesson, we will approximate the arc length of a curve using a process similar to the method that you applied to approximate the rate of change at a point. Recall that in Chapter 1, you calculated the slopes of secant lines to approximate the rates of change for graphs at specific points. Decreasing the distance between the two points on the secant lines increased the accuracy of the approximations as shown in the graph at the right.



Activity 1 Approximate Arc Length

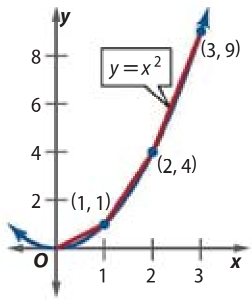
Approximate the arc length of the graph of $y = x^2$ for $0 \le x \le 3$.

Step 1 Graph $y = x^2$ for $0 \le x \le 3$ as shown.

Step 2 Graph points on the curve at $x = 1, 2,$ and 3. Connect the points using line segments as shown.

Step 3 Use the Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.

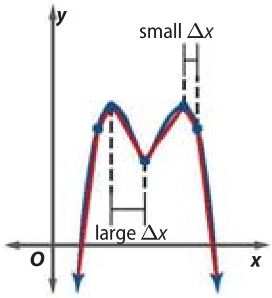


Analyze the Results

1. Is your approximation *greater* or *less* than the actual length? Explain your reasoning.
2. Approximate the arc length a second time using 6 line segments that are formed by the points $x = 0, 0.5, 1.0, 1.5, 2.0, 2.5,$ and 3.0. Include a sketch of the graph with your approximation.
3. Describe what happens to the approximation for the arc length as shorter line segments are used.
4. For the two approximations, the endpoints of the line segments were equally spaced along the x-axis. Do you think this will always produce the most accurate approximation? Explain your reasoning.

Notice that for the first activity, the endpoints of the line segments were equally spaced 0.5 units apart along the x-axis. When using advanced methods of calculus to find *exact* arc length, a constant difference between a pair of endpoints along the x-axis is essential. This difference is denoted Δx.

Accurately approximating arc length by using a constant Δx to create the line segments may not always be the most efficient method. The shape of the arc will dictate the spacing of the endpoints, thus creating different values for Δx. For example, if a graph shows an increase or decrease over a large interval for x, a large line segment may be used for the approximation. If a graph includes a turning point, it is better to use small line segments to account for the curve in the graph.



In Lesson 9-1, you learned how to calculate the distance between polar coordinates. This formula can be used to approximate the arc length of a curve represented by a polar equation.

Activity 2 Approximate Arc Length

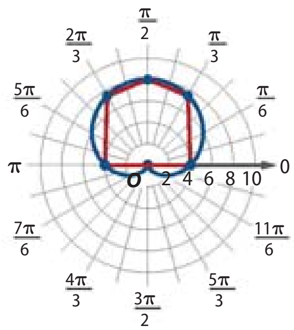
Approximate the arc length of the graph of $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$.

Step 1 Graph $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$ as shown.

Step 2 Draw 6 points on the curve at $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$, and $\frac{3\pi}{2}$. Connect the points using line segments as shown.

Step 3 Use the Polar Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.



StudyTip

Polar Graphs Create a table of values for r and θ when calculating the arc length for a polar graph. This will help to reduce errors created by functions that produce negative values for r.

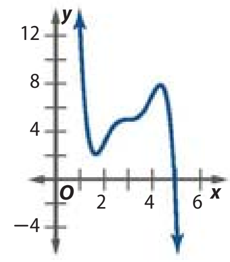
Analyze the Results

5. Explain how symmetry can be used to reduce the number of calculations in Step 3.
6. Approximate the arc length using at least 10 segments. Include a sketch of the graph.
7. Let n be the number of line segments used in an approximation and $\Delta\theta$ be a constant difference in θ between the endpoints of a line segment. Make a conjecture regarding the relationship between n, θ, and the approximation for an arc length.

Model and Apply

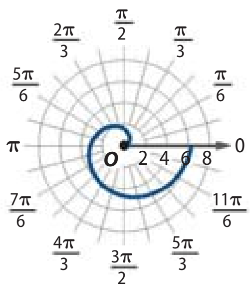
Approximate the arc length for each graph. Include a sketch of your graph.

8.



$y = -(x - 3)^5 + 3(x - 3)^3 + 5$ for $1 \leq x \leq 5$

9.



$r = \theta$ for $0 \leq \theta \leq 2\pi$

CHAPTER 10

Sequences and Series

Then

In **Chapters 1–4**, you modeled data using various types of functions.

Now

In **Chapter 10,** you will:

- Relate sequences and functions.
- Represent and calculate sums of series with sigma notation.
- Use arithmetic and geometric sequences and series.
- Prove statements by using mathematical induction.
- Expand powers by using the Binomial Theorem.

Why?

MARCHING BAND Sequences and series can be used to predict patterns. For example, arithmetic sequences can be used to determine the number of band members in a specified row of a pyramid formation.

PREREAD Use the text on this page to predict the organization of Chapter 10.