CHAPTER 8

Vectors

Then

In **Chapter 4,** you used trigonometry to solve triangles.

Now

In **Chapter 8,** you will:

- Represent and operate with vectors algebraically in the two- and three-dimensional coordinate systems.
- Find the projection of one vector onto another.
- Find cross products of vectors in space and find volumes of parallelepipeds.
- Find the dot products of and angles between vectors.

Why?

ROWING Vectors are often used to model changes in direction due to water and air currents. For example, a vector can be used to determine the resultant speed and direction of a kayak that is traveling 8 miles per hour against a 3-mile-per-hour river current.

PREREAD Scan the lesson titles and Key Concept boxes in Chapter 8. Use this information to predict what you will learn in this chapter.

Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

Find the distance between each given pair of points and the midpoint of the segment connecting the given points. (Prerequisite Skill)

1. (1, 4), (−2, 4)

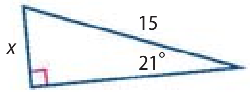
2. (−5, 3), (−5, 8)

3. (2, −9), (−3, −7)

4. (−4, −1), (−6, −8)

Find the value of *x*. Round to the nearest tenth if necessary. (Lesson 4-1)

5.



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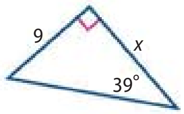

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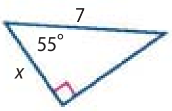

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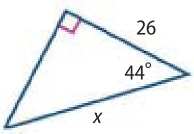


9. **BALLOON** A hot air balloon is being held in place by two people holding ropes and standing 35 feet apart. The angle formed between the ground and the rope held by each person is 40°. Determine the length of each rope to the nearest tenth of a foot. (Lesson 4-7)

Find all solutions for the given triangle, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measures to the nearest degree. (Lesson 4-7)

10. $a = 10$, $b = 7$, $A = 128°$

11. $a = 15$, $b = 16$, $A = 127°$

12. $a = 15$, $b = 18$, $A = 52°$

13. $a = 30$, $b = 19$, $A = 91°$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

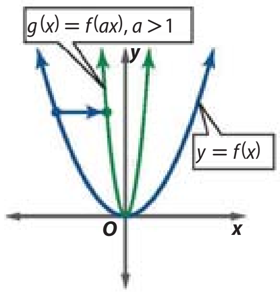
NewVocabulary

English		Español
vector	p. 482	vector
initial point	p. 482	punto inicial
terminal point	p. 482	punto terminal
standard position	p. 482	posición estándar
direction	p. 482	dirección
magnitude	p. 482	magnitud
quadrant bearing	p. 483	porte de cuadrante
true bearing	p. 483	porte verdadero
parallel vectors	p. 483	vectores paralelos
equivalent vectors	p. 483	vectores equivalentes
opposite vectors	p. 483	vectores de enfrente
resultant	p. 484	resultado
zero vector	p. 485	vector cero
component form	p. 492	forma componente
unit vector	p. 494	vector de unidad
dot product	p. 500	producto de punto
orthogonal	p. 500	ortogonal
z-axis	p. 510	*z*-eje
octants	p. 510	octants
ordered triple	p. 510	pedido triple
cross product	p. 519	producto enfadado
triple scalar product	p. 521	triplice el producto escalar

ReviewVocabulary

scalar p. P25 escalar a quantity with magnitude only

dilation p.49 dilatación a transformation in which the graph of a function is compressed or expanded vertically or horizontally



Horizontal dilation

LESSON 8-1 Introduction to Vectors

Then

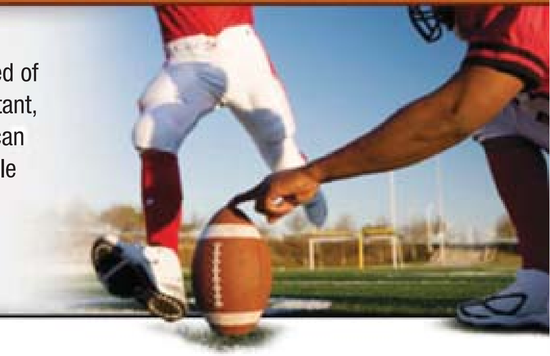
- You used trigonometry to solve triangles. (Lesson 5-4)

Now

1. Represent and operate with vectors geometrically.
2. Solve vector problems, and resolve vectors into their rectangular components.

Why?

- A successful field goal attempt in football depends on several factors. While the speed of the ball after it is kicked is certainly important, the direction the ball takes is as well. We can describe both of these factors using a single quantity called a *vector*.



NewVocabulary

vector
initial point
terminal point
standard position
direction
magnitude
quadrant bearing
true bearing
parallel vectors
equivalent vectors
opposite vectors
resultant
triangle method
parallelogram method
zero vector
components
rectangular components

1 Vectors

Many physical quantities, such as speed, can be completely described by a single real number called a *scalar*. This number indicates the *magnitude* or *size* of the quantity. A **vector** is a quantity that has both magnitude and *direction*. The velocity of a football is a vector that describes both the speed and direction of the ball .

Example 1 Identify Vector Quantities

State whether each quantity described is a *vector* quantity or a *scalar* quantity.

a. a boat traveling at 15 miles per hour

This quantity has a magnitude of 15 miles per hour, but no direction is given. Speed is a scalar quantity.

b. a hiker walking 25 paces due west

This quantity has a magnitude of 25 paces and a direction of due west. This directed distance is a vector quantity.

c. a person's weight on a bathroom scale

Weight is a vector quantity that is calculated using a person's mass and the downward pull due to gravity. (Acceleration due to gravity is a vector.)

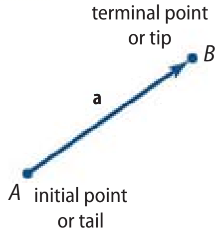
GuidedPractice

1A. a car traveling 60 miles per hour 15° east of south

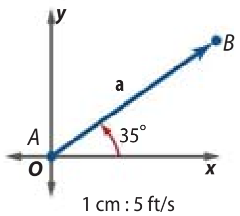
1B. a parachutist falling straight down at 12.5 miles per hour

1C. a child pulling a sled with a force of 40 newtons

A vector can be represented geometrically by a directed line segment, or arrow diagram, that shows both magnitude and direction. Consider the directed line segment with an **initial point** A (also known as the *tail*) and **terminal point** B (also known as the *head* or *tip*) shown. This vector is denoted by $\overrightarrow{AB}$, $\vec{a}$, or $\mathbf{a}$.

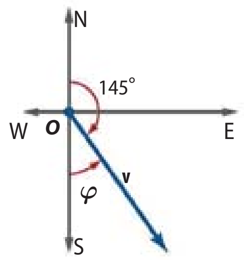


If a vector has its initial point at the origin, it is in **standard position**. The **direction** of a vector is the directed angle between the vector and the horizontal line that could be used to represent the positive x-axis. The direction of $\mathbf{a}$ is 35°.



The length of the line segment represents, and is proportional to, the **magnitude** of the vector. If the scale of the arrow diagram for $\mathbf{a}$ is 1 cm = 5 ft/s, then the magnitude of $\mathbf{a}$, denoted $|\mathbf{a}|$, is 2.6×5 or 13 feet per second.

The direction of a vector can also be given as a bearing. A **quadrant bearing** φ, or *phi*, is a directional measurement between 0° and 90° east or west of the north-south line. The quadrant bearing of vector **v** shown is 35° east of south or southeast, written S35°E.



A **true bearing** is a directional measurement where the angle is measured clockwise from north. True bearings are always given using three digits. So, a direction that measures 25° clockwise from north would be written as a true bearing of 025°.

StudyTip

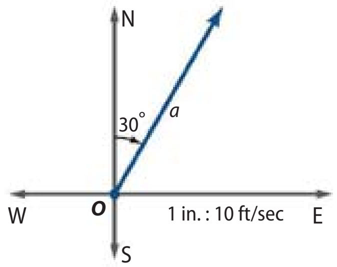
True Bearing When a degree measure is given without any additional directional components, it is assumed to be a true bearing. The true bearing of **v** is 145°.

Example 2 Represent a Vector Geometrically

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram.

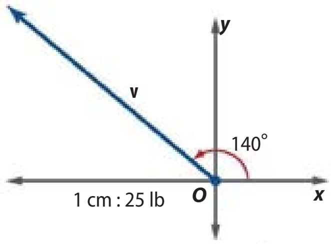
a. a = 20 feet per second at a bearing of 030°

Using a scale of 1 in. : 10 ft/sec, draw and label a 20 ÷ 10 or 2-inch arrow at an angle of 30° clockwise from the north.



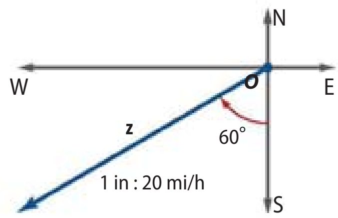
b. v = 75 pounds of force at 140° to the horizontal

Using a scale of 1 cm : 25 lb, draw and label a 75 ÷ 25 or 3-centimeter arrow in standard position at a 140° angle to the x-axis.



c. z = 30 miles per hour at a bearing of S60°W

Using a scale of 1 in. : 20 mi/h, draw and label a 30 ÷ 20 or 1.5-inch arrow 60° west of south.



GuidedPractice

2A. **t** = 20 feet per second at a bearing of 065°

2B. **u** = 15 miles per hour at a bearing of S25°E

2C. **m** = 60 pounds of force at 80° to the horizontal

WatchOut!

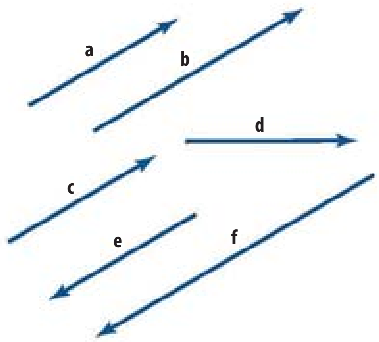
Magnitude The magnitude of a vector can represent distance, speed, or force. When a vector represents velocity, the length of the vector does not imply distance traveled.

In your operations with vectors, you will need to be familiar with the following vector types.

- **Parallel vectors** have the same or opposite direction but not necessarily the same magnitude. In the figure, **a** ∥ **b** ∥ **c** ∥ **e** ∥ **f**.

- **Equivalent vectors** have the same magnitude and direction. In the figure, **a** = **c** because they have the same magnitude and direction. Notice that **a** ≠ **b**, since |**a**| ≠ |**b**|, and **a** ≠ **d**, since **a** and **d** do not have the same direction.

- **Opposite vectors** have the same magnitude but opposite direction. The vector opposite **a** is written −**a**. In the figure, **e** = −**a**.



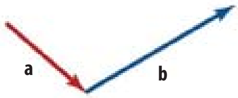
When two or more vectors are added, their sum is a single vector called the **resultant**. The resultant vector has the same effect as applying one vector after the other. Geometrically, the resultant can be found using either the **triangle method** or the **parallelogram method**.

KeyConcept Finding Resultants

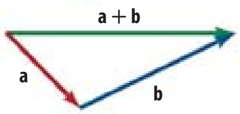
Triangle Method (Tip-to-Tail)

To find the resultant of **a** and **b**, follow these steps.

Step 1 Translate **b** so that the tail of **b** touches the tip of **a**.



Step 2 The resultant is the vector from the tail of **a** to the tip of **b**.

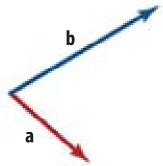


Parallelogram Method (Tail-to-Tail)

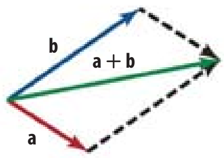
To find the resultant of **a** and **b**, follow these steps.

Step 1 Translate **b** so that the tail of **b** touches the tail of **a**.

Step 2 Complete the parallelogram that has **a** and **b** as two of its sides.



Step 3 The resultant is the vector that forms the indicated diagonal of the parallelogram.

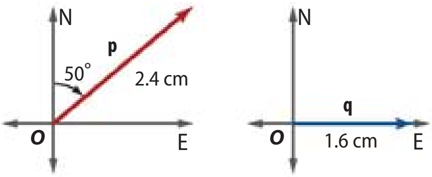


Real-World Example 3 Find the Resultant of Two Vectors

ORIENTEERING In an orienteering competition, Tia walks N50°E for 120 feet and then walks 80 feet due east. How far and at what quadrant bearing is Tia from her starting position?

Let **p** = walking 120 feet N50°E and **q** = walking 80 feet due east. Draw a diagram to represent **p** and **q** using a scale of 1 cm : 50 ft.

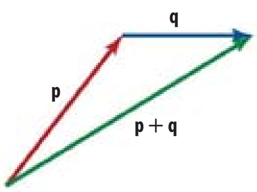
Use a ruler and a protractor to draw a 120 ÷ 50 or 2.4-centimeter arrow 50° east of north to represent **p** and an 80 ÷ 50 or 1.6-centimeter arrow due east to represent **q**.



1 cm : 50 ft

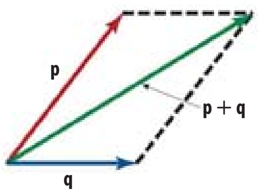
Method 1 Triangle Method

Translate **q** so that its tail touches the tip of **p**. Then draw the resultant vector **p** + **q** as shown.



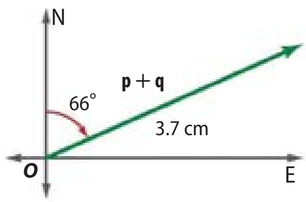
Method 2 Parallelogram Method

Translate **q** so that its tail touches the tail of **p**. Then complete the parallelogram and draw the diagonal, resultant **p** + **q**, as shown.



Both methods produce the same resultant vector **p** + **q**. Measure the length of **p** + **q** and then measure the angle this vector makes with the north-south line as shown.

The vector's length of approximately 3.7 centimeters represents 3.7 × 50 or 185 feet. Therefore, Tia is approximately 185 feet at a bearing of 66° east of north or N66°E from her starting position.



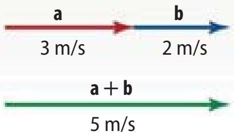
1 cm : 50 ft

StudyTip

Resultants The parallelogram method must be repeated in order to find the resultant of more than two vectors. The triangle method, however, is easier to use when finding the resultant of three or more vectors. Continue to place the initial point of subsequent vectors at the terminal point of the previous vector.

StudyTip

Parallel Vectors with Same Direction To add two or more parallel vectors with the *same direction*, add their magnitudes. The resultant has the same direction as the original vectors.

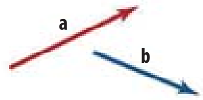


GuidedPractice

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest centimeter and its direction relative to the horizontal.

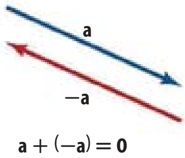
3A.

3B.



3C. **PINBALL** A pinball is struck by flipper and is sent 310° at a velocity of 7 inches per second. The ball then bounces off of a bumper and heads 055° at a velocity of 4 inches per second. Find the resulting direction and velocity of the pinball.

When you add two opposite vectors, the resultant is the **zero vector** or *null* vector, denoted by $\vec{0}$ or **0**, which has a magnitude of 0 and no specific direction. Subtracting vectors is similar to subtraction with integers. To find **p** − **q**, add the opposite of **q** to **p**. That is, **p** − **q** = **p** + (−**q**).



A vector can also be multiplied by a scalar.

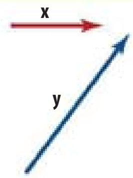
KeyConcept Multiplying Vectors by a Scalar

If a vector **v** is multiplied by a real number scalar k, the scalar multiple k**v** has a magnitude of $|k|\ |\mathbf{v}|$. Its direction is determined by the sign of k.

- If $k > 0$, k**v** has the same direction as **v**.
- If $k < 0$, k**v** has the opposite direction as **v**.

Example 4 Operations with Vectors

Draw a vector diagram of $3\mathbf{x} - \frac{3}{4}\mathbf{y}$.



Rewrite the expression as the addition of two vectors: $3\mathbf{x} - \frac{3}{4}\mathbf{y} = 3\mathbf{x} + \left(-\frac{3}{4}\mathbf{y}\right)$. To represent 3**x**, draw a vector 3 times as long as **x** in the same direction as **x** (Figure 8.1.1). To represent $-\frac{3}{4}\mathbf{y}$, draw a vector $\frac{3}{4}$ the length of **y** in the opposite direction from **y** (Figure 8.1.2). Then use the triangle method to draw the resultant vector (Figure 8.1.3).

Figure 8.1.1

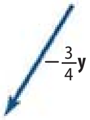


Figure 8.1.2

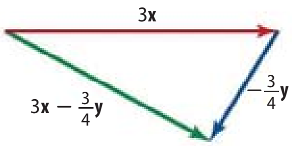
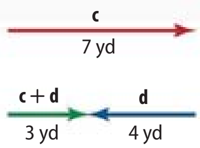


Figure 8.1.3

StudyTip

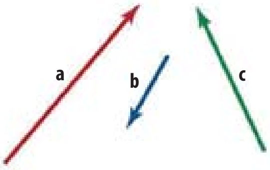
Parallel Vectors with Opposite Directions To add two parallel vectors with *opposite directions*, find the absolute value of the difference in their magnitudes. The resultant has the same direction as the vector with the greater magnitude.



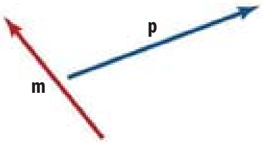
GuidedPractice

Draw a vector diagram of each expression.

4A. $\mathbf{a} - \mathbf{c} + 2\mathbf{b}$



4B. $\mathbf{m} - \frac{1}{4}\mathbf{p}$



2 Vector Applications

Vector Applications Vector addition and trigonometry can be used to solve vector problems involving triangles which are often oblique.

In navigation, a *heading* is the direction in which a vessel, such as an airplane or boat, is steered to overcome other forces, such as wind or current. The *relative velocity* of the vessel is the resultant when the heading velocity and other forces are combined.

Real-World Example 5 Use Vectors to Solve Navigation Problems

AVIATION **An airplane is flying with an airspeed of 310 knots on a heading of 050°. If a 78-knot wind is blowing from a true heading of 125°, determine the speed and direction of the plane relative to the ground.**

Step 1 Draw a diagram to represent the heading and wind velocities (Figure 8.1.4). Translate the wind vector as shown in Figure 8.1.5, and use the triangle method to obtain the resultant vector representing the plane's ground velocity **g**. In the triangle formed by these vectors (Figure 8.1.6), $\gamma = 125° - 50°$ or 75°.

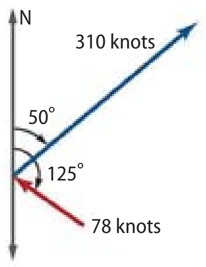


Figure 8.1.4

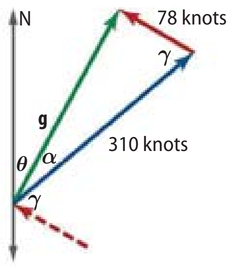


Figure 8.1.5

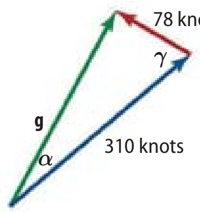


Figure 8.1.6

Step 2 Use the Law of Cosines to find $|\mathbf{g}|$, the plane's speed relative to the ground.

$c^2 = a^2 + b^2 - 2ab\cos\gamma$ — Law of Cosines

$|\mathbf{g}|^2 = 78^2 + 310^2 - 2(78)(310)\cos 75°$ — $c = |\mathbf{g}|$, $a = 78$, $b = 310$, and $\gamma = 75°$

$|\mathbf{g}| = \sqrt{78^2 + 310^2 - 2(78)(310)\cos 75°}$ — Take the positive square root of each side.

≈ 299.4 — Simplify.

The ground speed of the plane is about 299.4 knots.

Step 3 The heading of the resultant **g** is represented by angle θ, as shown in Figure 8.1.5. To find θ, first calculate α using the Law of Sines.

$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$ — Law of Sines

$\frac{\sin\alpha}{78} = \frac{\sin 75°}{299.4}$ — $c = |\mathbf{g}|$ or 299.4, $a = 78$, and $\gamma = 75°$

$\sin\alpha = \frac{78\sin 75°}{299.4}$ — Solve for $\sin\alpha$.

$\alpha = \sin^{-1}\frac{78\sin 75°}{299.4}$ — Apply the inverse sine function.

$\approx 14.6°$ — Simplify.

The measure of θ is $50° - \alpha$, which is $50° - 14.6°$ or 35.4°.

Therefore, the speed of the plane relative to the ground is about 299.4 knots at about 035°.

GuidedPractice

5. **SWIMMING** Mitchell swims due east at a speed of 3.5 feet per second across a river directly toward the opposite bank. At the same time, the current of the river is carrying him due south at a rate of 2 feet per second. Find Mitchell's speed and direction relative to the shore.

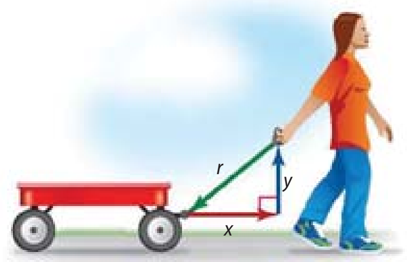
StudyTip

Alternate Interior Angles The translation of the tail of the wind vector to the tip of the vector representing the plane's heading produces two parallel vectors cut by a transversal. Since alternate interior angles of two parallel lines cut by a transversal are congruent, the angles made by these two vectors in both places in Figure 8.1.5 have the same measure.

WatchOut!

Wind Direction In Example 5, notice that the wind is blowing *from* a bearing of 125° and the vector is drawn so that the tip of the vector points *toward* the north-south line. Had the wind been blowing at a bearing of 125°, the vector would have pointed *away* from this line.

Two or more vectors with a sum that is a vector **r** are called **components** of **r**. While components can have any direction, it is often useful to express or *resolve* a vector into two perpendicular components. The **rectangular components** of a vector are horizontal and vertical.



In the diagram, the force **r** exerted to pull the wagon can be thought of as the sum of a horizontal component force **x** that moves the wagon forward and a vertical component force **y** that pulls the wagon upward.

Real-WorldLink

It takes a force of about 3 newtons to flip a light switch. The force due to gravity on a person is about 600 newtons. The force exerted by a weightlifter is about 2000 newtons.

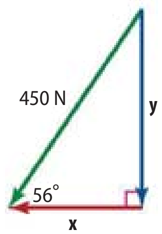
Source: *Contemporary College Physics*

Real-World Example 6 Resolve a Force into Rectangular Components

LAWN CARE **Heather is pushing the handle of a lawn mower with a force of 450 newtons at an angle of 56° with the ground.**

a. Draw a diagram that shows the resolution of the force that Heather exerts into its rectangular components.

Heather's push can be resolved into a horizontal push **x** forward and a vertical push **y** downward as shown.



b. Find the magnitudes of the horizontal and vertical components of the force.

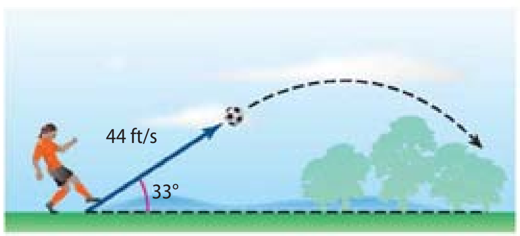
The horizontal and vertical components of the force form a right triangle. Use the sine or cosine ratios to find the magnitude of each force.

$\cos 56° = \frac{\lvert\mathbf{x}\rvert}{450}$	Right triangle definitions of cosine and sine	$\sin 56° = \frac{\lvert\mathbf{y}\rvert}{450}$
$\lvert\mathbf{x}\rvert = 450 \cos 56°$	Solve for *x* and *y*.	$\lvert\mathbf{y}\rvert = 450 \sin 56°$
$\lvert\mathbf{x}\rvert \approx 252$	Use a calculator.	$\lvert\mathbf{y}\rvert \approx 373$

The magnitude of the horizontal component is about 252 newtons, and the magnitude of the vertical component is about 373 newtons.

GuidedPractice

6. **SOCCER** A player kicks a soccer ball so that it leaves the ground with a velocity of 44 feet per second at an angle of 33° with the ground.



A. Draw a diagram that shows the resolution of this force into its rectangular components.

B. Find the magnitude of the horizontal and vertical components of the velocity.

Exercises

= Step-by-Step Solutions begin on page R29.

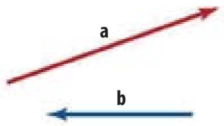
State whether each quantity described is a *vector* quantity or a *scalar* quantity. (Example 1)

1. a box being pushed with a force of 125 newtons
2. wind blowing at 20 knots
3. a deer running 15 meters per second due west
4. a baseball thrown with a speed of 85 miles per hour
5. a 15-pound tire hanging from a rope
6. a rock thrown straight up at a velocity of 50 feet per second

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram. (Example 2)

7. **h** = 13 inches per second at a bearing of 205°
8. **g** = 6 kilometers per hour at a bearing of N70°W
9. **j** = 5 feet per minute at 300° to the horizontal
10. **k** = 28 kilometers at 35° to the horizontal
11. **m** = 40 meters at a bearing of S55°E
12. **n** = 32 yards per second at a bearing of 030°

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal. (Example 3)

13.


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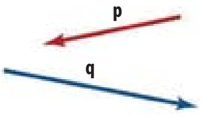

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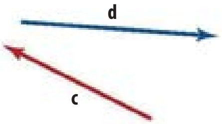

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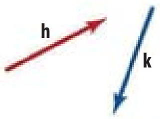

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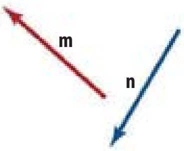
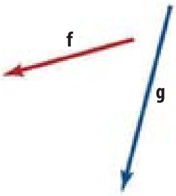


18.


19. **GOLF** While playing a golf video game, Ana hits a ball 35° above the horizontal at a speed of 40 miles per hour with a 5 miles per hour wind blowing, as shown. Find the resulting speed and direction of the ball. (Example 3)

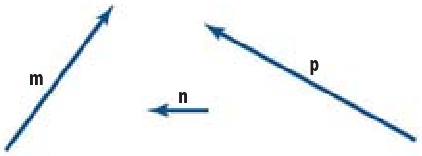
20. **BOATING** A charter boat leaves port on a heading of N60°W for 12 nautical miles. The captain changes course to a bearing of N25°E for the next 15 nautical miles. Determine the ship's distance and direction from port to its current location. (Example 3)

21. **HIKING** Nick and Lauren hiked 3.75 kilometers to a lake 55° east of south from their campsite. Then they hiked 33° west of north to the nature center 5.6 kilometers from the lake. Where is the nature center in relation to their campsite? (Example 3)

Determine the magnitude and direction of the resultant of each vector sum. (Example 3)

22. 18 newtons directly forward and then 20 newtons directly backward
23. 100 meters due north and then 350 meters due south
24. 10 pounds of force at a bearing of 025° and then 15 pounds of force at a bearing of 045°
25. 17 miles east and then 16 miles south
26. 15 meters per second squared at a 60° angle to the horizontal and then 9.8 meters per second squared downward

Use the set of vectors to draw a vector diagram of each expression. (Example 4)



27. $\mathbf{m} - 2\mathbf{n}$
28. $\mathbf{n} - \frac{3}{4}\mathbf{m}$
29. $\frac{1}{2}\mathbf{p} + 3\mathbf{n}$
30. $4\mathbf{n} + \frac{4}{5}\mathbf{p}$
31. $\mathbf{p} + 2\mathbf{n} - \mathbf{m}$
32. $-\frac{1}{3}\mathbf{m} + \mathbf{p} - 2\mathbf{n}$
33. $3\mathbf{n} - \frac{1}{2}\mathbf{p} + \mathbf{m}$
34. $\mathbf{m} - 3\mathbf{n} + \frac{1}{4}\mathbf{p}$

35. **RUNNING** A runner's resultant velocity is 8 miles per hour due west running with a wind of 3 miles per hour N28°W. What is the runner's speed, to the nearest mile per hour, without the effect of the wind? (Example 5)

36. **GLIDING** A glider is traveling at an air speed of 15 miles per hour due west. If the wind is blowing at 5 miles per hour in the direction N60°E, what is the resulting ground speed of the glider? (Example 5)

37. **CURRENT** Kaya is swimming due west at a rate of 1.5 meters per second. A strong current is flowing S20°E at a rate of 1 meter per second. Find Kaya's resulting speed and direction. (Example 5)

Draw a diagram that shows the resolution of each vector into its rectangular components. Then find the magnitudes of the vector's horizontal and vertical components. (Example 6)

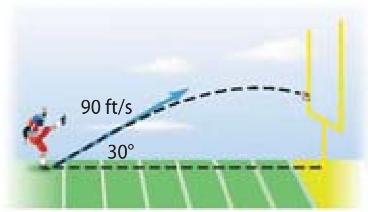
38. $2\frac{1}{8}$ inches at 310° to the horizontal

39. 1.5 centimeters at a bearing of N49°E

40. 3.2 centimeters per hour at a bearing of S78°W

41. $\frac{3}{4}$ inch per minute at a bearing of 255°

42. **FOOTBALL** For a field goal attempt, a football is kicked with the velocity shown in the diagram below.



a. Draw a diagram that shows the resolution of this force into its rectangular components.

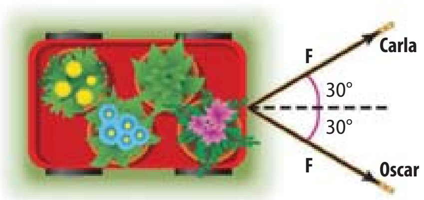
b. Find the magnitudes of the horizontal and vertical components. (Example 6)

43. **CLEANING** Aiko is pushing the handle of a push broom with a force of 190 newtons at an angle of 33° with the ground. (Example 6)

a. Draw a diagram that shows the resolution of this force into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components.

44. **GARDENING** Carla and Oscar are pulling a wagon full of plants. Each person pulls on the wagon with equal force at an angle of 30° with the axis of the wagon. The resultant force is 120 newtons.



a. How much force is each person exerting?

b. If each person exerts a force of 75 newtons, what is the resultant force?

c. How will the resultant force be affected if Carla and Oscar move closer together?

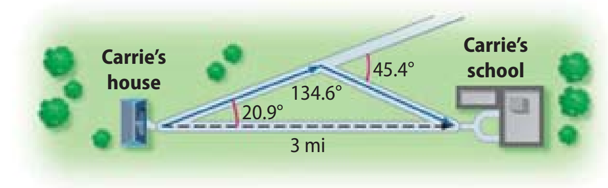
The magnitude and true bearings of three forces acting on an object are given. Find the magnitude and direction of the resultant of these forces.

45. 50 lb at 30°, 80 lb at 125°, and 100 lb at 220°

46. 8 newtons at 300°, 12 newtons at 45°, and 6 newtons at 120°

47. 18 lb at 190°, 3 lb at 20°, and 7 lb at 320°

48. **DRIVING** Carrie's school is on a direct path three miles from her house. She drives on two different streets on her way to school. She travels at an angle of 20.9° with the path on the first street and then turns 45.4° onto the second street.



a. How far does Carrie drive on the first street?

b. How far does she drive on the second street?

c. If it takes her 10 minutes to get to school and she averages 25 miles per hour on the first street, what speed does Carrie average after she turns onto the second street?

49. **SLEDDING** Irwin is pulling his sister on a sled. The direction of his resultant force is 31°, and the horizontal component of the force is 86 newtons.

a. What is the vertical component of the force?

b. What is the magnitude of the resultant force?

50. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate multiplication of a vector by a scalar.

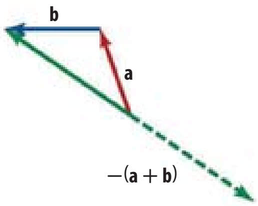
a. **GRAPHICAL** On a coordinate plane, draw a vector **a** so that the tail is located at the origin. Choose a value for a scalar k. Then draw the vector that results if you multiply the original vector by k on the same coordinate plane. Repeat the process for four additional vectors **b**, **c**, **d**, and **e**. Use the same value for k each time.

b. **TABULAR** Copy and complete the table below for each vector that you drew in part **a**.

Vector	Terminal Point of Vector	Terminal Point of Vector × k
a		
b		
c		
d		
e		

c. **ANALYTICAL** If the terminal point of a vector **a** is located at the point (a, b), what is the location of the terminal point of the vector k**a**?

An *equilibrant* vector is the opposite of a resultant vector. It balances a combination of vectors such that the sum of the vectors and the equilibrant is the zero vector. The equilibrant vector of **a** + **b** is −(**a** + **b**).



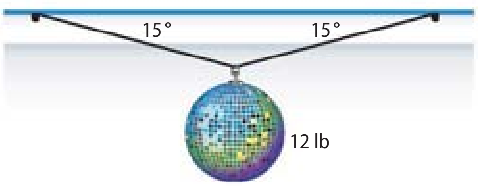
Find the magnitude and direction of the equilibrant vector for each set of vectors.

51. **a** = 15 miles per hour at a bearing of 125°
b = 12 miles per hour at a bearing of 045°

52. **a** = 4 meters at a bearing of N30W°
b = 6 meters at a bearing of N20E°

53. **a** = 23 feet per second at a bearing of 205°
b = 16 feet per second at a bearing of 345°

54. **PARTY PLANNING** A disco ball is suspended above a dance floor by two wires of equal length as shown.

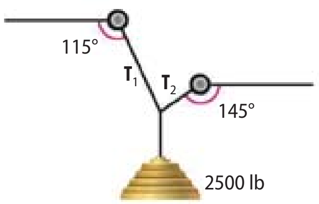


a. Draw a vector diagram of the situation that indicates that two tension vectors $\mathbf{T_1}$ and $\mathbf{T_2}$ with equal magnitude are keeping the disco ball stationary or at equilibrium.

b. Redraw the diagram using the triangle method to find $\mathbf{T_1} + \mathbf{T_2}$.

c. Use your diagram from part **b** and the fact that the equilibrant of the resultant $\mathbf{T_1} + \mathbf{T_2}$ and the vector representing the weight of the disco ball are equivalent vectors to calculate the magnitudes of $\mathbf{T_1}$ and $\mathbf{T_2}$.

55. **CABLE SUPPORT** Two cables with tensions $\mathbf{T_1}$ and $\mathbf{T_2}$ are tied together to support a 2500-pound load at equilibrium.



a. Write expressions to represent the horizontal and vertical components of $\mathbf{T_1}$ and $\mathbf{T_2}$.

b. Given that the equilibrant of the resultant $\mathbf{T_1} + \mathbf{T_2}$ and the vector representing the weight of the load are equivalent vectors, calculate the magnitudes of $\mathbf{T_1}$ and $\mathbf{T_2}$ to the nearest tenth of a pound.

c. Use your answers from parts **a** and **b** to find the magnitudes of the horizontal and vertical components of $\mathbf{T_1}$ and $\mathbf{T_2}$ to the nearest tenth of a pound.

Find the magnitude and direction of each vector given its vertical and horizontal components and the range of values for the angle of direction θ to the horizontal.

56. horizontal: 0.32 in., vertical: 2.28 in., $90° < \theta < 180°$

57. horizontal: 3.1 ft, vertical: 4.2 ft, $0° < \theta < 90°$

58. horizontal: 2.6 cm, vertical: 9.7 cm, $270° < \theta < 360°$

59. horizontal: 2.9 yd, vertical: 1.8 yd, $180° < \theta < 270°$

Draw any three vectors a, b, and c. Show geometrically that each of the following vector properties holds using these vectors.

60. Commutative Property: **a** + **b** = **b** + **a**

61. Associative Property: (**a** + **b**) + **c** = **a** + (**b** + **c**)

62. Distributive Property: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, for $k = 2, 0.5$, and -2

H.O.T. Problems Use Higher-Order Thinking Skills

63. **OPEN ENDED** Consider a vector of 5 units directed along the positive x-axis. Resolve the vector into two perpendicular components in which no component is horizontal or vertical.

64. **REASONING** Is it *sometimes*, *always*, or *never* possible to find the sum of two parallel vectors using the parallelogram method? Explain your reasoning.

65. **REASONING** Why is it important to establish a common reference for measuring the direction of a vector, for example, from the positive x-axis?

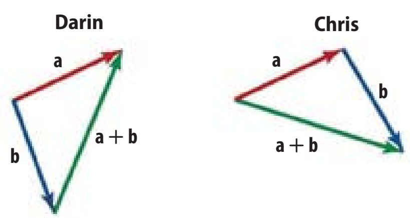
66. **CHALLENGE** The resultant of **a** + **b** is equal to the resultant of **a** − **b**. If the magnitude of **a** is $4x$, what is the magnitude of **b**?

67. **REASONING** Consider the statement $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$.

a. Express this statement using words.

b. Is this statement true or false? Justify your answer.

68. **ERROR ANALYSIS** Darin and Cris are finding the resultant of vectors **a** and **b**. Is either of them correct? Explain your reasoning.



69. **REASONING** Is it possible for the sum of two vectors to equal one of the vectors? Explain.

70. **WRITING IN MATH** Compare and contrast the parallelogram and triangle methods of finding the resultant of two or more vectors.

Spiral Review

71. KICKBALL Suppose a kickball player kicks a ball at a 32° angle to the horizontal with an initial speed of 20 meters per second. How far away will the ball land? (Lesson 7-5)

72. Graph $(x')^2 + y' - 5 = 1$ if it has been rotated 45° from its position in the xy-plane. (Lesson 7-4)

Write an equation for a circle that satisfies each set of conditions. Then graph the circle. (Lesson 7-2)

73. center at (4, 5), radius 4

74. center at (1, −4), diameter 7

Determine the equation of and graph the parabola with the given focus F and vertex V. (Lesson 7-1)

75. $F(2, 4)$, $V(2, 3)$

76. $F(1, 5)$, $V(-7, 5)$

77. CRAFTS Sanjay is selling wood carvings. He sells large statues for $60, clocks for $40, dollhouse furniture for $25, and chess pieces for $5. He takes the following number of items to the fair: 12 large statues, 25 clocks, 45 pieces of dollhouse furniture, and 50 chess pieces. (Lesson 6-2)

a. Write an inventory matrix for the number of each item and a cost matrix for the price of each item.

b. Find Sanjay's total income if he sells all of the items.

Solve each equation for all values of x. (Lesson 5-3)

78. $4 \sin x \cos x - 2 \sin x = 0$

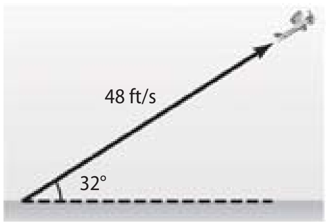
79. $\sin x - 2 \cos^2 x = -1$

Skills Review for Standardized Tests

80. SAT/ACT If town A is 12 miles from town B and town C is 18 miles from town A, then which of the following *cannot* be the distance from town B to town C?

A 5 miles
B 7 miles
C 10 miles
D 12 miles
E 18 miles

81. A remote control airplane flew along an initial path of 32° to the horizontal at a velocity of 48 feet per second as shown. Which of the following represent the magnitudes of the horizontal and vertical components of the velocity?



F 25.4 ft/s, 40.7 ft/s
G 40.7 ft/s, 25.4 ft/s
H 56.6 ft/s, 90.6 ft/s
J 90.6 ft/s, 56.6 ft/s

82. REVIEW Triangle ABC has vertices $A(-4, 2)$, $B(-4, -3)$, and $C(3, -3)$. After a dilation, triangle $A'B'C'$ has vertices $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -9)$. How many times as great is the area of $\triangle A'B'C'$ than the area of $\triangle ABC$?

A $\frac{1}{9}$
B $\frac{1}{3}$
C 3
D 9

83. REVIEW Holly is drawing a map of her neighborhood. Her house is represented by quadrilateral $ABCD$ with vertices $A(2, 2)$, $B(6, 2)$, $C(6, 6)$, and $D(2, 6)$. She wants to use the same coordinate system to make another map that is one half the size of the original map. What could be the new vertices of Holly's house?

F $A'(0, 0)$, $B'(2, 1)$, $C'(3, 3)$, $D'(0, 3)$
G $A'(0, 0)$, $B'(3, 1)$, $C'(2, 3)$, $D'(0, 2)$
H $A'(1, 1)$, $B'(3, 1)$, $C'(3, 3)$, $D'(1, 3)$
J $A'(1, 2)$, $B'(3, 0)$, $C'(2, 2)$, $D'(2, 3)$

LESSON

8-2 Vectors in the Coordinate Plane

Then

- You performed vector operations using scale drawings. (Lesson 8-1)

Now

1. Represent and operate with vectors in the coordinate plane.
2. Write a vector as a linear combination of unit vectors.

Why?

- Wind can impact the ground speed and direction of an airplane. While pilots can use scale drawings to determine the heading a plane should take to correct for wind, these calculations are more commonly calculated using vectors in the coordinate plane.

NewVocabulary
component form
unit vector
linear combination

1 Vectors in the Coordinate Plane

In Lesson 8-1, you found the magnitude and direction of the resultant of two or more forces geometrically by using a scale drawing. Since drawings can be inaccurate, an algebraic approach using a rectangular coordinate system is needed for situations where more accuracy is required or where the system of vectors is complex.

A vector $\overrightarrow{OP}$ in standard position on a rectangular coordinate system (as in Figure 8.2.1) can be uniquely described by the coordinates of its terminal point $P(x, y)$. We denote $\overrightarrow{OP}$ on the coordinate plane by $\langle x, y\rangle$. Notice that x and y are the rectangular components of $\overrightarrow{OP}$. For this reason, $\langle x, y\rangle$ is called the **component form** of a vector.

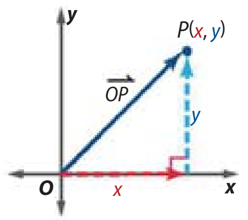


Figure 8.2.1

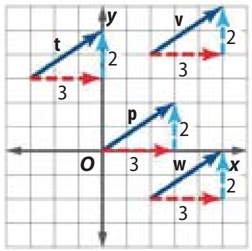


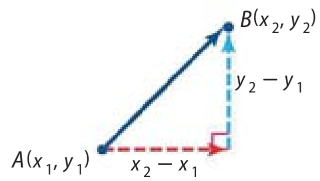
Figure 8.2.2

Since vectors with the same magnitude and direction are equivalent, many vectors can be represented by the same coordinates. For example, vectors **p**, **t**, **v**, and **w** in Figure 8.2.2 are *equivalent* because each can be denoted as $\langle 3, 2\rangle$. To find the component form of a vector that is not in standard position, you can use the coordinates of its initial and terminal points.

KeyConcept Component Form of a Vector

The component form of a vector $\overrightarrow{AB}$ with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1\rangle.$$



Example 1 Express a Vector in Component Form

Find the component form of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$	Component form
$= \langle 3 - (-4), -5 - 2\rangle$	$(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$
$= \langle 7, -7\rangle$	Subtract.

GuidedPractice

Find the component form of $\overrightarrow{AB}$ with the given initial and terminal points.

1A. $A(-2, -7), B(6, 1)$ **1B.** $A(0, 8), B(-9, -3)$

A. T. Willett / Alamy

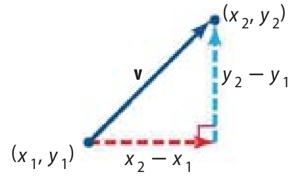
The magnitude of a vector in the coordinate plane is found by using the Distance Formula.

KeyConcept Magnitude of a Vector in the Coordinate Plane

If **v** is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2), then the magnitude of **v** is given by

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If **v** has a component form of $\langle a, b \rangle$, then $|\mathbf{v}| = \sqrt{a^2 + b^2}$.



ReadingMath

Norm The magnitude of a vector is sometimes called the *norm* of the vector.

Example 2 Find the Magnitude of a Vector

Find the magnitude of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{[3 - (-4)]^2 + (-5 - 2)^2}$ $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$

$= \sqrt{98}$ or about 9.9 Simplify.

CHECK From Example 1, you know that $\overrightarrow{AB} = \langle 7, -7 \rangle$. $|\overrightarrow{AB}| = \sqrt{7^2 + (-7)^2}$ or $\sqrt{98}$. ✓

GuidedPractice

Find the magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

2A. $A(-2, -7), B(6, 1)$ **2B.** $A(0, 8), B(-9, -3)$

Addition, subtraction, and scalar multiplication of vectors in the coordinate plane is similar to the same operations with matrices.

KeyConcept Vector Operations

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are vectors and k is a scalar, then the following are true.

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2 \rangle$

Example 3 Operations with Vectors

Find each of the following for $\mathbf{w} = \langle -4, 1 \rangle$, $\mathbf{y} = \langle 2, 5 \rangle$, and $\mathbf{z} = \langle -3, 0 \rangle$.

a. $\mathbf{w} + \mathbf{y}$

$\mathbf{w} + \mathbf{y} = \langle -4, 1 \rangle + \langle 2, 5 \rangle$ Substitute.

$= \langle -4 + 2, 1 + 5 \rangle$ or $\langle -2, 6 \rangle$ Vector addition

b. $\mathbf{z} - 2\mathbf{y}$

$\mathbf{z} - 2\mathbf{y} = \mathbf{z} + (-2)\mathbf{y}$ Rewrite subtraction as addition.

$= \langle -3, 0 \rangle + (-2)\langle 2, 5 \rangle$ Substitute.

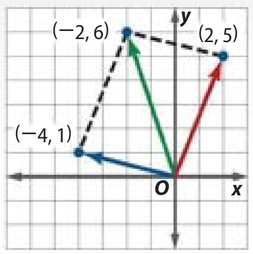
$= \langle -3, 0 \rangle + \langle -4, -10 \rangle$ or $\langle -7, -10 \rangle$ Scalar multiplication and vector addition

GuidedPractice

3A. $4\mathbf{w} + \mathbf{z}$ **3B.** $-3\mathbf{w}$ **3C.** $2\mathbf{w} + 4\mathbf{y} - \mathbf{z}$

StudyTip

Check Graphically A graphical check of Example 3a using the parallelogram method is shown below.



Math HistoryLink

William Rowan Hamilton
(1805–1865)
An Irish mathematician, Hamilton developed the theory of quaternions and published *Lectures on Quaternions.* Many basic concepts of vector analysis have their basis in this theory.

2 Unit Vectors

A vector that has a magnitude of 1 unit is called a **unit vector**. It is sometimes useful to describe a nonzero vector **v** as a scalar multiple of a unit vector **u** with the same direction as **v**. To find **u**, divide **v** by its magnitude $|\mathbf{v}|$.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{or} \quad \frac{1}{|\mathbf{v}|}\mathbf{v}$$

Example 4 Find a Unit Vector with the Same Direction as a Given Vector

Find a unit vector u with the same direction as v = ⟨−2, 3⟩.

$\mathbf{u} = \frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v}$	Unit vector with the same direction as **v**
$= \frac{1}{\lvert\langle -2, 3\rangle\rvert}\langle -2, 3\rangle$	Substitute.
$= \frac{1}{\sqrt{(-2)^2 + 3^2}}\langle -2, 3\rangle$	$\lvert\langle a, b\rangle\rvert = \sqrt{a^2 + b^2}$
$= \frac{1}{\sqrt{13}}\langle -2, 3\rangle$	Simplify.
$= \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$	Scalar multiplication
$= \left\langle -\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right\rangle$	Rationalize denominators.

CHECK Since **u** is a scalar multiple of **v**, it has the same direction as **v**. Verify that the magnitude of **u** is 1.

$\lvert\mathbf{u}\rvert = \sqrt{\left(-\frac{2\sqrt{13}}{13}\right)^2 + \left(\frac{3\sqrt{13}}{13}\right)^2}$	Distance Formula
$= \sqrt{\frac{52}{169} + \frac{117}{169}}$	Simplify.
$= \sqrt{1}$ or 1 ✓	Simplify.

GuidedPractice

Find a unit vector with the same direction as the given vector.

4A. $\mathbf{w} = \langle 6, -2\rangle$ **4B.** $\mathbf{x} = \langle -4, -8\rangle$

WatchOut!

Unit Vector i Do not confuse the unit vector **i** with the imaginary number *i*. The unit vector is denoted by a bold, nonitalic letter **i**. The imaginary number is denoted by a bold italic letter ***i***.

The unit vectors in the direction of the positive x-axis and positive y-axis are denoted by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, respectively (Figure 8.2.3). Vectors **i** and **j** are called *standard unit vectors*.

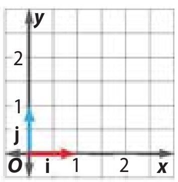


Figure 8.2.3

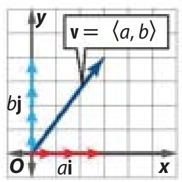


Figure 8.2.4

These vectors can be used to express any vector $\mathbf{v} = \langle a, b\rangle$ as $a\mathbf{i} + b\mathbf{j}$ as shown in Figure 8.2.4.

$\mathbf{v} = \langle a, b\rangle$	Component form of **v**
$= \langle a, 0\rangle + \langle 0, b\rangle$	Rewrite as the sum of two vectors.
$= a\langle 1, 0\rangle + b\langle 0, 1\rangle$	Scalar multiplication
$= a\mathbf{i} + b\mathbf{j}$	$\langle 1, 0\rangle = \mathbf{i}$ and $\langle 0, 1\rangle = \mathbf{j}$

The vector sum $a\mathbf{i} + b\mathbf{j}$ is called a **linear combination** of the vectors $\mathbf{i}$ and $\mathbf{j}$.

Example 5 Write a Vector as a Linear Combination of Unit Vectors

Let $\overrightarrow{DE}$ be the vector with initial point $D(-2, 3)$ and terminal point $E(4, 5)$. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

First, find the component form of $\overrightarrow{DE}$.

$\overrightarrow{DE} = \langle x_2 - x_1, y_2 - y_1 \rangle$ — Component form

$= \langle 4 - (-2), 5 - 3 \rangle$ — $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, 5)$

$= \langle 6, 2 \rangle$ — Simplify.

Then rewrite the vector as a linear combination of the standard unit vectors.

$\overrightarrow{DE} = \langle 6, 2 \rangle$ — Component form

$= 6\mathbf{i} + 2\mathbf{j}$ — $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$

GuidedPractice

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

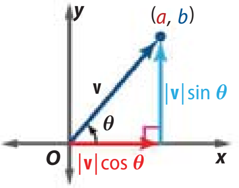
5A. $D(-6, 0)$, $E(2, 5)$

5B. $D(-3, -8)$, $E(-7, 1)$

StudyTip

Unit Vector From the statement that $\mathbf{v} = \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$, it follows that the unit vector in the direction of $\mathbf{v}$ has the form $\mathbf{v} = \langle 1 \cos \theta, 1 \sin \theta \rangle = \langle \cos \theta, \sin \theta \rangle$.

A way to specify the direction of a vector $\mathbf{v} = \langle a, b \rangle$ is to state the direction angle θ that $\mathbf{v}$ makes with the positive x-axis. From Figure 8.2.5, it follows that $\mathbf{v}$ can be written in component form or as a linear combination of $\mathbf{i}$ and $\mathbf{j}$ using the magnitude and direction angle of the vector.



$\mathbf{v} = \langle a, b \rangle$ — Component form

$= \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$ — Substitution

$= |\mathbf{v}| (\cos \theta)\mathbf{i} + |\mathbf{v}| (\sin \theta)\mathbf{j}$ — Linear combination of i and j

Example 6 Find Component Form

Find the component form of the vector v with magnitude 10 and direction angle 120°.

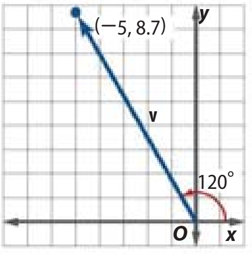
$\mathbf{v} = \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle$ — Component form of v in terms of $|\mathbf{v}|$ and θ

$= \langle 10 \cos 120°, 10 \sin 120° \rangle$ — $|\mathbf{v}| = 10$ and $\theta = 120°$

$= \left\langle 10\left(-\frac{1}{2}\right), 10\left(\frac{\sqrt{3}}{2}\right) \right\rangle$ — $\cos 120° = -\frac{1}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$

$= \langle -5, 5\sqrt{3} \rangle$ — Simplify.

CHECK Graph $\mathbf{v} = \langle -5, 5\sqrt{3} \rangle \approx \langle -5, 8.7 \rangle$. The measure of the angle $\mathbf{v}$ makes with the positive x-axis is about 120° as shown, and $|\mathbf{v}| = \sqrt{(-5)^2 + (5\sqrt{3})^2}$ or 10. ✓



GuidedPractice

Find the component form of v with the given magnitude and direction angle.

6A. $|\mathbf{v}| = 8$, $\theta = 45°$

6B. $|\mathbf{v}| = 24$, $\theta = 210°$

It also follows from Figure 8.2.5 on the previous page that the direction angle θ of vector $\mathbf{v} = \langle a, b \rangle$ can be found by solving the trigonometric equation $\tan \theta = \dfrac{|\mathbf{v}| \sin \theta}{|\mathbf{v}| \cos \theta}$ or $\tan \theta = \dfrac{b}{a}$.

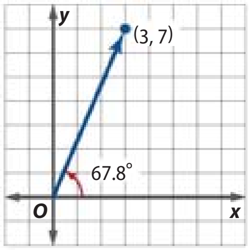


Figure 8.2.6

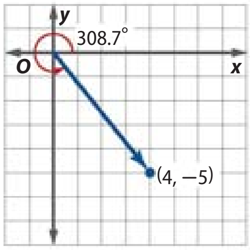


Figure 8.2.7

Example 7 Direction Angles of Vectors

Find the direction angle of each vector to the nearest tenth of a degree.

a. $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j}$

$\tan \theta = \dfrac{b}{a}$	Direction angle equation
$\tan \theta = \dfrac{7}{3}$	$a = 3$ and $b = 7$
$\theta = \tan^{-1} \dfrac{7}{3}$	Solve for θ.
$\theta \approx 66.8°$	Use a calculator.

So, the direction angle of vector **p** is about 67.8° as shown in Figure 8.2.6.

b. $\mathbf{r} = \langle 4, -5 \rangle$

$\tan \theta = \dfrac{b}{a}$	Direction angle equation
$\tan \theta = \dfrac{-5}{4}$	$a = 4$ and $b = -5$
$\theta = \tan^{-1} \left(-\dfrac{5}{4}\right)$	Solve for θ.
$\theta \approx -51.3°$	Use a calculator.

Since **r** lies in Quadrant IV as shown in Figure 8.2.7, $\theta = 360 + (-51.3)$ or 308.7°.

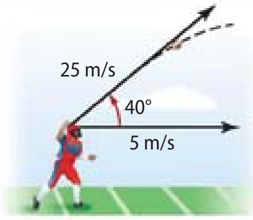
GuidedPractice

7A. $-6\mathbf{i} + 2\mathbf{j}$

7B. $\langle -3, -8 \rangle$

Real-World Example 8 Applied Vector Operations

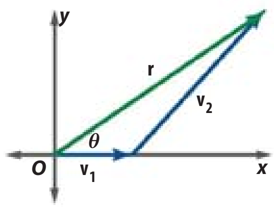
FOOTBALL **A quarterback running forward at 5 meters per second throws a football with a velocity of 25 meters per second at an angle of 40° with the horizontal. What is the resultant speed and direction of the pass?**



Since the quarterback moves straight forward, the component form of his velocity $\mathbf{v}_1$ is $\langle 5, 0 \rangle$. Use the magnitude and direction of the football's velocity $\mathbf{v}_2$ to write this vector in component form.

$\mathbf{v}_2 = \langle |\mathbf{v}_2| \cos \theta, |\mathbf{v}_2| \sin \theta \rangle$	Component form of v_2
$= \langle 25 \cos 40°, 25 \sin 40° \rangle$	$|v_2| = 25$ and $\theta = 40°$
$\approx \langle 19.2, 16.1 \rangle$	Simplify.

Add the algebraic vectors representing $\mathbf{v}_1$ and $\mathbf{v}_2$ to find the resultant velocity, vector **r**.



Note: Not drawn to scale.

$\mathbf{r} = \mathbf{v}_1 + \mathbf{v}_2$	Resultant vector
$= \langle 5, 0 \rangle + \langle 19.2, 16.1 \rangle$	Substitution
$= \langle 24.2, 16.1 \rangle$	Vector Addition

The magnitude of this resultant is $|\mathbf{r}| = \sqrt{24.2^2 + 16.1^2}$ or about 29.1. Next find the resultant direction angle θ.

$\tan \theta = \dfrac{16.1}{24.2}$	$\tan \theta = \frac{b}{a}$ where $\langle a, b \rangle = \langle 24.2, 16.1 \rangle$
$\theta = \tan^{-1} \dfrac{16.1}{24.2}$ or about 33.6°	Solve for θ.

Therefore, the resultant velocity of the pass is about 29.1 meters per second at an angle of about 33.6° with the horizontal.

GuidedPractice

8. **FOOTBALL** What would the resultant velocity of the football be if the quarterback made the same pass running 5 meters per second backward?

Exercises

= Step-by-Step Solutions begin on page R29.

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. (Examples 1 and 2)

1. $A(-3, 1), B(4, 5)$

2. $A(2, -7), B(-6, 9)$

3. $A(10, -2), B(3, -5)$

4. $A(-2, 7), B(-9, -1)$

5. $A(-5, -4), B(8, -2)$

6. $A(-2, 6), B(1, 10)$

7. $A(2.5, -3), B(-4, 1.5)$

8. $A(-4.3, 1.8), B(9.4, -6.2)$

9. $A\left(\frac{1}{2}, -9\right), B\left(6, \frac{5}{2}\right)$

10. $A\left(\frac{3}{5}, -\frac{2}{5}\right), B(-1, 7)$

Find each of the following for $\mathbf{f} = \langle 8, 0\rangle$, $\mathbf{g} = \langle -3, -5\rangle$, and $\mathbf{h} = \langle -6, 2\rangle$. (Example 3)

11. $4\mathbf{h} - \mathbf{g}$

12. $\mathbf{f} + 2\mathbf{h}$

13. $3\mathbf{g} - 5\mathbf{f} + \mathbf{h}$

14. $2\mathbf{f} + \mathbf{g} - 3\mathbf{h}$

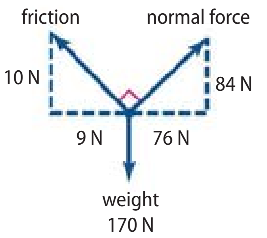
15. $\mathbf{f} - 2\mathbf{g} - 2\mathbf{h}$

16. $\mathbf{h} - 4\mathbf{f} + 5\mathbf{g}$

17. $4\mathbf{g} - 3\mathbf{f} + \mathbf{h}$

18. $6\mathbf{h} + 5\mathbf{f} - 10\mathbf{g}$

19. PHYSICS In physics, force diagrams are used to show the effects of all the different forces acting upon an object. The following force diagram could represent the forces acting upon a child sliding down a slide. (Example 3)



a. Using the blue dot representing the child as the origin, express each force as a vector in component form.

b. Find the component form of the resultant vector representing the force that causes the child to move down the slide.

Find a unit vector u with the same direction as v. (Example 4)

20. $\mathbf{v} = \langle -2, 7\rangle$

21. $\mathbf{v} = \langle 9, -3\rangle$

22. $\mathbf{v} = \langle -8, -5\rangle$

23. $\mathbf{v} = \langle 6, 3\rangle$

24. $\mathbf{v} = \langle -2, 9\rangle$

25. $\mathbf{v} = \langle -1, -5\rangle$

26. $\mathbf{v} = \langle 1, 7\rangle$

27. $\mathbf{v} = \langle 3, -4\rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j. (Example 5)

28. $D(4, -1), E(5, -7)$

29. $D(9, -6), E(-7, 2)$

30. $D(3, 11), E(-2, -8)$

31. $D(9.5, 1), E(0, -7.3)$

32. $D(-3, -5.7), E(6, -8.1)$

33. $D(-4, -6), E(9, 5)$

34. $D\left(\frac{1}{8}, 3\right), E\left(-4, \frac{2}{7}\right)$

35. $D(-3, 1.5), E(-3, 1.5)$

36. COMMUTE To commute to school, Larisa leaves her house and drives north on Pepper Lane for 2.4 miles. She turns left on Cinnamon Drive for 3.1 miles and then turns right on Maple Street for 5.8 miles. Express Larisa's commute as a linear combination of unit vectors **i** and **j**. (Example 5)

37. ROWING Nadia is rowing across a river at a speed of 5 miles per hour perpendicular to the shore. The river has a current of 3 miles per hour heading downstream. (Example 5)

a. At what speed is she traveling?

b. At what angle is she traveling with respect to the shore?

Find the component form of v with the given magnitude and direction angle. (Example 6)

38. $|\mathbf{v}| = 12, \theta = 60°$

39. $|\mathbf{v}| = 4, \theta = 135°$

40. $|\mathbf{v}| = 6, \theta = 240°$

41. $|\mathbf{v}| = 16, \theta = 330°$

42. $|\mathbf{v}| = 28, \theta = 273°$

43. $|\mathbf{v}| = 15, \theta = 125°$

Find the direction angle of each vector to the nearest tenth of a degree. (Example 7)

44. $3\mathbf{i} + 6\mathbf{j}$

45. $-2\mathbf{i} + 5\mathbf{j}$

46. $8\mathbf{i} - 2\mathbf{j}$

47. $-4\mathbf{i} - 3\mathbf{j}$

48. $\langle -5, 9\rangle$

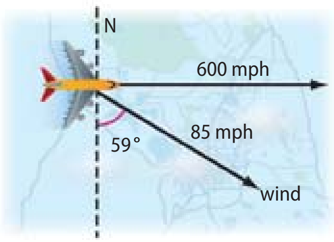
49. $\langle 7, 7\rangle$

50. $\langle -6, -4\rangle$

51. $\langle 3, -8\rangle$

52. SLEDDING Maggie is pulling a sled with a force of 275 newtons by holding its rope at a 58° angle. Her brother is going to help by pushing the sled with a force of 320 newtons. Determine the magnitude and direction of the total resultant force on the sled. (Example 8)

53. NAVIGATION An airplane is traveling due east with a speed of 600 miles per hour. The wind blows at 85 miles per hour at an angle of S59°E. (Example 8)



a. Determine the speed of the airplane's flight.

b. Determine the angle of the airplane's flight.

54. **HEADING** A pilot needs to plot a course that will result in a velocity of 500 miles per hour in a direction of due west. If the wind is blowing 100 miles per hour from the directed angle of 192°, find the direction and the speed the pilot should set to achieve this resultant.

Determine whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ with the initial and terminal points given are equivalent. If so, prove that $\overrightarrow{AB} = \overrightarrow{CD}$. If not, explain why not.

55. $A(3, 5)$, $B(6, 9)$, $C(-4, -4)$, $D(-2, 0)$

56. $A(-4, -5)$, $B(-8, 1)$, $C(3, -3)$, $D(1, 0)$

57. $A(1, -3)$, $B(0, -10)$, $C(11, 8)$, $D(10, 1)$

58. **RAFTING** The Soto family is rafting across a river. Suppose that they are on a stretch of the river that is 150 meters wide, flowing south at a rate of 1.0 meter per second. In still water, their raft travels 5.0 meters per second.

a. What is the speed of the raft?

b. How far downriver will the raft land?

c. How long does it take them to travel from one bank to the other if they head directly across the river?

59. **NAVIGATION** A jet is flying with an air speed of 480 miles per hour at a bearing of N82°E. Because of the wind, the ground speed of the plane is 518 miles per hour at a bearing of N79°E.

a. Draw a diagram to represent the situation.

b. What are the speed and direction of the wind?

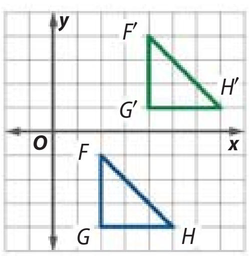
c. If the pilot increased the air speed of the plane to 500 miles per hour, what would be the resulting ground speed and direction of the plane?

60. **TRANSLATIONS** You can translate a figure along a translation vector $\langle a, b\rangle$ by adding a to each x-coordinate and b to each y-coordinate. Consider the triangles shown below.

a. Describe the translation from $\triangle FGH$ to $\triangle F'G'H'$ using a translation vector.

b. Graph $\triangle F'G'H'$ and its translated image $\triangle F''G''H''$ along $\langle -3, -6\rangle$.

c. Describe the translation from $\triangle FGH$ to $\triangle F''G''H''$ using a translation vector.

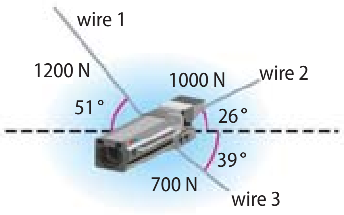


Given the initial point and magnitude of each vector, determine a possible terminal point of the vector.

61. $(-1, 4)$; $\sqrt{37}$

62. $(-3, -7)$; 10

63. **CAMERA** A video camera that follows the action at a sporting event is supported by three wires. The tension in each wire can be modeled by a vector.

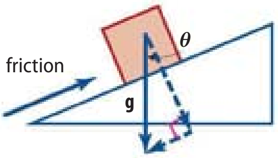


a. Find the component form of each vector.

b. Find the component form of the resultant vector acting on the camera.

c. Find the magnitude and direction of the resulting force.

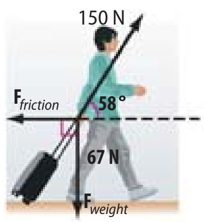
64. **FORCE** A box is stationary on a ramp. Both gravity **g** and friction are exerted on the box. The components of gravity are shown in the diagram. What must be true of the force of friction for this scenario to be possible?



H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** If vectors **a** and **b** are parallel, write a vector equation relating **a** and **b**.

66. **CHALLENGE** To pull luggage, Greg exerts a force of 150 newtons at an angle of 58° with the horizontal. If the resultant force on the luggage is 72 newtons at an angle of 56.7° with the horizontal, what is the magnitude of the resultant of $\mathbf{F}_{friction}$ and $\mathbf{F}_{weight}$?



67. **REASONING** If given the initial point of a vector and its magnitude, describe the locus of points that represent possible locations for the terminal point.

68. **WRITING IN MATH** Explain how to find the direction angle of a vector in the fourth quadrant.

69 **CHALLENGE** The direction angle of $\langle x, y\rangle$ is $(4y)°$. Find x in terms of y.

PROOF **Prove each vector property. Let $\mathbf{a} = \langle x_1, y_1\rangle$, $\mathbf{b} = \langle x_2, y_2\rangle$, and $\mathbf{c} = \langle x_3, y_3\rangle$.**

70. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

71. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

72. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, where k is a scalar

73. $|k\mathbf{a}| = |k|\,|\mathbf{a}|$, where k is a scalar

Spiral Review

74. TOYS Roman is pulling a toy by exerting a force of 1.5 newtons on a string attached to the toy. (Lesson 8-1)

a. The string makes an angle of 52° with the floor. Find the horizontal and vertical components of the force.

b. If Roman raises the string so that it makes a 78° angle with the floor, what are the magnitudes of the horizontal and vertical components of the force?

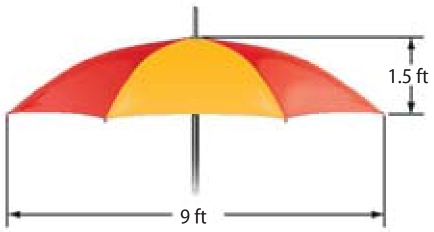
Write each pair of parametric equations in rectangular form. (Lesson 7-5)

75. $y = t^2 + 2,\ x = 3t - 6$

76. $y = t^2 - 5,\ x = 2t + 8$

77. $y = 7t,\ x = t^2 - 1$

78. UMBRELLAS A beach umbrella has an arch in the shape of a parabola. Write an equation to model the arch, assuming that the origin and the vertex are at the point where the pole and the canopy of the umbrella meet. Express y in terms of x. (Lesson 7-1)



Decompose each expression into partial fractions. (Lesson 6-4)

79. $\dfrac{5z - 11}{2z^2 + z - 6}$

80. $\dfrac{7x^2 + 18x - 1}{(x^2 - 1)(x + 2)}$

81. $\dfrac{9 - 9x}{x^2 - 9}$

Verify each identity. (Lesson 5-4)

82. $\sin(\theta + 180°) = -\sin\theta$

83. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$

Express each logarithm in terms of ln 3 and ln 7. (Lesson 3-3)

84. $\ln 189$

85. $\ln 5.\overline{4}$

86. $\ln 441$

87. $\ln \dfrac{9}{343}$

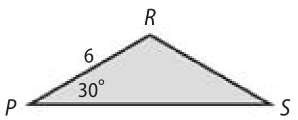
Find each $f(c)$ using synthetic substitution. (Lesson 2-3)

88. $f(x) = 6x^6 - 9x^4 + 12x^3 - 16x^2 + 8x + 24;\ c = 6$

89. $f(x) = 8x^5 - 12x^4 + 18x^3 - 24x^2 + 36x - 48,\ c = 4$

Skills Review for Standardized Tests

90. SAT/ACT If $PR = RS$, what is the area of triangle PRS?



A $9\sqrt{2}$
B $9\sqrt{3}$
C $18\sqrt{2}$
D $18\sqrt{3}$
E $36\sqrt{3}$

91. REVIEW Dalton has made a game for his younger sister's birthday party. The playing board is a circle divided evenly into 8 sectors. If the circle has a radius of 18 inches, what is the approximate area of one of the sectors?

F 4 in^2
G 32 in^2
H 127 in^2
J 254 in^2

92. Paramedics Lydia Gonzalez and Theo Howard are moving a person on a stretcher. Ms. Gonzalez is pushing the stretcher from behind with a force of 135 newtons at 58° with the horizontal, while Mr. Howard is pulling the stretcher from the front with a force of 214 newtons at 43° with the horizontal. What is the magnitude of the horizontal force exerted on the stretcher?

A 228 newtons
B 260 newtons
C 299 newtons
D 346 newtons

93. REVIEW Find the center and radius of the circle with equation $(x - 4)^2 + y^2 - 16 = 0$.

F $C(-4, 0);\ r = 4$ units
G $C(-4, 0);\ r = 16$ units
H $C(4, 0);\ r = 4$ units
J $C(4, 0);\ r = 16$ units

LESSON 8-3 Dot Products and Vector Projections

Then

- You found the magnitudes of and operated with algebraic vectors. (Lesson 8-2)

Now

1. Find the dot product of two vectors, and use the dot product to find the angle between them.
2. Find the projection of one vector onto another.

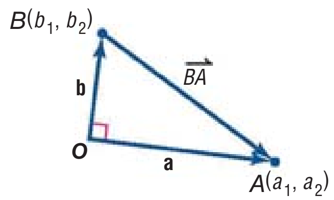
Why?

- The word *work* can have different meanings in everyday life; but in physics, its definition is very specific. Work is the magnitude of a force applied to an object multiplied by the distance through which the object moves parallel to this applied force. Work, such as that done to push a car a specific distance, can also be calculated using a vector operation called a *dot product*.

NewVocabulary
dot product
orthogonal
vector projection
work

1 Dot Product

In Lesson 8-2, you studied the vector operations of vector addition and scalar multiplication. In this lesson, you will use a third vector operation. Consider two perpendicular vectors in standard position **a** and **b**. Let $\overrightarrow{BA}$ be the vector between their terminal points as shown in the figure.



By the Pythagorean Theorem, we know that

$$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Using the definition of the magnitude of a vector, we can find $|\overrightarrow{BA}|^2$.

$$|\overrightarrow{BA}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$ Definition of vector magnitude

$$|\overrightarrow{BA}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$ Square each side.

$$|\overrightarrow{BA}|^2 = a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2$$ Expand each binomial square.

$$|\overrightarrow{BA}|^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1b_1 + a_2b_2)$$ Group the squared terms.

$$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$$ $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ so $|\mathbf{a}|^2 = a_1^2 + a_2^2$ and $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$, so $|\mathbf{b}|^2 = b_1^2 + b_2^2$.

Notice that the expressions $|\mathbf{a}|^2 + |\mathbf{b}|^2$ and $|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$ are equivalent if and only if $a_1b_1 + a_2b_2 = 0$. The expression $a_1b_1 + a_2b_2$ is called the **dot product** of **a** and **b**, denoted $\mathbf{a} \cdot \mathbf{b}$ and read as *a dot b*.

KeyConcept Dot Product of Vectors in a Plane

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.

Notice that unlike vector addition and scalar multiplication, the dot product of two vectors yields a scalar, not a vector. As demonstrated above, two nonzero vectors are perpendicular if and only if their dot product is 0. Two vectors with a dot product of 0 are said to be **orthogonal**.

KeyConcept Orthogonal Vectors

The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The terms *perpendicular* and *orthogonal* have essentially the same meaning, except when **a** or **b** is the zero vector. The zero vector is orthogonal to any vector **a**, since $\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0a_1 + 0a_2$ or 0. However, since the zero vector has no magnitude or direction, it cannot be perpendicular to **a**.

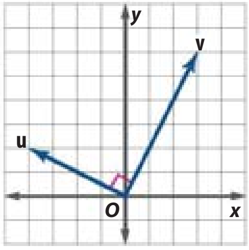


Figure 8.3.1

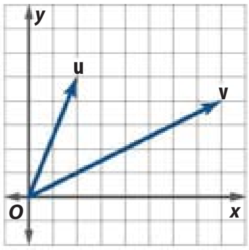


Figure 8.3.2

Example 1 Find the Dot Product to Determine Orthogonal Vectors

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle 3, 6\rangle$, $\mathbf{v} = \langle -4, 2\rangle$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 3(-4) + 6(2)\\ &= 0\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$, **u** and **v** are orthogonal, as illustrated in Figure 8.3.1.

b. $\mathbf{u} = \langle 2, 5\rangle$, $\mathbf{v} = \langle 8, 4\rangle$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 2(8) + 5(4)\\ &= 36\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, **u** and **v** are not orthogonal, as illustrated in Figure 8.3.2.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -2\rangle$, $\mathbf{v} = \langle -5, 1\rangle$

1B. $\mathbf{u} = \langle -2, -3\rangle$, $\mathbf{v} = \langle 9, -6\rangle$

Dot products have the following properties.

KeyConcept Properties of the Dot Product

If u, v, and w are vectors and k is a scalar, then the following properties hold.

Commutative Property	$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
Distributive Property	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
Scalar Multiplication Property	$k(\mathbf{u}\cdot\mathbf{v}) = k\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot k\mathbf{v}$
Zero Vector Dot Product Property	$\mathbf{0}\cdot\mathbf{u} = 0$
Dot Product and Vector Magnitude Relationship	$\mathbf{u}\cdot\mathbf{u} = \lvert\mathbf{u}\rvert^2$

Proof

Proof $\quad \mathbf{u}\cdot\mathbf{u} = \lvert\mathbf{u}\rvert^2$

Let $\mathbf{u} = \langle u_1, u_2\rangle$.

$\mathbf{u}\cdot\mathbf{u} = {u_1}^2 + {u_2}^2$ — Dot product

$= \left(\sqrt{({u_1}^2 + {u_2}^2)}\right)^2$ — Write as the square of the square root of $u_1^2 + u_2^2$.

$= \lvert\mathbf{u}\rvert^2$ — $\sqrt{u_1^2 + u_2^2} = \lvert\mathbf{u}\rvert$

You will prove the first three properties in Exercises 70–72.

> **ReadingMath**
> **Inner and Scalar Products**
> The dot product is also called the *inner product* or the *scalar product.*

Example 2 Use the Dot Product to Find Magnitude

Use the dot product to find the magnitude of a = ⟨−5, 12⟩.

Since $\lvert\mathbf{a}\rvert^2 = \mathbf{a}\cdot\mathbf{a}$, then $\lvert\mathbf{a}\rvert = \sqrt{\mathbf{a}\cdot\mathbf{a}}$.

$\lvert\langle -5, 12\rangle\rvert = \sqrt{\langle -5, 12\rangle\cdot\langle -5, 12\rangle}$ — $\mathbf{a} = \langle -5, 12\rangle$

$= \sqrt{(-5)^2 + 12^2}$ or 13 — Simplify.

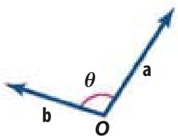
GuidedPractice

Use the dot product to find the magnitude of the given vector.

2A. $\mathbf{b} = \langle 12, 16\rangle$

2B. $\mathbf{c} = \langle -1, -7\rangle$

The angle θ between any two nonzero vectors **a** and **b** is the corresponding angle between these vectors when placed in standard position, as shown. This angle is always measured such that $0 \leq \theta \leq \pi$ or $0° \leq \theta \leq 180°$. The dot product can be used to find the angle between two nonzero vectors.



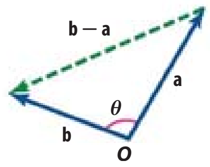
StudyTip

Parallel and Perpendicular Vectors Two vectors are perpendicular if the angle between them is 90°. Two vectors are parallel if the angle between them is 0° or 180°.

KeyConcept Angle Between Two Vectors

If θ is the angle between nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$



Proof

Consider the triangle determined by **a**, **b**, and **b** − **a** in the figure above.

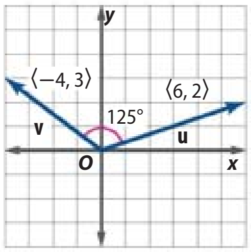
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos \theta = \lvert\mathbf{b} - \mathbf{a}\rvert^2$	Law of Cosines
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos \theta = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$	$\lvert\mathbf{u}\rvert^2 = \mathbf{u} \cdot \mathbf{u}$
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos \theta = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a}$	Distributive Property for Dot Products
$\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos \theta = \lvert\mathbf{b}\rvert^2 - 2\mathbf{a} \cdot \mathbf{b} + \lvert\mathbf{a}\rvert^2$	$\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$
$-2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert \cos \theta = -2\mathbf{a} \cdot \mathbf{b}$	Subtract $\lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2$ from each side.
$\cos \theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert}$	Divide each side by $-2\lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert$.

Example 3 Find the Angle Between Two Vectors

Find the angle θ between vectors u and v to the nearest tenth of a degree.

a. $\mathbf{u} = \langle 6, 2\rangle$ and $\mathbf{v} = \langle -4, 3\rangle$

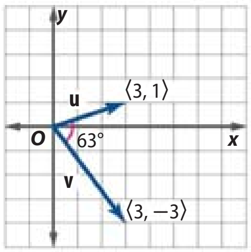
$\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert}$	Angle between two vectors
$\cos \theta = \dfrac{\langle 6, 2\rangle \cdot \langle -4, 3\rangle}{\lvert\langle 6, 2\rangle\rvert\,\lvert\langle -4, 3\rangle\rvert}$	$\mathbf{u} = \langle 6, 2\rangle$ and $\mathbf{v} = \langle -4, 3\rangle$
$\cos \theta = \dfrac{-24 + 6}{\sqrt{40}\,\sqrt{25}}$	Evaluate.
$\cos \theta = \dfrac{-9}{5\sqrt{10}}$	Simplify.
$\theta = \cos^{-1} \dfrac{-9}{5\sqrt{10}}$ or about 124.7°	Solve for θ.



The measure of the angle between **u** and **v** is about 124.7°.

b. $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 3, -3\rangle$

$\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\,\lvert\mathbf{v}\rvert}$	Angle between two vectors
$\cos \theta = \dfrac{\langle 3, 1\rangle \cdot \langle 3, -3\rangle}{\lvert\langle 3, 1\rangle\rvert\,\lvert\langle 3, -3\rangle\rvert}$	$\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 3, -3\rangle$
$\cos \theta = \dfrac{9 + (-3)}{\sqrt{10}\,\sqrt{18}}$	Evaluate.
$\cos \theta = \dfrac{1}{\sqrt{5}}$	Simplify.
$\theta = \cos^{-1} \dfrac{1}{\sqrt{5}}$ or about 63.4°	Solve for θ.



The measure of the angle between **u** and **v** is about 63.4°.

GuidedPractice

3A. $\mathbf{u} = \langle -5, -2\rangle$ and $\mathbf{v} = \langle 4, 4\rangle$

3B. $\mathbf{u} = \langle 9, 5\rangle$ and $\mathbf{v} = \langle -6, 7\rangle$

2 Vector Projection

In Lesson 8-1, you learned that a vector can be resolved or decomposed into two perpendicular components. While these components are often horizontal and vertical, it is sometimes useful instead for one component to be parallel to another vector.

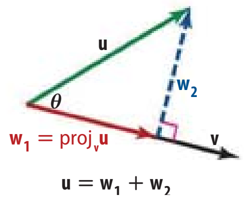
StudyTip

Perpendicular Component The vector $\mathbf{w}_2$ is called the *component of* $\mathbf{u}$ *perpendicular to* $\mathbf{v}$.

KeyConcept Projection of u onto v

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vector components of $\mathbf{u}$ such that $\mathbf{w}_1$ is parallel to $\mathbf{v}$ as shown. Then vector $\mathbf{w}_1$ is called the **vector projection** of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, and

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$



Proof

Since $\text{proj}_{\mathbf{v}}\mathbf{u}$ is parallel to $\mathbf{v}$, it can be written as a scalar multiple of $\mathbf{v}$. As a scalar multiple of a unit vector $\mathbf{v}_x$ with the same direction as $\mathbf{v}$, $\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\,\mathbf{v}_x$. Use the right triangle formed by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{u}$ and the cosine ratio to find an expression for $|\mathbf{w}_1|$.

$\cos\theta = \dfrac{|\mathbf{w}_1|}{|\mathbf{u}|}$ — Cosine ratio

$|\mathbf{u}||\mathbf{v}|\cos\theta = \cancel{|\mathbf{u}|}\,|\mathbf{v}|\,\dfrac{|\mathbf{w}_1|}{\cancel{|\mathbf{u}|}}$ — Multiply each side by the scalar quantity $|\mathbf{u}|\,|\mathbf{v}|$.

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{v}|\,|\mathbf{w}_1|$ — $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$, so $|\mathbf{u}|\,|\mathbf{v}|\cos\theta = \mathbf{u} \cdot \mathbf{v}$.

$|\mathbf{w}_1| = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ — Solve for $|\mathbf{w}_1|$.

Now use $\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\,\mathbf{v}_x$ to find $\text{proj}_{\mathbf{v}}\mathbf{u}$ as a scalar multiple of $\mathbf{v}$.

$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{w}_1|\,\mathbf{v}_x$

$= \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \cdot \dfrac{\mathbf{v}}{|\mathbf{v}|}$ — $|\mathbf{w}_1| = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ and $\mathbf{v}_x = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ — Multiply magnitudes.

Example 4 Find the Projection of u onto v

Find the projection of $\mathbf{u} = \langle 3, 2\rangle$ onto $\mathbf{v} = \langle 5, -5\rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is the projection of $\mathbf{u}$ onto $\mathbf{v}$.

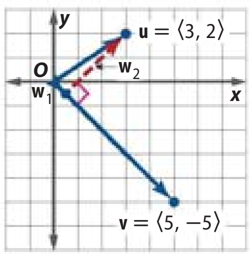


Figure 8.3.5

Step 1 Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 3, 2\rangle \cdot \langle 5, -5\rangle}{|\langle 5, -5\rangle|^2}\langle 5, -5\rangle$$

$$= \frac{5}{50}\langle 5, -5\rangle$$

$$= \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

$$= \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$$

$$= \langle 3, 2\rangle - \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$= \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$$

Therefore, $\text{proj}_{\mathbf{v}}\mathbf{u}$ is $\mathbf{w}_1 = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$ as shown in Figure 8.3.5, and $\mathbf{u} = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$

GuidedPractice

4. Find the projection of $\mathbf{u} = \langle 1, 2\rangle$ onto $\mathbf{v} = \langle 8, 5\rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is the projection of $\mathbf{u}$ onto $\mathbf{v}$.

While the projection of **u** onto **v** is a vector parallel to **v**, this vector will not necessarily have the same direction as **v**, as illustrated in the next example.

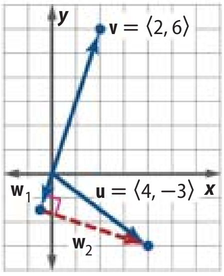


Figure 8.3.6

Example 5 Projection with Direction Opposite v

Find the projection of $\mathbf{u} = \langle 4, -3\rangle$ onto $\mathbf{v} = \langle 2, 6\rangle$. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

Notice that the angle between **u** and **v** is obtuse, so the projection of **u** onto **v** lies on the vector opposite **v** or $-\mathbf{v}$, as shown in Figure 8.3.6.

Step 1 Find the projection of **u** onto **v**.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 4, -3\rangle \cdot \langle 2, 6\rangle}{|\langle 2, 6\rangle|^2}\langle 2, 6\rangle$$

$$= \frac{-10}{40}\langle 2, 6\rangle \text{ or } \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ or $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 4, -3\rangle - \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

$$= \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$$

Therefore, $\text{proj}_{\mathbf{v}}\mathbf{u}$, $\mathbf{w}_1 = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$ and $\mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle + \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$.

GuidedPractice

5. Find the projection of $\mathbf{u} = \langle -3, 4\rangle$ onto $\mathbf{v} = \langle 6, 1\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.

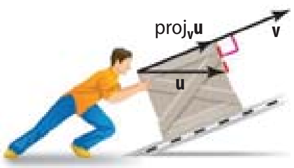
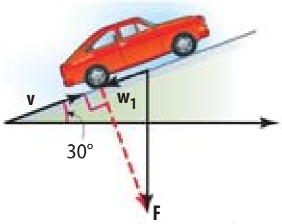


Figure 8.3.7

If the vector **u** represents a force, then $\text{proj}_{\mathbf{v}}\mathbf{u}$ represents the effect of that force acting in the direction of **v**. For example, if you push a box uphill (in the direction **v**) with a force **u** (Figure 8.3.7), the effective force is a component push in the direction of **v**, $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Real-World Example 6 Use a Vector Projection to Find a Force

CARS **A 3000-pound car sits on a hill inclined at 30° as shown. Ignoring the force of friction, what force is required to keep the car from rolling down the hill?**



Note: Not drawn to scale

The weight of the car is the force exerted due to gravity, $\mathbf{F} = \langle 0, -3000\rangle$. To find the force $-\mathbf{w}_1$ required to keep the car from rolling down the hill, project **F** onto a unit vector **v** in the direction of the side of the hill.

Step 1 Find a unit vector **v** in the direction of the hill.

$\mathbf{v} = \langle |\mathbf{v}| (\cos\theta), |\mathbf{v}| (\sin\theta)\rangle$ — Component form of **v** in terms of $|\mathbf{v}|$ and θ

$= \langle 1(\cos 30°), 1(\sin 30°)\rangle$ or $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$ — $|\mathbf{v}| = 1$ and $\theta = 30°$

Step 2 Find $\mathbf{w}_1$, the projection of **F** onto unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

$\text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ — Projection of **F** onto **v**

$= (\mathbf{F}\cdot\mathbf{v})\mathbf{v}$ — Since **v** is a unit vector, $|\mathbf{v}| = 1$. Simplify.

$= \left(\langle 0, -3000\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right)\mathbf{v}$ — $\mathbf{F} = \langle 0, -3000\rangle$ and $\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

$= -1500\mathbf{v}$ — Find the dot product.

The force required is $-\mathbf{w}_1 = -(-1500\,\mathbf{v})$ or $1500\mathbf{v}$. Since **v** is a unit vector, this means that this force has a magnitude of 1500 pounds and is in the direction of the side of the hill.

GuidedPractice

6. **SLEDDING** Mary sits on a sled on the side of a hill inclined at 60°. What force is required to keep the sled from sliding down the hill if the weight of Mary and the sled is 125 pounds?

Another application of vector projection is the calculation of the work done by a force. Consider a constant force **F** acting on an object to move it from point A to point B as shown in Figure 8.3.8. If **F** is parallel to $\overrightarrow{AB}$, then the **work** W done by **F** is the magnitude of the force times the distance from A to B or $W = |\mathbf{F}||\overrightarrow{AB}|$.

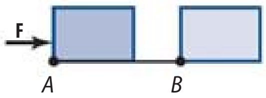


Figure 8.3.8

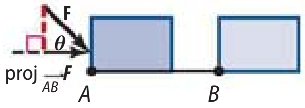


Figure 8.3.9

To calculate the work done by a constant force **F** in *any* direction to move an object from point A to B, as shown in Figure 8.3.9 you can use the vector projection of **F** onto $\overrightarrow{AB}$.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}|\,|\overrightarrow{AB}|$ — Projection formula for work

$= |\mathbf{F}|\,(\cos\theta)\,|\overrightarrow{AB}|$ — $\cos\theta = \dfrac{|\text{proj}_{\overrightarrow{AB}}\mathbf{F}|}{|\mathbf{F}|}$, so $|\text{proj}_{\overrightarrow{AB}}\mathbf{F}| = |\mathbf{F}|\cos\theta$.

$= \mathbf{F} \cdot \overrightarrow{AB}$ — $\cos\theta = \dfrac{\mathbf{F}\cdot\overrightarrow{AB}}{|\mathbf{F}|\,|\overrightarrow{AB}|}$, so $|\mathbf{F}|\,|\overrightarrow{AB}|\cos\theta = \mathbf{F}\cdot\overrightarrow{AB}$.

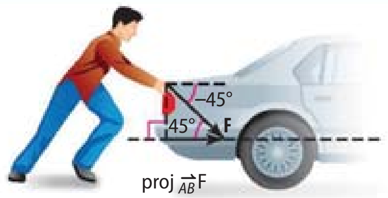
Therefore, this work can be calculated by finding the dot product of the constant force **F** and the directed distance $\overrightarrow{AB}$.

StudyTip

Units for Work Work is measured in foot-pounds in the customary system of measurement and in newton-meters (N·m) or joules (J) in the metric system.

Real-World Example 7 Calculate Work

ROADSIDE ASSISTANCE **A person pushes a car with a constant force of 120 newtons at a constant angle of 45° as shown. Find the work done in joules moving the car 10 meters.**



Method 1 Use the projection formula for work.

The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 120\cos 45°$. The magnitude of the directed distance $\overrightarrow{AB}$ is 10.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$ — Projection formula for work

$= (120\cos 45°)(10)$ or about 848.5 — Substitution

Method 2 Use the dot product formula for work.

The component form of the force vector **F** in terms of magnitude and direction angle given is $\langle 120\cos(-45°), 120\sin(-45°)\rangle$. The component form of the directed distance the car is moved is $\langle 10, 0\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB}$ — Dot product formula for work

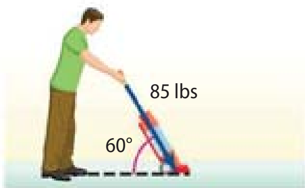
$= \langle 120\cos(-45°), 120\sin(-45°)\rangle \cdot \langle 10, 0\rangle$ — Substitution

$= [120\cos(-45°)](10)$ or about 848.5 — Dot product

Therefore, the person does about 848.5 joules of work pushing the car.

GuidedPractice

7. **CLEANING** Rick is pushing a vacuum cleaner with a force of 85 pounds. The handle of the vacuum cleaner makes a 60° angle with the floor. How much work in foot-pounds does he do if he pushes the vacuum cleaner 6 feet?



Exercises

= Step-by-Step Solutions begin on page R29.

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -5\rangle, \mathbf{v} = \langle 6, 2\rangle$
2. $\mathbf{u} = \langle -10, -16\rangle, \mathbf{v} = \langle -8, 5\rangle$
3. $\mathbf{u} = \langle 9, -3\rangle, \mathbf{v} = \langle 1, 3\rangle$
4. $\mathbf{u} = \langle 4, -4\rangle, \mathbf{v} = \langle 7, 5\rangle$
5. $\mathbf{u} = \langle 1, -4\rangle, \mathbf{v} = \langle 2, 8\rangle$
6. $\mathbf{u} = 11\mathbf{i} + 7\mathbf{j}; \mathbf{v} = -7\mathbf{i} + 11\mathbf{j}$
7. $\mathbf{u} = \langle -4, 6\rangle, \mathbf{v} = \langle -5, -2\rangle$
8. $\mathbf{u} = 8\mathbf{i} + 6\mathbf{j}; \mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

9. **SPORTING GOODS** The vector $\mathbf{u} = \langle 406, 297\rangle$ gives the numbers of men's basketballs and women's basketballs, respectively, in stock at a sporting goods store. The vector $\mathbf{v} = \langle 27.5, 15\rangle$ gives the prices in dollars of the two types of basketballs, respectively. (Example 1)
   a. Find the dot product $\mathbf{u} \cdot \mathbf{v}$.
   b. Interpret the result in the context of the problem.

Use the dot product to find the magnitude of the given vector. (Example 2)

10. $\mathbf{m} = \langle -3, 11\rangle$
11. $\mathbf{r} = \langle -9, -4\rangle$
12. $\mathbf{n} = \langle 6, 12\rangle$
13. $\mathbf{v} = \langle 1, -18\rangle$
14. $\mathbf{p} = \langle -7, -2\rangle$
15. $\mathbf{t} = \langle 23, -16\rangle$

Find the angle θ between u and v to the nearest tenth of a degree. (Example 3)

16. $\mathbf{u} = \langle 0, -5\rangle, \mathbf{v} = \langle 1, -4\rangle$
17. $\mathbf{u} = \langle 7, 10\rangle, \mathbf{v} = \langle 4, -4\rangle$
18. $\mathbf{u} = \langle -2, 4\rangle, \mathbf{v} = \langle 2, -10\rangle$
19. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}, \mathbf{v} = -4\mathbf{i} - 2\mathbf{j}$
20. $\mathbf{u} = \langle -9, 0\rangle, \mathbf{v} = \langle -1, -1\rangle$
21. $\mathbf{u} = -\mathbf{i} - 3\mathbf{j}, \mathbf{v} = -7\mathbf{i} - 3\mathbf{j}$
22. $\mathbf{u} = \langle 6, 0\rangle, \mathbf{v} = \langle -10, 8\rangle$
23. $\mathbf{u} = -10\mathbf{i} + \mathbf{j}, \mathbf{v} = 10\mathbf{i} - 5\mathbf{j}$

24. **CAMPING** Regina and Luis set off from their campsite to search for firewood. The path that Regina takes can be represented by $\mathbf{u} = \langle 3, -5\rangle$. The path that Luis takes can be represented by $\mathbf{v} = \langle -7, 6\rangle$. Find the angle between the pair of vectors. (Example 3)

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Examples 4 and 5)

25. $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j}, \mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$
26. $\mathbf{u} = \langle 5, 7\rangle, \mathbf{v} = \langle -4, 4\rangle$
27. $\mathbf{u} = \langle 8, 2\rangle, \mathbf{v} = \langle -4, 1\rangle$
28. $\mathbf{u} = 6\mathbf{i} + \mathbf{j}, \mathbf{v} = -3\mathbf{i} + 9\mathbf{j}$
29. $\mathbf{u} = \langle 2, 4\rangle, \mathbf{v} = \langle -3, 8\rangle$
30. $\mathbf{u} = \langle -5, 9\rangle, \mathbf{v} = \langle 6, 4\rangle$
31. $\mathbf{u} = 5\mathbf{i} - 8\mathbf{j}, \mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$
32. $\mathbf{u} = -2\mathbf{i} - 5\mathbf{j}, \mathbf{v} = 9\mathbf{i} + 7\mathbf{j}$

33. **WAGON** Malcolm is pulling his sister in a wagon up a small slope at an incline of 15°. If the combined weight of Malcolm's sister and the wagon is 78 pounds, what force is required to keep her from rolling down the slope? (Example 6)

34. **SLIDE** Isabel is going down a slide but stops herself when she notices that another student is lying hurt at the bottom of the slide. What force is required to keep her from sliding down the slide if the incline is 53° and she weighs 62 pounds? (Example 6)

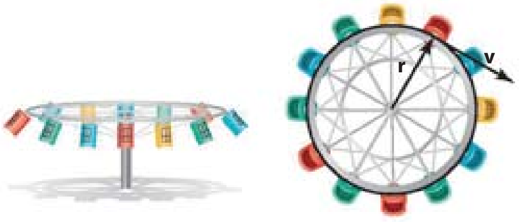
35. **PHYSICS** Alexa is pushing a construction barrel up a ramp 1.5 meters long into the back of a truck. She is using a force of 534 newtons and the ramp is 25° from the horizontal. How much work in joules is Alexa doing? (Example 7)

36. **SHOPPING** Sophia is pushing a shopping cart with a force of 125 newtons at a downward angle, or angle of depression, of 52°. How much work in joules would Sophia do if she pushed the shopping cart 200 meters? (Example 7)

Find a vector orthogonal to each vector.

37. $\langle -2, -8\rangle$
38. $\langle 3, 5\rangle$
39. $\langle 7, -4\rangle$
40. $\langle -1, 6\rangle$

41. **RIDES** For a circular amusement park ride, the position vector **r** is perpendicular to the tangent velocity vector **v** at any point on the circle, as shown below.



Front View **Top View**

   a. If the radius of the ride is 20 feet and the speed of the ride is constant at 40 feet per second, write the component forms of the position vector **r** and the tangent velocity vector **v** when **r** is at a directed angle of 35°.
   b. What method can be used to prove that the position vector and the velocity vector that you developed in part **a** are perpendicular? Show that the two vectors are perpendicular.

Given v and u • v, find u.

42. $\mathbf{v} = \langle 3, -6\rangle$, $\mathbf{u} \cdot \mathbf{v} = 33$

43. $\mathbf{v} = \langle 4, 6\rangle$, $\mathbf{u} \cdot \mathbf{v} = 38$

44. $\mathbf{v} = \langle -5, -1\rangle$, $\mathbf{u} \cdot \mathbf{v} = -8$

45. $\mathbf{v} = \langle -2, 7\rangle$, $\mathbf{u} \cdot \mathbf{v} = -43$

46. **SCHOOL** A student rolls her backpack from her Chemistry classroom to her English classroom using a force of 175 newtons.

a. If she exerts 3060 joules to pull her backpack 31 meters, what is the angle of her force?

b. If she exerts 1315 joules at an angle of 60°, how far did she pull her backpack?

Determine whether each pair of vectors are *parallel*, *perpendicular*, or *neither*. Explain your reasoning.

47. $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{3}{4}\right\rangle$, $\mathbf{v} = \langle 9, 8\rangle$

48. $\mathbf{u} = \langle -1, -4\rangle$, $\mathbf{v} = \langle 3, 6\rangle$

49. $\mathbf{u} = \langle 5, 7\rangle$, $\mathbf{v} = \langle -15, -21\rangle$

50. $\mathbf{u} = \langle \sec\theta, \csc\theta\rangle$, $\mathbf{v} = \langle \csc\theta, -\sec\theta\rangle$

Find the angle between the two vectors in radians.

51. $\mathbf{u} = \cos\left(\frac{\pi}{3}\right)\mathbf{i} + \sin\left(\frac{\pi}{3}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$

52. $\mathbf{u} = \cos\left(\frac{7\pi}{6}\right)\mathbf{i} + \sin\left(\frac{7\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{4}\right)\mathbf{i} + \sin\left(\frac{5\pi}{4}\right)\mathbf{j}$

53. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j}$

54. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{6}\right)\mathbf{i} + \sin\left(\frac{5\pi}{6}\right)\mathbf{j}$

55. **WORK** Tommy lifts his nephew, who weighs 16 kilograms, a distance of 0.9 meter. The force of weight in newtons can be calculated using $F = mg$, where m is the mass in kilograms and g is 9.8 meters per second squared. How much work did Tommy do to lift his nephew?

The vertices of a triangle on the coordinate plane are given. Find the measures of the angles of each triangle using vectors. Round to the nearest tenth of a degree.

56. (2, 3), (4, 7), (8, 1)

57. (−3, −2), (−3, −7), (3, −7)

58. (−4, −3), (−8, −2), (2, 1)

59. (1, 5), (4, −3), (−4, 0)

Given u, |v|, and θ, the angle between u and v, find possible values of v. Round to the nearest hundredth.

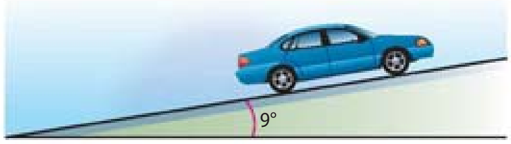
60. $\mathbf{u} = \langle 4, -2\rangle$, $|\mathbf{v}| = 10$, 45°

61. $\mathbf{u} = \langle 3, 4\rangle$, $|\mathbf{v}| = \sqrt{29}$, 121°

62. $\mathbf{u} = \langle -1, -6\rangle$, $|\mathbf{v}| = 7$, 96°

63. $\mathbf{u} = \langle -2, 5\rangle$, $|\mathbf{v}| = 12$, 27°

64. **CARS** A car is stationary on a 9° incline. Assuming that the only forces acting on the car are gravity and the 275 newton force applied by the brakes, about how much does the car weigh?



H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** Determine whether the statement below is *true* or *false*. Explain.

If |**d**|, |**e**|, and |**f**| form a Pythagorean triple, and the angles between **d** and **e** and between **e** and **f** are acute, then the angle between **d** and **f** must be a right angle. Explain your reasoning.

66. **ERROR ANALYSIS** Beng and Ethan are studying the properties of the dot product. Beng concludes that the dot product is associative because it is commutative; that is, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$. Ethan disagrees. Is either of them correct? Explain your reasoning.

67. **REASONING** Determine whether the statement below is *true* or *false*.
If **a** and **b** are both orthogonal to **v** in the plane, then **a** and **b** are parallel. Explain your reasoning.

68. **CHALLENGE** If **u** and **v** are perpendicular, what is the projection of **u** onto **v**?

69. **PROOF** Show that if the angle between vectors **u** and **v** is 90°, $\mathbf{u} \cdot \mathbf{v} = 0$ using the formula for the angle between two nonzero vectors.

PROOF Prove each dot product property. Let $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$, and $\mathbf{w} = \langle w_1, w_2\rangle$.

70. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

71. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

72. $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$

73. **WRITING IN MATH** Explain how to find the dot product of two nonzero vectors.

Spiral Review

Find each of the following for a = ⟨10, 1⟩, b = ⟨−5, 2.8⟩, and c = $\left\langle \frac{3}{4}, -9 \right\rangle$. (Lesson 8-2)

74. $\mathbf{b} - \mathbf{a} + 4\mathbf{c}$

75. $\mathbf{c} - 3\mathbf{a} + \mathbf{b}$

76. $2\mathbf{a} - 4\mathbf{b} + \mathbf{c}$

77. **GOLF** Jada drives a golf ball with a velocity of 205 feet per second at an angle of 32° with the ground. On the same hole, James drives a golf ball with a velocity of 190 feet per second at an angle of 41°. Find the magnitudes of the horizontal and vertical components for each force. (Lesson 8-1)

Graph the hyperbola given by each equation. (Lesson 7-3)

78. $\frac{(x-5)^2}{48} - \frac{y^2}{5} = 1$

79. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

80. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

Find the exact value of each expression, if it exists. (Lesson 4-6)

81. $\arcsin\left(\sin \frac{\pi}{6}\right)$

82. $\arctan\left(\tan \frac{1}{2}\right)$

83. $\sin\left(\cos^{-1} \frac{3}{4}\right)$

Solve each equation. (Lesson 3-4)

84. $\log_{12}(x^3 + 2) = \log_{12} 127$

85. $\log_2 x = \log_2 6 + \log_2 (x - 5)$

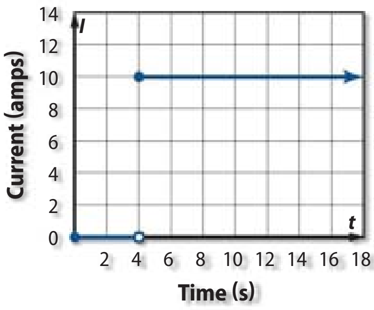
86. $e^{5x-4} = 70$

87. **ELECTRICITY** A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown at the right. I represents current in amps, and t represents time in seconds. (Lesson 1-3)

a. At what t-value is this function discontinuous?

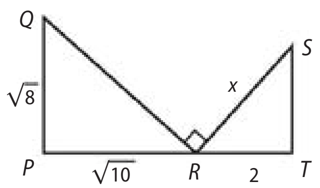
b. When was the power supply turned on?

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be?



Skills Review for Standardized Tests

88. **SAT/ACT** In the figure below, $\triangle PQR \sim \triangle TRS$. What is the value of x?



A $\sqrt{2}$

B $\sqrt{5}$

C 3

D $3\sqrt{2}$

E 6

89. **REVIEW** Consider $C(-9, 2)$ and $D(-4, -3)$. Which of the following is the component form and magnitude of $\overrightarrow{CD}$?

F $\langle 5, -5\rangle, 5\sqrt{2}$

G $\langle 5, -5\rangle, 6\sqrt{2}$

H $\langle 6, -5\rangle, 5\sqrt{2}$

J $\langle 6, -6\rangle, 6\sqrt{2}$

90. A snow sled is pulled by exerting a force of 25 pounds on a rope that makes a 20° angle with the horizontal, as shown in the figure. What is the approximate work done in pulling the sled 50 feet?

A 428 foot-pounds

B 1093 foot-pounds

C 1175 foot-pounds

D 1250 foot-pounds

91. **REVIEW** If $\mathbf{s} = \langle 4, -3\rangle$ $\mathbf{t} = \langle -6, 2\rangle$, which of the following represents $\mathbf{t} - 2\mathbf{s}$?

F $\langle 14, 8\rangle$

G $\langle 14, 6\rangle$

H $\langle -14, 8\rangle$

J $\langle -14, -8\rangle$

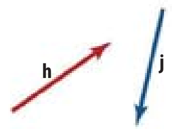
CHAPTER 8

Mid-Chapter Quiz

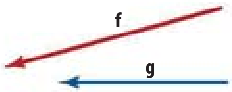
Lessons 8-1 through 8-3

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal. (Lesson 8-1)

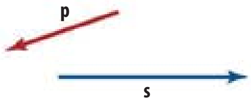
1.



2.



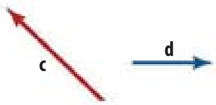
3.



4.

5. **SLEDDING** Alvin pulls a sled through the snow with a force of 50 newtons at an angle of 35° with the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 8-1)

6. Draw a vector diagram of $\frac{1}{2}\mathbf{c} - 3\mathbf{d}$. (Lesson 8-1)



Let $\overrightarrow{BC}$ be the vector with the given initial and terminal points. Write $\overrightarrow{BC}$ as a linear combination of the vectors **i** and **j**. (Lesson 8-2)

7. $B(3, -1), C(4, -7)$

8. $B(10, -6), C(-8, 2)$

9. $B(1, 12), C(-2, -9)$

10. $B(4, -10), C(4, -10)$

11. **MULTIPLE CHOICE** Which of the following is the component form of $\overrightarrow{AB}$ with initial point $A(-5, 3)$ and terminal point $B(2, -1)$? (Lesson 8-2)

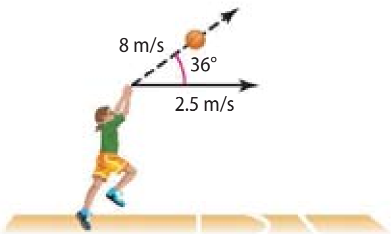
A $\langle 4, -1 \rangle$

B $\langle 7, -4 \rangle$

C $\langle 7, 4 \rangle$

D $\langle -6, 4 \rangle$

12. **BASKETBALL** With time running out in a game, Rachel runs towards the basket at a speed of 2.5 meters per second and from half-court, launches a shot at a speed of 8 meters per second at an angle of 36° to the horizontal. (Lesson 8-2)



a. Write the component form of the vectors representing Rachel's velocity and the path of the ball.

b. What is the resultant speed and direction of the shot?

Find the component form and magnitude of the vector with each initial and terminal point. (Lesson 8-2)

13. $A(-4, 2), B(3, 6)$

14. $Q(1, -5), R(-7, 8)$

15. $X(-3, -5), Y(2, 5)$

16. $P(9, -2), S(2, -5)$

Find the angle θ between **u** and **v** to the nearest tenth of a degree. (Lesson 8-3)

17. $\mathbf{u} = \langle 9, -4 \rangle, \mathbf{v} = \langle -1, -2 \rangle$

18. $\mathbf{u} = \langle 5, 2 \rangle, \mathbf{v} = \langle -4, 10 \rangle$

19. $\mathbf{u} = \langle 8, 4 \rangle, \mathbf{v} = \langle -2, 4 \rangle$

20. $\mathbf{u} = \langle 2, -2 \rangle, \mathbf{v} = \langle 3, 8 \rangle$

21. **MULTIPLE CHOICE** If $\mathbf{u} = \langle 2, 3 \rangle, \mathbf{v} = \langle -1, 4 \rangle$, and $\mathbf{w} = \langle 8, -5 \rangle$, find $(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{v})$. (Lesson 8-3)

F -18

G -2

H 15

J 38

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal. (Lesson 8-3)

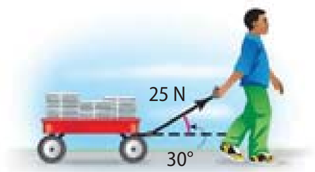
22. $\langle 2, -5 \rangle \cdot \langle 4, 2 \rangle$

23. $\langle 4, -3 \rangle \cdot \langle 7, 4 \rangle$

24. $\langle 1, -6 \rangle \cdot \langle 5, 8 \rangle$

25. $\langle 3, -6 \rangle \cdot \langle 10, 5 \rangle$

26. **WAGON** Henry uses a wagon to carry newspapers for his paper route. He is pulling the wagon with a force of 25 newtons at an angle of 30° with the horizontal. (Lesson 8-3)



a. How much work in joules is Henry doing when he pulls the wagon 150 meters?

b. If the handle makes an angle of 40° with the ground and he pulls the wagon with the same distance and force, is Henry doing more or less work? Explain your answer.

Find the projection of **u** onto **v**. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**. (Lesson 8-3)

27. $\mathbf{u} = \langle 7, -3 \rangle, \mathbf{v} = \langle 2, 5 \rangle$

28. $\mathbf{u} = \langle 2, 4 \rangle, \mathbf{v} = \langle 1, 3 \rangle$

29. $\mathbf{u} = \langle 3, 8 \rangle, \mathbf{v} = \langle -9, 2 \rangle$

30. $\mathbf{u} = \langle -1, 4 \rangle, \mathbf{v} = \langle -6, 1 \rangle$

LESSON 8-4 Vectors in Three-Dimensional Space

Then

- You represented vectors both geometrically and algebraically in two-dimensions. (Lessons 8-2 and 8-3)

Now

1. Plot points and vectors in the three-dimensional coordinate system.
2. Express algebraically and operate with vectors in space.

Why?

- The direction of a rocket after takeoff is given in terms of both a two-dimensional bearing and a third-dimensional angle relative to the horizontal. Since directed distance, velocities, and forces are not restricted to the plane, the concept of vectors must extend from two- to three-dimensional space.

NewVocabulary

three-dimensional coordinate system
z-axis
octant
ordered triple

1 Coordinates in Three Dimensions

The Cartesian plane is a two-dimensional coordinate system made up of the x- and y-axes that allows you to identify and locate points in a plane. We need a **three-dimensional coordinate system** to represent a point in space.

Start with the xy-plane and position it so that it gives the appearance of depth (Figure 8.4.1). Then add a third axis called the **z-axis** that passes through the origin and is perpendicular to both the x- and y-axes (Figure 8.4.2). The additional axis divides space into eight regions called **octants**. To help visualize the first octant, look at the corner of a room (Figure 8.4.3). The floor represents the xy-plane, and the walls represent the xz- and yz-planes.

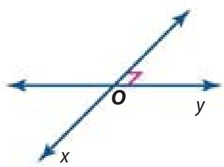


Figure 8.4.1

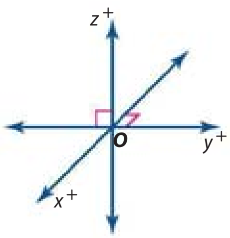


Figure 8.4.2

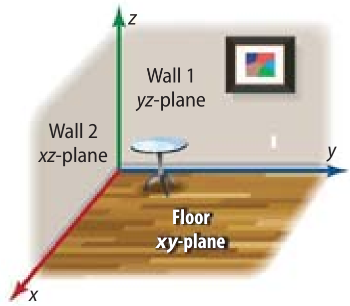


Figure 8.4.3

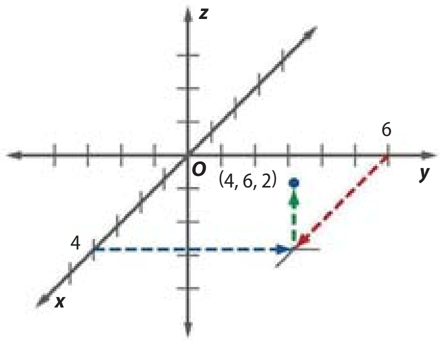
A point in space is represented by an **ordered triple** of real numbers (x, y, z). To plot such a point, first locate the point (x, y) in the xy-plane and move up or down parallel to the z-axis according to the directed distance given by z.

Example 1 Locate a Point in Space

Plot each point in a three-dimensional coordinate system.

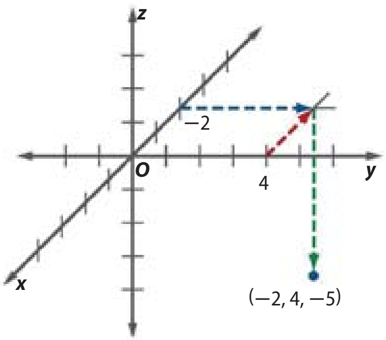
a. (4, 6, 2)

Locate (4, 6) in the xy-plane and mark it with a cross. Then plot a point **2** units up from this location parallel to the z-axis.



b. (−2, 4, −5)

Locate (−2, 4) in the xy-plane and mark it with a cross. Then plot a point **5** units down from this location parallel to the z-axis.



GuidedPractice

1A. (−3, −4, 2) **1B.** (3, 2, −3) **1C.** (5, −4, −1)

Finding the distance between points and the midpoint of a segment in space is similar to finding distance and a midpoint in the coordinate plane.

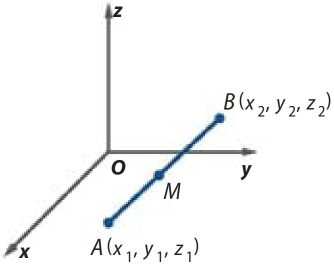
KeyConcept Distance and Midpoint Formulas in Space

The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The midpoint M of $\overline{AB}$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



You will prove these formulas in Exercise 66.

Real-WorldLink

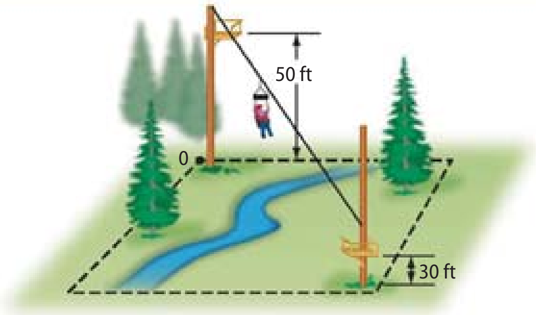
A tour at Monteverde, Costa Rica, allows visitors to view nature from a system of trails, suspension bridges, and zip-lines. The zip-lines allow the guests to view the surroundings from as much as 456 feet above the ground.

Source: Monteverde Info

Jerry Driendl/The Image Bank/Getty Images

Real-World Example 2 Distance and Midpoint of Points in Space

ZIP-LINE A tour of the Sierra Madre Mountains lets guests experience nature by zip-lining from one platform to another over the scenic surroundings. Two platforms that are connected by a zip-line are represented by the coordinates (10, 12, 50) and (70, 92, 30), where the coordinates are given in feet.



a. Find the length of the zip-line needed to connect the two platforms.

Use the Distance Formula for points in space.

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ — Distance Formula

$= \sqrt{(70 - 10)^2 + (92 - 12)^2 + (30 - 50)^2}$ — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

≈ 101.98 — Simplify.

The zip-line needs to be about 102 feet long to connect the two towers.

b. An additional platform is to be built halfway between the existing platforms. Find the coordinates of the new platform.

Use the Midpoint Formula for points in space.

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ — Midpoint Formula

$= \left(\frac{10 + 70}{2}, \frac{12 + 92}{2}, \frac{50 + 30}{2}\right)$ or $(40, 52, 40)$ — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

The coordinates of the new platform will be (40, 52, 40).

GuidedPractice

2. **AIRPLANES** Safety regulations require airplanes to be at least a half a mile apart when in the sky. Two planes are flying above Cleveland with the coordinates (300, 150, 30000) and (450, −250, 28000), where the coordinates are given in feet.

   A. Are the two planes in violation of the safety regulations? Explain.

   B. If a firework was launched and exploded directly in between the two planes, what are the coordinates of the firework explosion?

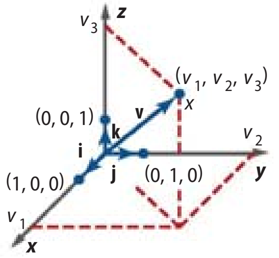


Figure 8.4.4

2 Vectors in Space

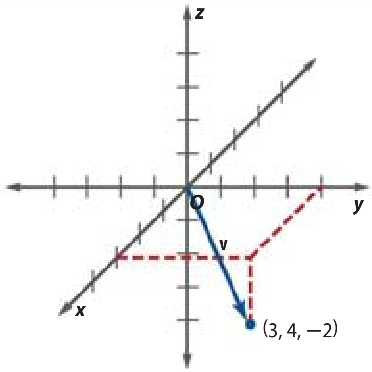
In space, a vector **v** in standard position with a terminal point located at (v_1, v_2, v_3) is denoted by $\langle v_1, v_2, v_3 \rangle$. The zero vector is $\mathbf{0} = \langle 0, 0, 0 \rangle$, and the standard unit vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ as shown in Figure 8.4.4. The component form of **v** can be expressed as a linear combination of these unit vectors, $\langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Example 3 Locate a Vector in Space

Locate and graph $\mathbf{v} = \langle 3, 4, -2 \rangle$.

Plot the point (3, 4, −2).

Draw **v** with terminal point at (3, 4, −2).



GuidedPractice

Locate and graph each vector in space.

3A. $\mathbf{u} = \langle -4, 2, -3 \rangle$

3B. $\mathbf{w} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

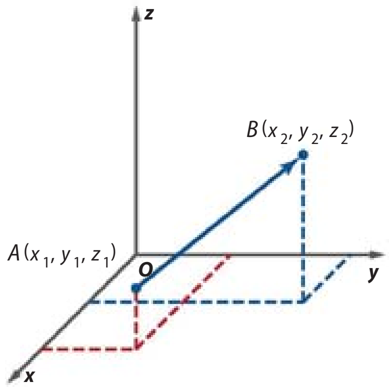
As with two-dimensional vectors, to find the component form of the directed line segment from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$, you subtract the coordinates of its initial point from its terminal point.

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Then $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ or

if $\overrightarrow{AB} = \langle a_1, a_2, a_3 \rangle$, then $|\overrightarrow{AB}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

A unit vector **u** in the direction of $\overrightarrow{AB}$ is still $\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.



Example 4 Express Vectors in Space Algebraically

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(-4, -2, 1)$ and terminal point $B(3, 6, -6)$. Then find a unit vector in the direction of $\overrightarrow{AB}$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ — Component form of vector

$= \langle 3 - (-4), 6 - (-2), -6 - 1 \rangle$ or $\langle 7, 8, -7 \rangle$ — $(x_1, y_1, z_1) = (-4, -2, 1)$ and $(x_2, y_2, z_2) = (3, 6, -6)$

Using the component form, the magnitude of $\overrightarrow{AB}$ is

$|\overrightarrow{AB}| = \sqrt{7^2 + 8^2 + (-7)^2}$ or $9\sqrt{2}$. — $\overrightarrow{AB} = \langle 7, 8, -7 \rangle$

Using this magnitude and component form, find a unit vector **u** in the direction of $\overrightarrow{AB}$.

$\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ — Unit vector in the direction of $\overrightarrow{AB}$

$= \dfrac{\langle 7, 8, -7 \rangle}{9\sqrt{2}}$ or $\left\langle \dfrac{7\sqrt{2}}{18}, \dfrac{4\sqrt{2}}{9}, -\dfrac{7\sqrt{2}}{18} \right\rangle$ — $\overrightarrow{AB} = \langle 7, 8, -7 \rangle$ and $|\overrightarrow{AB}| = 9\sqrt{2}$

GuidedPractice

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$.

4A. $A(-2, -5, -5)$, $B(-1, 4, -2)$

4B. $A(-1, 4, 6)$, $B(3, 3, 8)$

As with vectors in the plane, when vectors in space are in component form or expressed as a linear combination of unit vectors, they can be added, subtracted, or multiplied by a scalar.

KeyConcept Vector Operations in Space

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and any scalar k, then

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

StudyTip

Vector Operations The properties for vector operations in space are the same as those for operations in the plane.

Example 5 Operations with Vectors in Space

Find each of the following for $\mathbf{y} = \langle 3, -6, 2 \rangle$, $\mathbf{w} = \langle -1, 4, -4 \rangle$, and $\mathbf{z} = \langle -2, 0, 5 \rangle$.

a. $4\mathbf{y} + 2\mathbf{z}$

$4\mathbf{y} + 2\mathbf{z} = 4\langle 3, -6, 2 \rangle + 2\langle -2, 0, 5 \rangle$ — Substitute.

$= \langle 12, -24, 8 \rangle + \langle -4, 0, 10 \rangle$ or $\langle 8, -24, 18 \rangle$ — Scalar multiplication and vector addition

b. $2\mathbf{w} - \mathbf{z} + 3\mathbf{y}$

$2\mathbf{w} - \mathbf{z} + 3\mathbf{y} = 2\langle -1, 4, -4 \rangle - \langle -2, 0, 5 \rangle + 3\langle 3, -6, 2 \rangle$ — Substitute.

$= \langle -2, 8, -8 \rangle + \langle 2, 0, -5 \rangle + \langle 9, -18, 6 \rangle$ — Scalar multiplication

$= \langle 9, -10, -7 \rangle$ — Vector addition

GuidedPractice

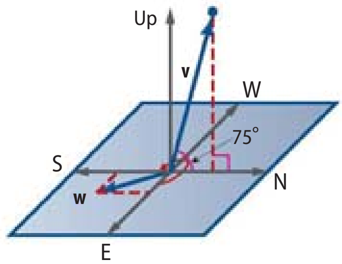
5A. $4\mathbf{w} - 8\mathbf{z}$

5B. $3\mathbf{y} + 3\mathbf{z} - 6\mathbf{w}$

Real-World Example 6 Use Vectors in Space

ROCKETS After liftoff, a model rocket is headed due north and climbing at an angle of 75° relative to the horizontal at 200 miles per hour. If the wind blows from the northwest at 5 miles per hour, find a vector for the resultant velocity of the rocket relative to the point of liftoff.

Let **i** point east, **j** point north, and **k** point up. Vector **v** representing the rocket's velocity and vector **w** representing the wind's velocity are shown. Notice that **w** points toward the southeast, since the wind is blowing *from* the northwest.



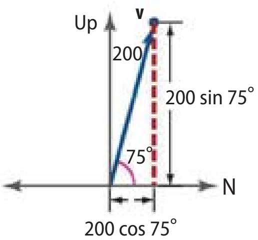


Figure 8.4.5

Since **v** has a magnitude of 200 and a direction angle of 75°, we can find the component form of **v**, as shown in Figure 8.4.5.

$\mathbf{v} = \langle 0, 200 \cos 75°, 200 \sin 75° \rangle$ or about $\langle 0, 51.8, 193.2 \rangle$

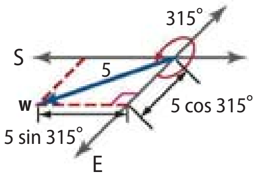


Figure 8.4.6

With east as the positive x-axis, **w** has direction angle of 315°. Since $|\mathbf{w}| = 5$, the component form of this vector is $\mathbf{w} = \langle 5 \cos 315°, 5 \sin 315°, 0 \rangle$ or about $\langle 3.5, -3.5, 0 \rangle$, as shown in Figure 8.4.6.

The resultant velocity of the rocket is $\mathbf{v} + \mathbf{w}$.

$\mathbf{v} + \mathbf{w} = \langle 0, 51.8, 193.2 \rangle + \langle 3.5, -3.5, 0 \rangle$

$= \langle 3.5, 48.3, 193.2 \rangle$ or $3.5\mathbf{i} + 48.3\mathbf{j} + 193.2\mathbf{k}$

GuidedPractice

6. AVIATION After takeoff, an airplane is headed east and is climbing at an angle of 18° relative to the horizontal. Its air speed is 250 miles per hour. If the wind blows from the northeast at 10 miles per hour, find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up.

Exercises

= Step-by-Step Solutions begin on page R29.

Plot each point in a three-dimensional coordinate system. (Example 1)

1. (1, −2, −4)
2. (3, 2, 1)
3. (−5, −4, −2)
4. (−2, −5, 3)
5. (−5, 3, 1)
6. (2, −2, 3)
7. (4, −10, −2)
8. (−16, 12, −13)

Find the length and midpoint of the segment with the given endpoints. (Example 2)

9. (−4, 10, 4), (1, 0, 9)
10. (−6, 6, 3), (−9, −2, −2)
11. (6, 1, 10), (−9, −10, −4)
12. (8, 3, 4), (−4, −7, 5)
13. (−3, 2, 8), (9, 6, 0)
14. (−7, 2, −5), (−2, −5, −8)

15. **VACATION** A family from Wichita, Kansas, is using a GPS device to plan a vacation to Castle Rock, Colorado. According to the device, the coordinates for the family's home are (37.7°, 97.2°, 433 m), and the coordinates to Castle Rock are (39.4°, 104.8°, 1981 m). Determine the longitude, latitude, and altitude of the halfway point between Wichita and Castle Rock. (Example 2)

16. **FIGHTER PILOTS** During a training session, the location of two F-18 fighter jets are represented by the coordinates (675, −121, 19,300) and (−289, 715, 16,100), where the coordinates are given in feet. (Example 2)
    a. Determine the distance between the two jets.
    b. To what location would one of the fighter pilots have to fly the F-18 in order to reduce the distance between the two jets by half?

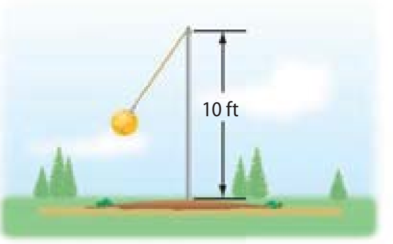
Locate and graph each vector in space. (Example 3)

17. $\mathbf{a} = \langle 0, -4, 4\rangle$
18. $\mathbf{b} = \langle -3, -3, -2\rangle$
19. $\mathbf{c} = \langle -1, 3, -4\rangle$
20. $\mathbf{d} = \langle 4, -2, -3\rangle$
21. $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$
22. $\mathbf{w} = -10\mathbf{i} + 5\mathbf{k}$
23. $\mathbf{m} = 7\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$
24. $\mathbf{n} = \mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$. (Example 4)

25. $A(-5, -5, -9)$, $B(11, -3, -1)$
26. $A(-4, 0, -3)$, $B(-4, -8, 9)$
27. $A(3, 5, 1)$, $B(0, 0, -9)$
28. $A(-3, -7, -12)$, $B(-7, 1, 8)$
29. $A(2, -5, 4)$, $B(1, 3, -6)$
30. $A(8, 12, 7)$, $B(2, -3, 11)$
31. $A(3, 14, -5)$, $B(7, -1, 0)$
32. $A(1, -18, -13)$, $B(21, 14, 29)$
33. $A(-5, 12, 17)$, $B(6, -11, 4)$
34. $A(9, 3, 7)$, $B(-5, -7, 2)$

35. **TETHERBALL** In the game of tetherball, a ball is attached to a 10-foot pole by a length of rope. Two players hit the ball in opposing directions in an attempt to wind the entire length of rope around the pole. To serve, a certain player holds the ball so that its coordinates are (5, 3.6, 4.7) and the coordinates of the end of the rope connected to the pole are (0, 0, 9.8), where the coordinates are given in feet. Determine the magnitude of the vector representing the length of the rope. (Example 4)



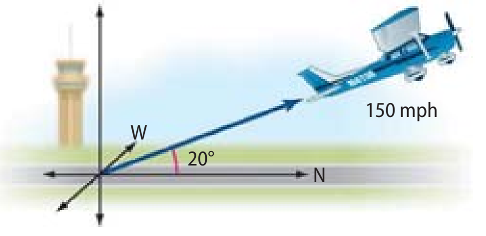
Find each of the following for $\mathbf{a} = \langle -5, -4, 3\rangle$, $\mathbf{b} = \langle 6, -2, -7\rangle$, and $\mathbf{c} = \langle -2, 2, 4\rangle$. (Example 5)

36. $6\mathbf{a} - 7\mathbf{b} + 8\mathbf{c}$
37. $7\mathbf{a} - 5\mathbf{b}$
38. $2\mathbf{a} + 5\mathbf{b} - 9\mathbf{c}$
39. $6\mathbf{b} + 4\mathbf{c} - 4\mathbf{a}$
40. $8\mathbf{a} - 5\mathbf{b} - \mathbf{c}$
41. $-6\mathbf{a} + \mathbf{b} + 7\mathbf{c}$

Find each of the following for $\mathbf{x} = -9\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{y} = 6\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$, and $\mathbf{z} = -2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. (Example 5)

42. $7\mathbf{x} + 6\mathbf{y}$
43. $3\mathbf{x} - 5\mathbf{y} + 3\mathbf{z}$
44. $4\mathbf{x} + 3\mathbf{y} + 2\mathbf{z}$
45. $-8\mathbf{x} - 2\mathbf{y} + 5\mathbf{z}$
46. $-6\mathbf{y} - 9\mathbf{z}$
47. $-\mathbf{x} - 4\mathbf{y} - \mathbf{z}$

48. **AIRPLANES** An airplane is taking off headed due north with an air speed of 150 miles per hour at an angle of 20° relative to the horizontal. The wind is blowing with a velocity of 8 miles per hour from the southwest. Find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up. (Example 6)



49. **TRACK AND FIELD** Lena throws a javelin due south at a speed of 18 miles per hour and at an angle of 48° relative to the horizontal. If the wind is blowing with a velocity of 12 miles per hour at an angle of S15°E, find a vector that represents the resultant velocity of the javelin. Let **i** point east, **j** point north, and **k** point up. (Example 6)

50. **SUBMARINE** A submarine heading due west dives at a speed of 25 knots and an angle of decline of 55°. The current is moving with a velocity of 4 knots at an angle of S20°W. Find a vector that represents the resultant velocity of the submarine relative to the initial point of the dive. Let **i** point east, **j** point north, and **k** point up. (Example 6)

If N is the midpoint of $\overline{MP}$, find P.

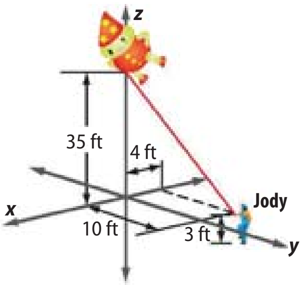
51. $M(3, 4, 5)$; $N\left(\frac{7}{2}, 1, 2\right)$

52. $M(-1, -4, -9)$; $N(-2, 1, -5)$

53. $M(7, 1, 5)$; $N\left(5, -\frac{1}{2}, 6\right)$

54. $M\left(\frac{3}{2}, -5, 9\right)$; $N\left(-2, -\frac{13}{2}, \frac{11}{2}\right)$

55. **VOLUNTEERING** Jody is volunteering to help guide a balloon in a parade. If the balloon is 35 feet high and she is holding the tether three feet above the ground as shown, how long is the tether to the nearest foot?



Determine whether the triangle with the given vertices is *isosceles* or *scalene*.

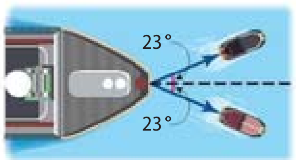
56. $A(3, 1, 2)$, $B(5, -1, 1)$, $C(1, 3, 1)$

57. $A(4, 3, 4)$, $B(4, 6, 4)$, $C(4, 3, 6)$

58. $A(-1, 4, 3)$, $B(2, 5, 1)$, $C(0, -6, 6)$

59. $A(-2.2, 4.3, 5.6)$, $B(0.7, 9.3, 15.6)$, $C(3.6, 14.3, 5.6)$

60. **TUGBOATS** Two tugboats are pulling a disabled supertanker. One of the tow lines makes an angle 23° west of north and the other makes an angle 23° east of north. Each tug exerts a constant force of 2.5×10^6 newtons depressed 15° below the point where the lines attach to the supertanker. They pull the supertanker two miles due north.



a. Write a three-dimensional vector to describe the force from each tugboat.

b. Find the vector that describes the total force on the supertanker.

c. If each tow line is 300 feet long, about how far apart are the tugboats?

61. **SPHERES** Use the distance formula for two points in space to prove that the standard form of the equation of a sphere with center (h, k, ℓ) and radius r is $r^2 = (x - h)^2 + (y - k)^2 + (z - \ell)^2$.

Use the formula from Exercise 61 to write an equation for the sphere with the given center and radius.

62. center $= (-4, -2, 3)$; radius $= 4$

63. center $= (6, 0, -1)$; radius $= \frac{1}{2}$

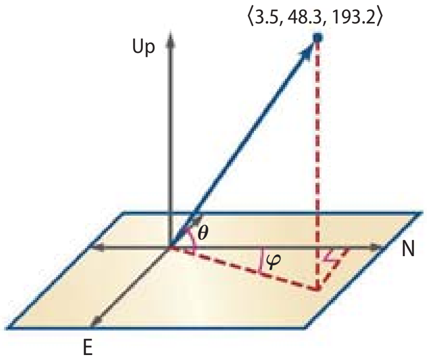
64. center $= (5, -3, 4)$; radius $= \sqrt{3}$

65. center $= (0, 7, -1)$; radius $= 12$

H.O.T. Problems Use Higher-Order Thinking Skills

66. **REASONING** Prove the Distance Formula in Space. (*Hint*: Use the Pythagorean Theorem twice.)

67. **CHALLENGE** Refer to Example 6.

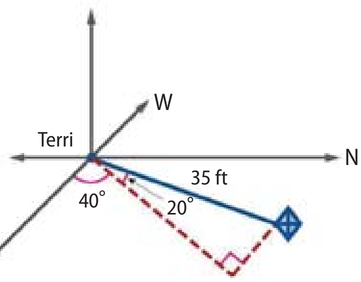


a. Calculate the resultant speed of the rocket.

b. Find the quadrant bearing φ of the rocket.

c. Calculate the resultant angle of incline θ of the rocket relative to the horizontal.

68. **CHALLENGE** Terri is standing in an open field facing N50°E. She is holding a kite on a 35-foot string that is flying at a 20° angle with the field. Find the components of the vector from Terri to the kite. (*Hint*: Use trigonometric ratios and two right triangles to find **x**, **y**, and **z**.)



69. **WRITING IN MATH** Describe a situation where it is more reasonable to use a two-dimensional coordinate system and one where it is more reasonable to use a three-dimensional coordinate system.

Spiral Review

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Lesson 8-3)

70. $\mathbf{u} = \langle 6, 8 \rangle, \mathbf{v} = \langle 2, -1 \rangle$ **71.** $\mathbf{u} = \langle -1, 4 \rangle, \mathbf{v} = \langle 5, 1 \rangle$ **72.** $\mathbf{u} = \langle 5, 4 \rangle, \mathbf{v} = \langle 4, -2 \rangle$

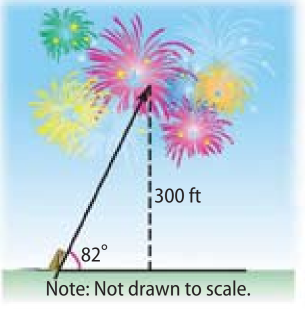
Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. (Lesson 8-2)

73. $A(6, -4), B(-7, -7)$ **74.** $A(-4, -8), B(1, 6)$ **75.** $A(-5, -12), B(1, 6)$

76. ENTERTAINMENT The Independence Day fireworks at Memorial Park are fired at an angle of 82° with the horizontal. The technician firing the shells expects them to explode about 300 feet in the air 4.8 seconds after they are fired. (Lesson 7-5)

a. Find the initial speed of a shell fired from ground level.

b. Safety barriers will be placed around the launch area to protect spectators. If the barriers are placed 100 yards from the point directly below the explosion of the shells, how far should the barriers be from the point where the fireworks are launched?



77. CONSTRUCTION A stone fireplace that was designed as an arch in the shape of a semi-ellipse will have an opening with a height of 3 feet at the center and a width of 8 feet along the base. To sketch an outline of the fireplace, a contractor uses a string tied to two thumbtacks. (Lesson 7-2)

a. At what locations should the thumbtacks be placed?

b. What length of string needs to be used? Explain your reasoning.

Solve each equation for all values of θ. (Lesson 5-3)

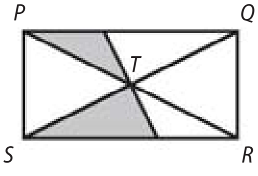
78. $\csc \theta + 2 \cot \theta = 0$ **79.** $\sec^2 \theta - 9 = 0$ **80.** $2 \csc \theta - 3 = 5 \sin \theta$

Sketch the graph of each function. (Lesson 4-6)

81. $y = \cos^{-1}(x - 2)$ **82.** $y = \arccos x + 3$ **83.** $y = \sin^{-1} 3x$

Skills Review for Standardized Tests

84. SAT/ACT What percent of the area of rectangle $PQRS$ is shaded?



A 22%
B 25%
C 30%
D $33\frac{1}{3}$%
E 35%

85. REVIEW A ship leaving port sails for 75 miles in a direction of 35° north of east. At that point, how far north of its starting point is the ship?

F 43 miles
G 55 miles
H 61 miles
J 72 miles

86. During a storm, the force of the wind blowing against a skyscraper can be expressed by the vector $\langle 132, 3454, -76 \rangle$, where the force of the wind is measured in newtons. What is the approximate magnitude of this force?

A 3457 N
B 3568 N
C 3692 N
D 3717 N

87. REVIEW An airplane is flying due west at a velocity of 100 meters a second. The wind is blowing from the south at 30 meters a second. What is the approximate magnitude of the airplane's resultant velocity?

F 4 m/s
G 95.4 m/s
H 100 m/s
J 104.4 m/s

Graphing Technology Lab
Vector Transformations with Matrices

Objective

- Use a graphing calculator to transform vectors using matrices.

In Lesson 8-4, you learned that a vector in space can be transformed when written in component form or when expressed as a linear combination. A vector in space can also be transformed when written as a 3×1 or 1×3 matrix.

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } [x \;\; y \;\; z]$$

Once in matrix form, a vector can be transformed using matrix-vector multiplication.

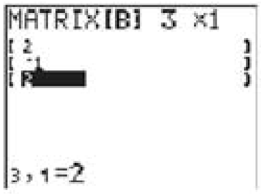
Activity Matrix-Vector Multiplication

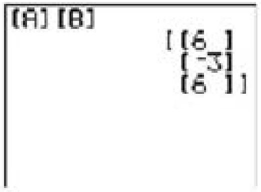
Multiply the vector $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ by the transformation matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and graph both vectors.

Step 1 Write **B** as a matrix.

$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Step 2 Enter **B** and A in a graphing calculator and find $A\mathbf{B}$. Convert to vector form.

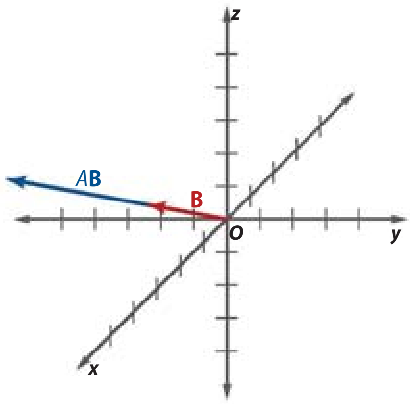




$A\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

Step 3 Graph **B** and A**B** on a coordinate plane.

A**B** is a dilation of **B** by a factor of 3.



Exercises

Multiply each vector by the transformation matrix. Graph both vectors.

1. $\mathbf{h} = 4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$

$$B = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

2. $\mathbf{e} = 5\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$

$$V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. $\mathbf{f} = \mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

$$W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. **REASONING** Multiply $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ by the transformation matrix $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and graph both vectors. Explain the type of transformation that was performed.

LESSON 8-5 Dot and Cross Products of Vectors in Space

Then

- You found the dot product of two vectors in the plane. (Lesson 8-3)

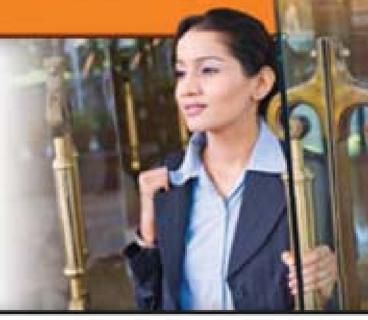
Now

1. Find dot products of and angles between vectors in space.
2. Find cross products of vectors in space, and use cross products to find area and volume.

Why?

- The tendency of a hinged door to rotate when pushed is affected by the distance between the location of the push and the hinge, the magnitude of the push, and the direction of the push.

A quantity called *torque* measures how effectively a force applied to a lever causes rotation about an axis.



NewVocabulary
cross product
torque
parallelepiped
triple scalar product

1 Dot Products in Space

Calculating the dot product of two vectors in space is similar to calculating the dot product of two vectors in a plane. As with vectors in a plane, nonzero vectors in space are perpendicular if and only if their dot product equals zero.

KeyConcept Dot Product and Orthogonal Vectors in Space

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 1 Find the Dot Product to Determine Orthogonal Vectors in Space

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle -7, 3, -3\rangle$, $\mathbf{v} = \langle 5, 17, 5\rangle$

$$\mathbf{u} \cdot \mathbf{v} = -7(5) + 3(17) + (-3)(5)$$
$$= -35 + 51 + (-15) \text{ or } 1$$

Since $\mathbf{u} \cdot \mathbf{v} \neq 0$, $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

b. $\mathbf{u} = \langle 3, -3, 3\rangle$, $\mathbf{v} = \langle 4, 7, 3\rangle$

$$\mathbf{u} \cdot \mathbf{v} = 3(4) + (-3)(7) + 3(3)$$
$$= 12 + (-21) + 9 \text{ or } 0$$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -5, 4\rangle$, $\mathbf{v} = \langle 5, 7, 5\rangle$

1B. $\mathbf{u} = \langle 4, -2, -3\rangle$, $\mathbf{v} = \langle 1, 3, -2\rangle$

As with vectors in a plane, if θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Example 2 Angle Between Two Vectors in Space

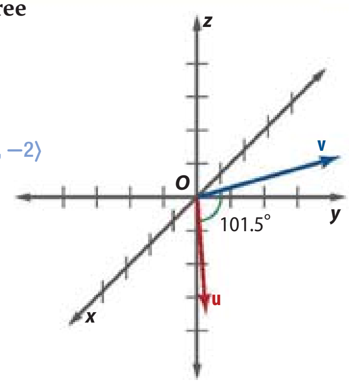
Find the angle θ between u and v to the nearest tenth of a degree if $\mathbf{u} = \langle 3, 2, -1\rangle$ and $\mathbf{v} = \langle -4, 3, -2\rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$ Angle between two vectors

$$\cos\theta = \frac{\langle 3, 2, -1\rangle \cdot \langle -4, 3, -2\rangle}{|\langle 3, 2, -1\rangle|\,|\langle -4, 3, -2\rangle|}$$ $u = \langle 3, 2, -1\rangle$ and $v = \langle -4, 3, -2\rangle$

$$\cos\theta = \frac{-4}{\sqrt{14}\sqrt{29}}$$ Evaluate the dot product and magnitudes.

$$\theta = \cos^{-1}\frac{-4}{\sqrt{406}} \text{ or about } 101.5°$$ Simplify and solve for θ.



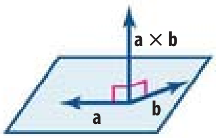
The measure of the angle between $\mathbf{u}$ and $\mathbf{v}$ is about 101.5°.

GuidedPractice

2. Find the angle between $\mathbf{u} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$ to the nearest tenth of a degree.

2 Cross Products

Cross Products Another important product involving vectors in space is the cross product. Unlike the dot product, the **cross product** of two vectors **a** and **b** in space, denoted **a** × **b** and read *a cross b*, is a vector, not a scalar. The vector **a** × **b** is perpendicular to the plane containing vectors **a** and **b**.



KeyConcept Cross Product of Vectors in Space

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of **a** and **b** is the vector

$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

ReviewVocabulary

2 × 2 Determinant The determinant of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$. (Lesson 6-2)

If we apply the formula for calculating the determinant of a 3 × 3 matrix to the following *determinant form* involving **i**, **j**, **k**, and the components of **a** and **b**, we arrive at the same formula for **a** × **b**.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Put the unit vectors **i**, **j**, and **k** in Row 1.
Put the components of **a** in Row 2.
Put the components of **b** in Row 3.

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Apply the formula for a 3 × 3 determinant.

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Compute each 2 × 2 determinant.

Example 3 Find the Cross Product of Two Vectors

WatchOut!

Cross Product The cross product definition applies only to vectors in three-dimensional space. The cross product is not defined for vectors in the two-dimensional coordinate system.

Find the cross product of u = ⟨3, −2, 1⟩ and v = ⟨−3, 3, 1⟩. Then show that u × v is orthogonal to both u and v.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -3 & 3 & 1 \end{vmatrix}$$

$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$= \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{k}$$

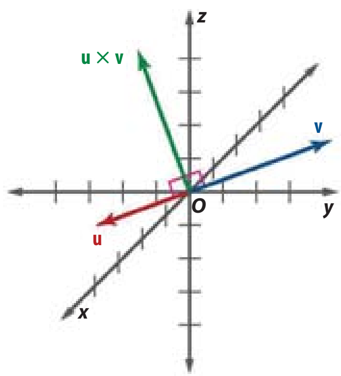
Determinant of a 3 × 3 matrix

$= (-2 - 3)\mathbf{i} - [3 - (-3)]\mathbf{j} + (9 - 6)\mathbf{k}$ — Determinants of 2 × 2 matrices

$= -5\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ — Simplify.

$= \langle -5, -6, 3 \rangle$ — Component form

In the graph of **u**, **v**, and **u** × **v**, **u** × **v** is orthogonal to **u** and **u** × **v** is orthogonal to **v**.



To show that **u** × **v** is orthogonal to both **u** and **v**, find the dot product of **u** × **v** with **u** and **u** × **v** with **v**.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$	$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$
$= \langle -5, -6, 3 \rangle \cdot \langle 3, -2, 1 \rangle$	$= \langle -5, -6, 3 \rangle \cdot \langle -3, 3, 1 \rangle$
$= -5(3) + (-6)(-2) + 3(1)$	$= -5(-3) + (-6)(3) + 3(1)$
$= -15 + 12 + 3$ or 0 ✓	$= 15 + (-18) + 3$ or 0 ✓

Because both dot products are zero, the vectors are orthogonal.

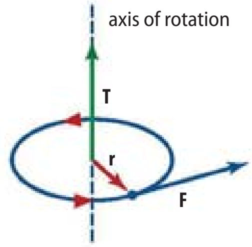
GuidedPractice

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

3A. $\mathbf{u} = \langle 4, 2, -1 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

3B. $\mathbf{u} = \langle -2, -1, -3 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

You can use the cross product to find a vector quantity called **torque**. Torque measures how effectively a force applied to a lever causes rotation along the axis of rotation. The torque vector $\mathbf{T}$ is perpendicular to the plane containing the directed distance $\mathbf{r}$ from the axis of rotation to the point of the applied force and the applied force $\mathbf{F}$ as shown. Therefore, the torque vector is $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ and is measured in newton-meters (N • m).

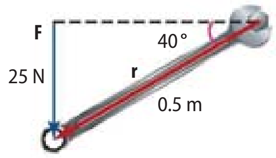


Real-World Career

Automotive Mechanic
Automotive mechanics perform repairs ranging from simple mechanical problems to high-level, technology–related repairs. They should have good problem-solving skills, mechanical aptitude, and knowledge of electronics and mathematics. Most mechanics complete a vocational training program in automotive service technology.

Real-World Example 4 Torque Using Cross Product

AUTO REPAIR **Robert uses a lug wrench to tighten a lug nut. The wrench he uses is 50 centimeters or 0.5 meter long. Find the magnitude and direction of the torque about the lug nut if he applies a force of 25 newtons straight down to the end of the handle when it is 40° below the positive *x*-axis as shown.**



Step 1 Graph each vector in standard position (Figure 8.5.1).

Step 2 Determine the component form of each vector.

The component form of the vector representing the directed distance from the axis of rotation to the end of the handle can be found using the triangle in Figure 8.5.2 and trigonometry. Vector $\mathbf{r}$ is therefore $\langle 0.5 \cos 40°, 0, -0.5 \sin 40° \rangle$ or about $\langle 0.38, 0, -0.32 \rangle$. The vector representing the force applied to the end of the handle is 25 newtons straight down, so $\mathbf{F} = \langle 0, 0, -25 \rangle$.

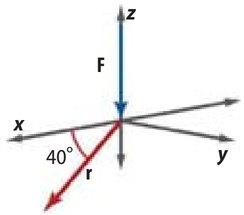


Figure 8.5.1

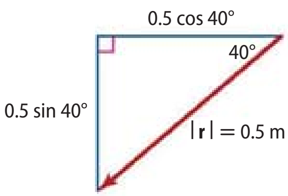


Figure 8.5.2

Step 3 Use the cross product of these vectors to find the vector representing the torque about the lug nut.

$\mathbf{T} = \mathbf{r} \times \mathbf{F}$ — Torque Cross Product Formula

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.38 & 0 & -0.32 \\ 0 & 0 & -25 \end{vmatrix}$$ Cross product of r and F

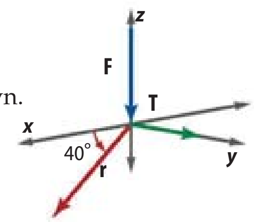
$$= \begin{vmatrix} 0 & -0.32 \\ 0 & -25 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0.38 & -0.32 \\ 0 & -25 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0.38 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k}$$ Determinant of a 3 × 3 matrix

$= 0\mathbf{i} - (-9.5)\mathbf{j} + 0\mathbf{k}$ — Determinants of 2 × 2 matrices

$= \langle 0, 9.5, 0 \rangle$ — Component form

Step 4 Find the magnitude and direction of the torque vector.

The component form of the torque vector $\langle 0, 9.5, 0 \rangle$ tells us that the magnitude of the vector is about 9.5 newton-meters parallel to the positive *y*-axis as shown.



Guided Practice

4. **AUTO REPAIR** Find the magnitude of the torque if Robert applied the same amount of force to the end of the handle straight down when the handle makes a 40° angle above the positive *x*-axis as shown in Figure 8.5.3.

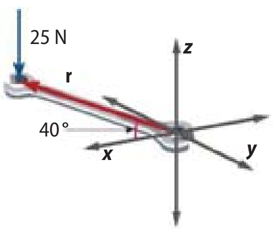


Figure 8.5.3

The cross product of two vectors has several geometric applications. One is that the magnitude of $\mathbf{u} \times \mathbf{v}$ represents the area of the parallelogram that has **u** and **v** as its adjacent sides (Figure 8.5.4).

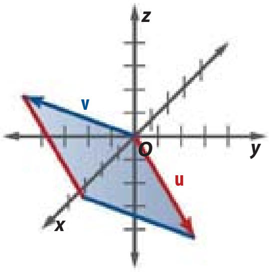


Figure 8.5.4

Example 5 Area of a Parallelogram in Space

Find the area of the parallelogram with adjacent sides $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

Step 1 Find $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix}$$ $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}\mathbf{k}$$ Determinant of a 3×3 matrix

$$= -3\mathbf{i} - 9\mathbf{j} - 14\mathbf{k}$$ Determinants of 2×2 matrices

Step 2 Find the magnitude of $\mathbf{u} \times \mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-3)^2 + (-9)^2 + (-14)^2}$$ Magnitude of a vector in space

$$= \sqrt{286} \text{ or about } 16.9$$ Simplify.

The area of the parallelogram shown in Figure 8.5.4 is about 16.9 square units.

GuidedPractice

5. Find the area of the parallelogram with adjacent sides $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Three vectors that lie in different planes but share the same initial point determine the adjacent edges of a **parallelepiped** (par-uh-lel-uh-PIE-ped), a polyhedron with faces that are all parallelograms (Figure 8.5.5). The absolute value of the **triple scalar product** of these vectors represents the volume of the parallelepiped.

StudyTip

Triple Scalar Product Notice that to find the triple scalar product of **t**, **u**, and **v**, you write the determinant representing $\mathbf{u} \times \mathbf{v}$ and replace the top row with the values for the vector **t**.

KeyConcept Triple Scalar Product

If $\mathbf{t} = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the triple scalar product is given by $\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

Example 6 Volume of a Parallelepiped

Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} 4 & -2 & -2 \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix}$$ $\mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}(4) - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}(-2) + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}(-2)$$ Determinant of a 3×3 matrix

$$= -12 + 18 + 28 \text{ or } 34$$ Simplify.

The volume of the parallelepiped shown in Figure 8.5.5 is $|\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v})|$ or 34 cubic units.

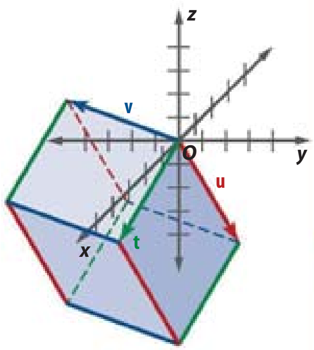


Figure 8.5.5

GuidedPractice

6. Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Exercises

= Step-by-Step Solutions begin on page R29.

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -9, 6\rangle, \mathbf{v} = \langle -8, 2, 7\rangle$
2. $\mathbf{u} = \langle 5, 0, -4\rangle, \mathbf{v} = \langle 6, -1, 4\rangle$
3. $\mathbf{u} = \langle 2, -8, -7\rangle, \mathbf{v} = \langle 5, 9, -7\rangle$
4. $\mathbf{u} = \langle -7, -3, 1\rangle, \mathbf{v} = \langle -4, 5, -13\rangle$
5. $\mathbf{u} = \langle 11, 4, -2\rangle, \mathbf{v} = \langle -1, 3, 8\rangle$
6. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
7. $\mathbf{u} = 3\mathbf{i} - 10\mathbf{j} + \mathbf{k}, \mathbf{v} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
8. $\mathbf{u} = 9\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}, \mathbf{v} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

9. **CHEMISTRY** A water molecule, in which the oxygen atom is centered at the origin, has one hydrogen atom centered at $\langle 55.5, 55.5, -55.5\rangle$ and the second hydrogen atom centered at $\langle -55.5, -55.5, -55.5\rangle$. Determine the bond angle between the vectors formed by the oxygen-hydrogen bonds. (Example 2)

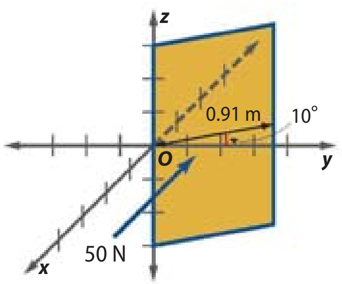
Find the angle θ between vectors u and v to the nearest tenth of a degree. (Example 2)

10. $\mathbf{u} = \langle 3, -2, 2\rangle, \mathbf{v} = \langle 1, 4, -7\rangle$
11. $\mathbf{u} = \langle 6, -5, 1\rangle, \mathbf{v} = \langle -8, -9, 5\rangle$
12. $\mathbf{u} = \langle -8, 1, 12\rangle, \mathbf{v} = \langle -6, 4, 2\rangle$
13. $\mathbf{u} = \langle 10, 0, -8\rangle, \mathbf{v} = \langle 3, -1, -12\rangle$
14. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}, \mathbf{v} = 4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}$
15. $\mathbf{u} = -6\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \mathbf{v} = -4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

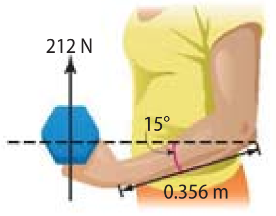
Find the cross product of u and v. Then show that u × v is orthogonal to both u and v. (Example 3)

16. $\mathbf{u} = \langle -1, 3, 5\rangle, \mathbf{v} = \langle 2, -6, -3\rangle$
17. $\mathbf{u} = \langle 4, 7, -2\rangle, \mathbf{v} = \langle -5, 9, 1\rangle$
18. $\mathbf{u} = \langle 3, -6, 2\rangle, \mathbf{v} = \langle 1, 5, -8\rangle$
19. $\mathbf{u} = \langle 5, -8, 0\rangle, \mathbf{v} = \langle -4, -2, 7\rangle$
20. $\mathbf{u} = -2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}, \mathbf{v} = 7\mathbf{i} + \mathbf{j} - 6\mathbf{k}$
21. $\mathbf{u} = -4\mathbf{i} + \mathbf{j} + 8\mathbf{k}, \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

22. **RESTAURANTS** A restaurant server applies 50 newtons of force to open a door. Find the magnitude and direction of the torque about the door's hinge. (Example 4)



23. **WEIGHTLIFTING** A weightlifter doing bicep curls applies 212 newtons of force to lift the dumbbell. The weightlifter's forearm is 0.356 meters long and she begins the bicep curl with her elbow bent at a 15° angle below the horizontal in the direction of the positive x-axis. (Example 4)



a. Find the vector representing the torque about the weightlifter's elbow in component form.
b. Find the magnitude and direction of the torque.

Find the area of the parallelogram with adjacent sides u and v. (Example 5)

24. $\mathbf{u} = \langle 2, -5, 3\rangle, \mathbf{v} = \langle 4, 6, -1\rangle$
25. $\mathbf{u} = \langle -9, 1, 2\rangle, \mathbf{v} = \langle 6, -5, 3\rangle$
26. $\mathbf{u} = \langle 4, 3, -1\rangle, \mathbf{v} = \langle 7, 2, -2\rangle$
27. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}, \mathbf{v} = 5\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$
28. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 8\mathbf{k}, \mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$
29. $\mathbf{u} = -3\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}, \mathbf{v} = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

Find the volume of the parallelepiped having t, u, and v as adjacent edges. (Example 6)

30. $\mathbf{t} = \langle -1, -9, 2\rangle, \mathbf{u} = \langle 4, -7, -5\rangle, \mathbf{v} = \langle 3, -2, 6\rangle$
31. $\mathbf{t} = \langle -6, 4, -8\rangle, \mathbf{u} = \langle -3, -1, 6\rangle, \mathbf{v} = \langle 2, 5, -7\rangle$
32. $\mathbf{t} = \langle 2, -3, -1\rangle, \mathbf{u} = \langle 4, -6, 3\rangle, \mathbf{v} = \langle -9, 5, -4\rangle$
33. $\mathbf{t} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}, \mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$
34. $\mathbf{t} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}, \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}, \mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}$
35. $\mathbf{t} = 5\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}, \mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}, \mathbf{v} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

Find a vector that is orthogonal to each vector.

36. $\langle 3, -8, 4\rangle$
37. $\langle -1, -2, 5\rangle$
38. $\left\langle 6, -\frac{1}{3}, -3\right\rangle$
39. $\langle 7, 0, 8\rangle$

Given v and u · v, find u.

40. $\mathbf{v} = \langle 2, -4, -6\rangle, \mathbf{u} \cdot \mathbf{v} = -22$
41. $\mathbf{v} = \left\langle \frac{1}{2}, 0, 4\right\rangle, \mathbf{u} \cdot \mathbf{v} = \frac{31}{2}$
42. $\mathbf{v} = \langle -2, -6, -5\rangle, \mathbf{u} \cdot \mathbf{v} = 35$

Determine whether the points are collinear.

43. $(-1, 7, 7), (-3, 9, 11), (-5, 11, 13)$
44. $(11, 8, -1), (17, 5, -7), (8, 11, 5)$

Determine whether each pair of vectors are parallel.

45. $\mathbf{m} = \langle 2, -10, 6 \rangle$, $\mathbf{n} = \langle 3, -15, 9 \rangle$

46. $\mathbf{a} = \langle 6, 3, -7 \rangle$, $\mathbf{b} = \langle -4, -2, 3 \rangle$

47. $\mathbf{w} = \left\langle -\frac{3}{2}, \frac{3}{4}, -\frac{9}{8} \right\rangle$, $\mathbf{z} = \langle -4, 2, -3 \rangle$

Write the component form of each vector.

48. $\mathbf{u}$ lies in the yz-plane, has a magnitude of 8, and makes a 60° angle above the positive y-axis.

49. $\mathbf{v}$ lies in the xy-plane, has a magnitude of 11, and makes a 30° angle to the left of the negative x-axis.

Given the four vertices, determine whether quadrilateral *ABCD* is a parallelogram. If it is, find its area, and determine whether it is a rectangle.

50. $A(3, 0, -2)$, $B(0, 4, -1)$, $C(0, 2, 5)$, $D(3, 2, 4)$

51. $A(7, 5, 5)$, $B(4, 4, 4)$, $C(4, 6, 2)$, $D(7, 7, 3)$

52. **AIR SHOWS** In an air show, two airplanes take off simultaneously. The first plane starts at the position $(0, -2, 0)$ and is at the position $(6, -10, 15)$ after three seconds. The second plane starts at the position $(0, 2, 0)$ and is at the position $(6, 10, 15)$ after three seconds. Are the paths of the two planes parallel? Explain.

For $\mathbf{u} = \langle 3, 2, -2 \rangle$ and $\mathbf{v} = \langle -4, 4, 5 \rangle$, find each of the following, if possible.

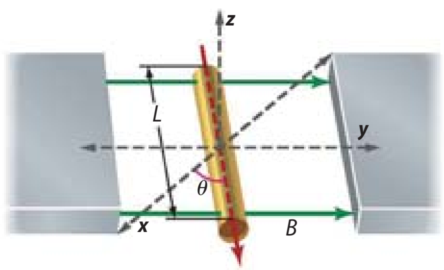
53. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

54. $\mathbf{v} \times (\mathbf{u} \cdot \mathbf{v})$

55. $\mathbf{u} \times \mathbf{u} \times \mathbf{v}$

56. $\mathbf{v} \cdot \mathbf{v} \cdot \mathbf{u}$

57. **ELECTRICITY** When a wire carrying an electric current is placed in a magnetic field, the force on the wire in newtons is given by $\vec{F} = I\vec{L} \times \vec{B}$, where I represents the current flowing through the wire in amps, $\vec{L}$ represents the vector length of the wire pointing in the direction of the current in meters, and $\vec{B}$ is the force of the magnetic field in teslas. In the figure below, the wire is rotated through an angle θ in the xy-plane.

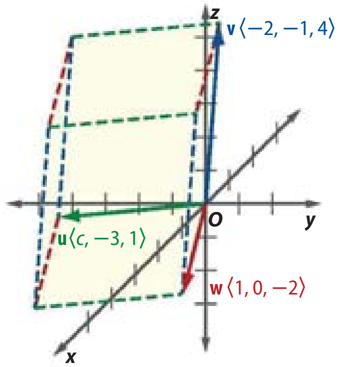


a. If the force of a magnetic field is 1.1 teslas, find the magnitude of the force on a wire in the xy-plane that is 0.15 meter in length carrying a current of 25 amps at an angle of 60°.

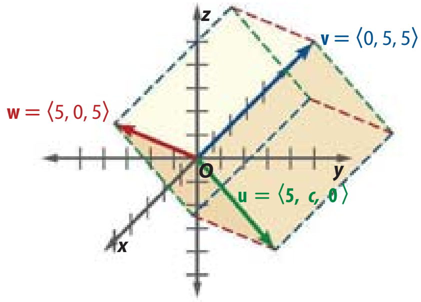
b. If the force on the wire is $\vec{F} = \langle 0, 0, -0.63 \rangle$, what is the angle of the wire?

Given v, w, and the volume of the parallelepiped having adjacent edges u, v, and w, find *c*.

58. $\mathbf{v} = \langle -2, -1, 4 \rangle$, $\mathbf{w} = \langle 1, 0, -2 \rangle$, $\mathbf{u} = \langle c, -3, 1 \rangle$, and $V = 7$ cubic units

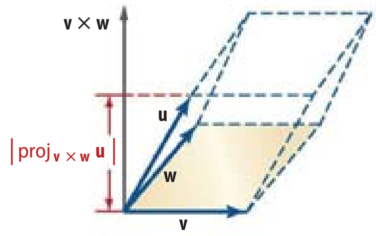


59. $\mathbf{v} = \langle 0, 5, 5 \rangle$, $\mathbf{w} = \langle 5, 0, 5 \rangle$, $\mathbf{u} = \langle 5, c, 0 \rangle$, and $V = 250$ cubic units



H.O.T. Problems Use Higher-Order Thinking Skills

60. **PROOF** Verify the formula for the volume of a parallelepiped. (*Hint*: Use the projection of $\mathbf{u}$ onto $\mathbf{v} \times \mathbf{w}$.)



61. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain.

For any two nonzero, nonparallel vectors in space, there is a vector that is perpendicular to both.

62. **REASONING** If $\mathbf{u}$ and $\mathbf{v}$ are parallel in space, then how many vectors are perpendicular to both? Explain.

63 **CHALLENGE** Given $\mathbf{u} = \langle 4, 6, c \rangle$ and $\mathbf{v} = \langle -3, -2, 5 \rangle$, find the value of c for which $\mathbf{u} \times \mathbf{v} = 34\mathbf{i} - 26\mathbf{j} + 10\mathbf{k}$.

64. **REASONING** Explain why the cross product is not defined for vectors in the two-dimensional coordinate system.

65. **WRITING IN MATH** Compare and contrast the methods for determining whether vectors in space are parallel or perpendicular.

Spiral Review

Find the length and midpoint of the segment with the given endpoints. (Lesson 8-4)

66. $(1, 10, 13), (-2, 22, -6)$

67. $(12, -1, -14), (21, 19, -23)$

68. $(-22, 24, -9), (10, 10, 2)$

Find the dot product of u and v. Then determine if u and v are orthogonal. (Lesson 8-3)

69. $\langle -8, -7 \rangle \cdot \langle 1, 2 \rangle$

70. $\langle -4, -6 \rangle \cdot \langle 7, 5 \rangle$

71. $\langle 6, -3 \rangle \cdot \langle -3, 5 \rangle$

72. BAKERY Hector's bakery has racks that can hold up to 900 bagels and muffins. Due to costs, the number of bagels produced must be less than or equal to 300 plus twice the number of muffins produced. The demand for bagels is at least three times that of muffins. Hector makes a profit of \$3 per muffin sold and \$1.25 per bagel sold. How many of each item should he make to maximize profit? (Lesson 6-5)

73. Decompose $\frac{2m + 16}{m^2 - 16}$ into partial fractions. (Lesson 6-4)

Verify each identity. (Lesson 5-2)

74. $\tan^2 \theta + \cos^2 \theta + \sin^2 \theta = \sec^2 \theta$

75. $\sec^2 \theta \cot^2 \theta - \cot^2 \theta = 1$

76. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$

Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree. (Lesson 4-7)

77. $a = 20, c = 24, B = 47°$

78. $A = 25°, B = 78°, a = 13.7$

79. $a = 21.5, b = 16.7, c = 10.3$

Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth. (Lesson 4-2)

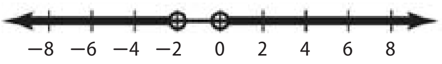
80. $-72.775°$

81. $29° 6' 6''$

82. $132° 18' 31''$

Skills Review for Standardized Tests

83. SAT/ACT The graph represents the set of all possible solutions to which of the following statements?



A $|x - 1| > 1$

B $|x - 1| < 1$

C $|x + 1| < 1$

D $|x + 1| > 1$

84. What is the cross product of $\mathbf{u} = \langle 3, 8, 0 \rangle$ and $\mathbf{v} = \langle -4, 2, 6 \rangle$?

F $48\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

G $48\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

H $46\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

J $46\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

85. FREE RESPONSE A batter hits a ball at a 30° angle with the ground at an initial speed of 90 feet per second.

a. Find the magnitude of the horizontal and vertical components of the velocity.

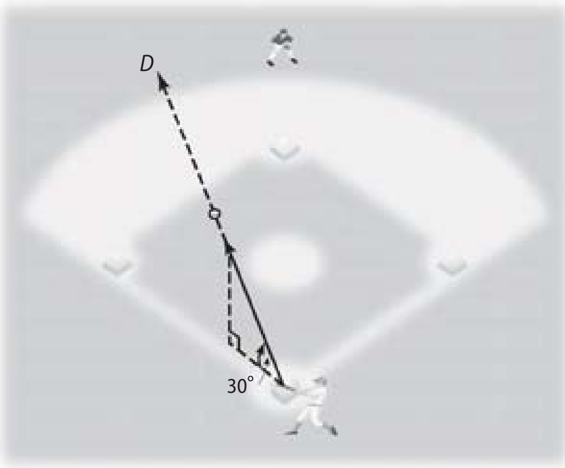
b. Are the values in part **a** vectors or scalars?

c. Assume that the ball is not caught and the player hit it one yard off the ground. How far will it travel in the air?

d. Assume that home plate is at the origin and second base lies due north. If the ball is hit at a bearing of N20°W and lands at point D, find the component form of $\overrightarrow{CD}$.

e. Determine the unit vector of $\overrightarrow{CD}$.

f. The fielder is standing at (0, 150) when the ball is hit. At what quadrant bearing should the fielder run in order to meet the ball where it will hit the ground?



Study Guide and Review

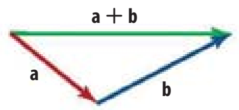
Chapter Summary

KeyConcepts

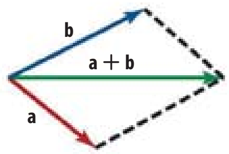
Introduction to Vectors (Lesson 8-1)

- The direction of a vector is the directed angle between the vector and a horizontal line. The magnitude of a vector is its length.
- When two or more vectors are combined, their sum is a single vector called the resultant, which can be found using the triangle or parallelogram method.

Triangle Method



Parallelogram Method



Vectors in the Coordinate Plane (Lesson 8-2)

- The component form of a vector with rectangular components x and y is $\langle x, y\rangle$.
- The component form of a vector that is not in standard position, with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, is given by $\langle x_2 - x_1, y_2 - y_1\rangle$.
- The magnitude of a vector $\mathbf{v} = \langle v_1, v_2\rangle$ is given by $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- If $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ are vectors and k is a scalar, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$, $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$, and $k\mathbf{a} = \langle ka_1, ka_2\rangle$.
- The standard unit vectors **i** and **j** can be used to express any vector $\mathbf{v} = \langle a, b\rangle$ as $a\mathbf{i} + b\mathbf{j}$.

Dot Products (Lesson 8-3)

- The dot product of $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.
- If θ is the angle between nonzero vectors **a** and **b**, then $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Vectors in Three-Dimensional Space (Lesson 8-4)

- The distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The midpoint of $\overline{AB}$ is given by $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Dot and Cross Products of Vectors in Space (Lesson 8-5)

- The dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of **a** and **b** is the vector $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

KeyVocabulary

component form (p. 492)
components (p. 487)
cross product (p. 519)
direction (p. 482)
dot product (p. 500)
equivalent vectors (p. 483)
initial point (p. 482)
linear combination (p. 495)
magnitude (p. 482)
octants (p. 510)
opposite vectors (p. 483)
ordered triple (p. 510)
orthogonal (p. 500)
parallelepiped (p. 521)
parallelogram method (p. 484)
parallel vectors (p. 483)
quadrant bearing (p. 483)
rectangular components (p. 487)
resultant (p. 484)
standard position (p. 482)
terminal point (p. 482)
three-dimensional coordinate system (p. 510)
torque (p. 520)
triangle method (p. 484)
triple scalar product (p. 521)
true bearing (p. 483)
unit vector (p. 494)
vector (p. 482)
vector projection (p. 503)
work (p. 505)
z-axis (p. 510)
zero vector (p. 485)

VocabularyCheck

Determine whether each statement is *true* or *false*. If false, replace the underlined term or expression to make the statement true.

1. The terminal point of a vector is where the vector begins.
2. If $\mathbf{a} = \langle -4, 1\rangle$ and $\mathbf{b} = \langle 3, 2\rangle$, the dot product is calculated by $\underline{-4(1) + 3(2)}$.
3. The midpoint of $\overline{AB}$ with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
4. The magnitude of **r** if the initial point is $A(-1, 2)$ and the terminal point is $B(2, -4)$ is $\langle 3, -6\rangle$.
5. Two vectors are equal only if they have the same direction and magnitude.
6. When two nonzero vectors are orthogonal, the angle between them is $\underline{180°}$.
7. The component of **u** onto **v** is the vector with direction that is parallel to **v** *and* with length that is the component of **u** along **v**.
8. To find at least one vector orthogonal to any two vectors in space, calculate the cross product of the two original vectors.
9. When a vector is subtracted, it is equivalent to adding the opposite vector.
10. If **v** is a unit vector in the same direction as **u**, then $\mathbf{v} = \underline{\frac{|\mathbf{u}|}{\mathbf{u}}}$.

Lesson-by-Lesson Review

8-1 Introduction to Vectors (pp. 482–491)

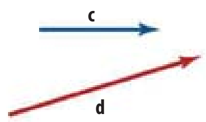
State whether each quantity described is a *vector* quantity or a *scalar* quantity.

11. a car driving 50 miles an hour due east

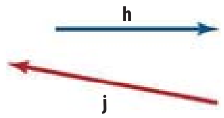
12. a gust of wind blowing 5 meters per second

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

13.



14.



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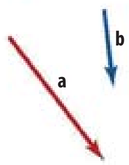
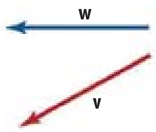


16.



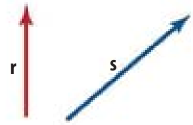
Determine the magnitude and direction of the resultant of each vector sum.

17. 70 meters due west and then 150 meters due east

18. 8 newtons directly backward and then 12 newtons directly backward

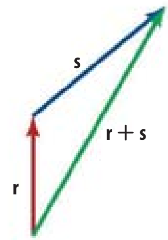
Example 1

Find the resultant of **r** and **s** using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal.



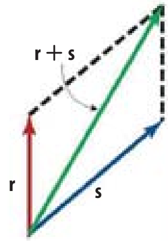
Triangle Method

Translate **r** so that the tip of **r** touches the tail of **s**. The resultant is the vector from the tail of **r** to the tip of **s**.



Parallelogram Method

Translate **s** so that the tail of **s** touches the tail of **r**. Complete the parallelogram that has **r** and **s** as two of its sides. The resultant is the vector that forms the indicated diagonal of the parallelogram.



The magnitude of the resultant is 3.4 cm and the direction is 59°.

8-2 Vectors in the Coordinate Plane (pp. 492–499)

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

19. $A(-1, 3), B(5, 4)$

20. $A(7, -2), B(-9, 6)$

21. $A(-8, -4), B(6, 1)$

22. $A(2, -10), B(3, -5)$

Find each of the following for $\mathbf{p} = \langle 4, 0\rangle$, $\mathbf{q} = \langle -2, -3\rangle$, and $\mathbf{t} = \langle -4, 2\rangle$.

23. $2\mathbf{q} - \mathbf{p}$

24. $\mathbf{p} + 2\mathbf{t}$

25. $\mathbf{t} - 3\mathbf{p} + \mathbf{q}$

26. $2\mathbf{p} + \mathbf{t} - 3\mathbf{q}$

Find a unit vector **u** with the same direction as **v**.

27. $\mathbf{v} = \langle -7, 2\rangle$

28. $\mathbf{v} = \langle 3, -3\rangle$

29. $\mathbf{v} = \langle -5, -8\rangle$

30. $\mathbf{v} = \langle 9, 3\rangle$

Example 2

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, -2)$ and terminal point $B(4, -1)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$ — Component form

$= \langle 4 - 3, -1 - (-2)\rangle$ — Substitute.

$= \langle 1, 1\rangle$ — Subtract.

Find the magnitude using the Distance Formula.

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ — Distance Formula

$= \sqrt{[(4 - 3)]^2 + [-1 - (-2)]^2}$ — Substitute.

$= \sqrt{2}$ or about 1.4 — Simplify.

8-3 Dot Products and Vector Projections (pp. 500–508)

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

31. $\mathbf{u} = \langle -3, 5\rangle, \mathbf{v} = \langle 2, 1\rangle$ **32.** $\mathbf{u} = \langle 4, 4\rangle, \mathbf{v} = \langle 5, 7\rangle$

33. $\mathbf{u} = \langle -1, 4\rangle, \mathbf{v} = \langle 8, 2\rangle$ **34.** $\mathbf{u} = \langle -2, 3\rangle, \mathbf{v} = \langle 1, 3\rangle$

Find the angle θ between **u** and **v** to the nearest tenth of a degree.

35. $\mathbf{u} = \langle 5, -1\rangle, \mathbf{v} = \langle -2, 3\rangle$ **36.** $\mathbf{u} = \langle -1, 8\rangle, \mathbf{v} = \langle 4, 2\rangle$

Example 3

Find the dot product of $\mathbf{x} = \langle 2, -5\rangle$ and $\mathbf{y} = \langle -4, 7\rangle$. Then determine if **x** and **y** are orthogonal.

$\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + x_2 y_2$ Dot product

$= 2(-4) + -5(7)$ Substitute.

$= -8 + (-35)$ or -43 Simplify.

Since $\mathbf{x} \bullet \mathbf{y} \neq 0$, **x** and **y** are not orthogonal.

8-4 Vectors in Three-Dimensional Space (pp. 510–516)

Plot each point in a three-dimensional coordinate system.

37. $(1, 2, -4)$ **38.** $(3, 5, 3)$

39. $(5, -3, -2)$ **40.** $(-2, -3, -2)$

Find the length and midpoint of the segment with the given endpoints.

41. $(-4, 10, 4), (2, 0, 8)$ **42.** $(-5, 6, 4), (-9, -2, -2)$

43. $(3, 2, 0), (-9, -10, 4)$ **44.** $(8, 3, 2), (-4, -6, 6)$

Locate and graph each vector in space.

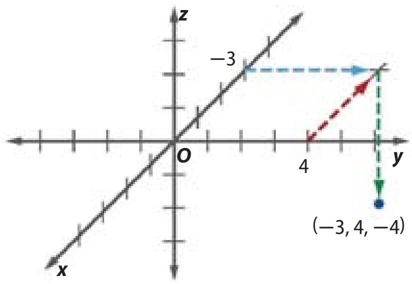
45. $\mathbf{a} = \langle 0, -3, 4\rangle$ **46.** $\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

47. $\mathbf{c} = -2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ **48.** $\mathbf{d} = \langle -4, -5, -3\rangle$

Example 4

Plot $(-3, 4, -4)$ in a three-dimensional coordinate system.

Locate the point $(-3, 4)$ in the xy-plane and mark it with a cross. Then plot a point 4 units down from this location parallel to the z-axis.



8-5 Vectors in Three-Dimensional Space (pp. 518–524)

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

49. $\mathbf{u} = \langle 2, 5, 2\rangle, \mathbf{v} = \langle 8, 2, -13\rangle$

50. $\mathbf{u} = \langle 5, 0, -6\rangle, \mathbf{v} = \langle -6, 1, 3\rangle$

Find the cross product of **u** and **v**. Then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

51. $\mathbf{u} = \langle 1, -3, -2\rangle, \mathbf{v} = \langle 2, 4, -3\rangle$

52. $\mathbf{u} = \langle 4, 1, -2\rangle, \mathbf{v} = \langle 5, -4, -1\rangle$

Example 5

Find the cross product of $\mathbf{u} = \langle -4, 2, -3\rangle$ and $\mathbf{v} = \langle 7, 11, 2\rangle$. Then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 2 & -3 \\ 11 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -3 \\ 7 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 2 \\ 7 & 11 \end{vmatrix}\mathbf{k}$$

$$= \langle 37, -13, -58\rangle$$

$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = \langle 37, -13, -58\rangle \cdot \langle -4, 2, -3\rangle$

$= -148 - 26 + 174$ or 0 ✓

$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = \langle 37, -13, -58\rangle \cdot \langle 7, 11, 2\rangle$

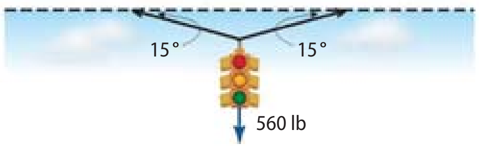
$= 259 - 143 - 116$ or 0 ✓

Applications and Problem Solving

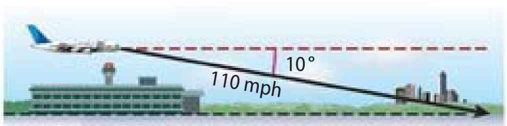
53. BASEBALL A player throws a baseball with an initial velocity of 55 feet per second at an angle of 25° above the horizontal, as shown below. Find the magnitude of the horizontal and vertical components. (Lesson 8-1)

54. STROLLER Miriam is pushing a stroller with a force of 200 newtons at an angle of 20° below the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 8-1)

55. LIGHTS A traffic light at an intersection is hanging from two wires of equal length at 15° below the horizontal as shown. If the traffic light weighs 560 pounds, what is the tension in each wire keeping the light at equilibrium? (Lesson 8-1)



56. AIRPLANE An airplane is descending at a speed of 110 miles per hour at an angle of 10° below the horizontal. Find the component form of the vector that represents the velocity of the airplane. (Lesson 8-2)

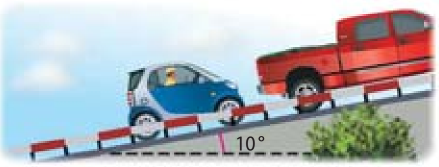


57. LIFEGUARD A lifeguard at a wave pool swims at a speed of 4 miles per hour at a 60° angle to the side of the pool as shown. (Lesson 8-2)

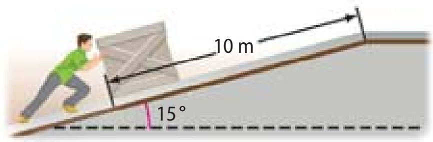
a. At what speed is the lifeguard traveling if the current in the pool is 2 miles per hour parallel to the side of the pool as shown?

b. At what angle is the lifeguard traveling with respect to the starting side of the pool?

58. TRAFFIC A 1500-pound car is stopped in traffic on a hill that is at an incline of 10°. Determine the force that is required to keep the car from rolling down the hill. (Lesson 8-3)



59. WORK At a warehouse, Phil pushes a box on sliders with a constant force of 80 newtons up a ramp that has an incline of 15° with the horizontal. Determine the amount of work in joules that Phil does if he pushes the dolly 10 meters. (Lesson 8-3)



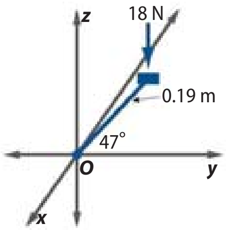
60. SATELLITES The positions of two satellites that are in orbit can be represented by the coordinates (28,625, 32,461, −38,426) and (−31,613, −29,218, 43,015), where (0, 0, 0) represents the center of Earth and the coordinates are given in miles. The radius of Earth is about 3963 miles. (Lesson 8-4)

a. Determine the distance between the two satellites.

b. If a third satellite were to be placed directly between the two satellites, what would the coordinates be?

c. Can a third satellite be placed at the coordinates found in part **b**? Explain your reasoning.

61. BICYCLES A bicyclist applies 18 newtons of force down on the pedal to put the bicycle in motion. The pedal has an initial position of 47° above the *y*-axis, and a length of 0.19 meters to the pedal's axle, as shown. (Lesson 8-5)

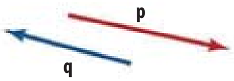


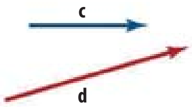
a. Find the vector representing the torque about the axle of the bicycle pedal in component form.

b. Find the magnitude and direction of the torque.

CHAPTER 8 Practice Test

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

1. 

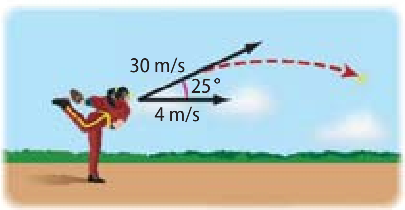
2. 

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

3. $A(1, -3), B(-5, 1)$

4. $A\left(\frac{1}{2}, \frac{3}{2}\right), B(-1, 7)$

5. **SOFTBALL** A batter on the opposing softball team hits a ground ball that rolls out to Libby in left field. She runs toward the ball at a velocity of 4 meters per second, scoops it, and proceeds to throw it to the catcher at a speed of 30 meters per second and at an angle of 25° with the horizontal in an attempt to throw out a runner. What is the resultant speed and direction of the throw?



Find a unit vector in the same direction as **u**.

6. $\mathbf{u} = \langle -1, 4 \rangle$

7. $\mathbf{u} = \langle 6, -3 \rangle$

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

8. $\mathbf{u} = \langle 2, -5 \rangle, \mathbf{v} = \langle -3, 2 \rangle$

9. $\mathbf{u} = \langle 4, -3 \rangle, \mathbf{v} = \langle 6, 8 \rangle$

10. $\mathbf{u} = 10\mathbf{i} - 3\mathbf{j}, \mathbf{v} = \mathbf{i} + 8\mathbf{j}$

11. **MULTIPLE CHOICE** Write **u** as the sum of two orthogonal vectors, one of which being the projection of **u** onto **v** if $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle -4, 2 \rangle$.

A $\mathbf{u} = \left\langle \frac{2}{5}, -\frac{3}{5} \right\rangle + \left\langle \frac{3}{5}, \frac{18}{5} \right\rangle$

B $\mathbf{u} = \left\langle \frac{2}{5}, \frac{3}{5} \right\rangle + \left\langle \frac{3}{5}, \frac{12}{5} \right\rangle$

C $\mathbf{u} = \left\langle -\frac{4}{5}, \frac{2}{5} \right\rangle + \left\langle \frac{9}{5}, \frac{13}{5} \right\rangle$

D $\mathbf{u} = \left\langle -\frac{2}{5}, \frac{1}{5} \right\rangle + \left\langle \frac{7}{5}, \frac{14}{5} \right\rangle$

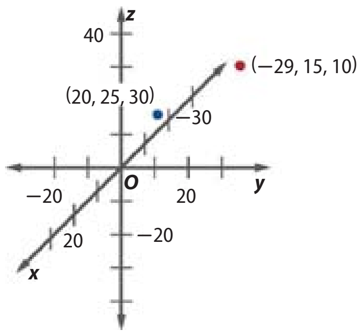
12. **MOVING** Tamera is pushing a box along a level floor with a force of 120 pounds at an angle of depression of 20°. Determine how much work is done if the box is moved 25 feet.

Find each of the following for $\mathbf{a} = \langle 2, 4, -3 \rangle$, $\mathbf{b} = \langle -5, -7, 1 \rangle$, and $\mathbf{c} = \langle 8, 5, -9 \rangle$.

13. $2\mathbf{a} + 5\mathbf{b} - 3\mathbf{c}$

14. $\mathbf{b} - 6\mathbf{a} + 2\mathbf{c}$

15. **HOT AIR BALLOONS** During a festival, twelve hot air balloons take off. A few minutes later, the coordinates of the first two balloons are (20, 25, 30) and (−29, 15, 10) as shown, where the coordinates are given in feet.



a. Determine the distance between the first two balloons that took off.

b. A third balloon is halfway between the first two balloons. Determine the coordinates of the third balloon.

c. Find a unit vector in the direction of the first balloon if it took off at (0, 0, 0).

Find the angle θ between vectors **u** and **v** to the nearest tenth of a degree.

16. $\mathbf{u} = \langle -2, 4, 6 \rangle, \mathbf{v} = \langle 3, 7, 12 \rangle$

17. $\mathbf{u} = -9\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}, \mathbf{v} = -5\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$

Find the cross product of **u** and **v**. Then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

18. $\mathbf{u} = \langle 1, 7, 3 \rangle, \mathbf{v} = \langle 9, 4, 11 \rangle$

19. $\mathbf{u} = -6\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{v} = 5\mathbf{i} - 3\mathbf{j} - 2k$

20. **BOATING** The tiller is a lever that controls the position of the rudder on a boat. When force is applied to the tiller, the boat will turn. Suppose the tiller on a certain boat is 0.75 meter in length and is currently resting in the *xy*-plane at a 15° angle from the positive *x*-axis. Find the magnitude of the torque that is developed about the axle of the tiller if 50 newtons of force is applied in a direction parallel to the positive *y*-axis.

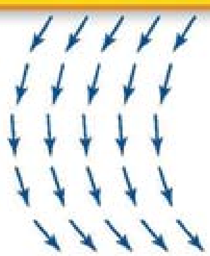
CHAPTER 8

Connect to AP Calculus
Vector Fields

Objective

- Graph vectors in and identify graphs of vector fields.

In Chapter 8, you examined the effects that wind and water currents have on a moving object. The force produced by the wind and current was represented by a single vector. However, we know that the current in a body of water or the force produced by wind is not necessarily constant; instead it differs from one region to the next. If we want to represent the entire current or air flow in an area, we would need to assign a vector to each point in space, thus creating a *vector field*.



Vector fields are commonly used in engineering and physics to model air resistance, magnetic and gravitational forces, and the motion of liquids. While these applications of vector fields require multiple dimensions, we will analyze vector fields in only two dimensions.

A vector field $\mathbf{F}(x, y)$ is a function that converts two-dimensional coordinates into sets of two-dimensional vectors.

$$\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y)\rangle, \text{ where } f_1(x, y) \text{ and } f_2(x, y) \text{ are scalar functions.}$$

To graph a vector field, evaluate $\mathbf{F}(x, y)$ at (x, y) and graph the vector using (x, y) as the initial point. This is done for several points.

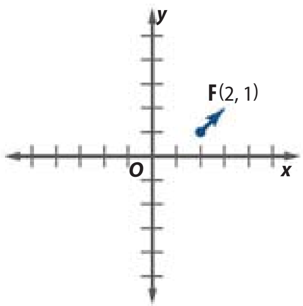
Activity 1 Vector Fields

Evaluate F(2, 1), F(−1, −1), F(1.5, −2), and F(−3, 2) for the vector field $\mathbf{F}(x, y) = \langle y^2, x - 1\rangle$. Graph each vector using (x, y) as the initial point.

Step 1 To evaluate $\mathbf{F}(2, 1)$, let $x = 2$ and $y = 1$.

$$\langle y^2, x - 1)\rangle = \langle 1^2, 2 - 1\rangle$$
$$= \langle 1, 1\rangle$$

Step 2 To graph, let (2, 1) represent the initial point of the vector.
This makes (2 + 1, 1 + 1) or (3, 2) the terminal point.

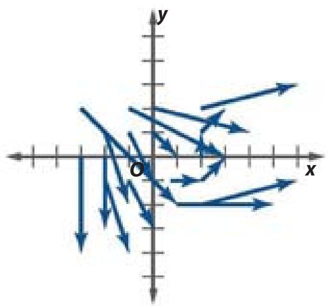


Step 3 Repeat Steps 1–2 for $\mathbf{F}(-1, -1)$, $\mathbf{F}(1.5, -2)$ and $\mathbf{F}(-3, 2)$.

Analyze the Results

1. Are the magnitudes and directions of the vectors the *same* or *different*?
2. Make a conjecture as to why a vector field can be defined as a function.
3. Is it possible to graph every vector in a vector field? Explain your reasoning.

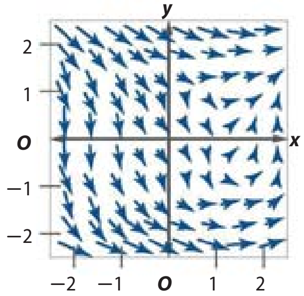
A graph of a vector field $\mathbf{F}(x, y)$ should include a variety of vectors all with initial points at (x, y). Graphing devices are typically used to graph vector fields because sketching vector fields by hand is often too difficult.



StudyTip

Graphs of Vector Fields Every point in a plane has a corresponding vector. The graphs of vector fields only show a selection of points.

To keep vectors from overlapping each other and to prevent the graph from looking too jumbled, the graphing devices proportionally reduce the lengths of the vectors and only construct vectors at certain intervals. For example, if we continue to graph more vectors for the vector field from Activity 1, the result would be the graph on the right.



Analyze the component functions of a vector field to identify the type of graph it will produce.

Activity 2 Vector Fields

Match each vector field to its graph.

$\mathbf{F}(x, y) = \langle 2, 1 + 2xy \rangle$ $\quad$ $\mathbf{G}(x, y) = \langle e^y, x \rangle$ $\quad$ $\mathbf{H}(x, y) = \langle e^y, y \rangle$

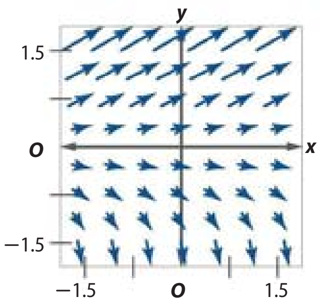


Figure 1

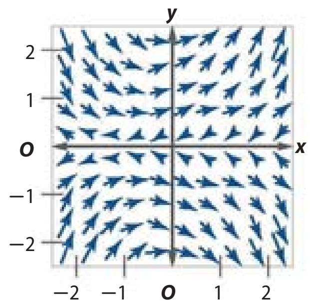


Figure 2

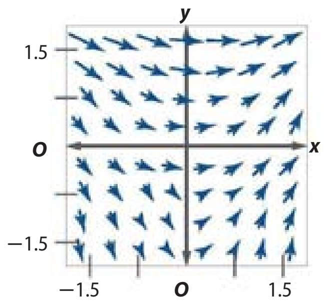


Figure 3

Step 1 Start by analyzing the components that make up $\mathbf{F}(x, y)$. The second component $(1 + 2xy)$ will produce a positive outcome when x and y have the same sign. The vertical component of the vectors in Quadrants I and III is positive, which makes the vectors in these quadrants point upward.

Step 2 The graph that has vectors pointing upward in Quadrants I and III is Figure 2.

Step 3 Repeat Steps 1–2 for the remaining vector fields.

Analyze the Results

4. Suppose the vectors in a vector field represent a force. What is the relationship between the force, the magnitude, and the length of a vector?

5. Representing wind by a single vector assumed that the force created remained constant for an entire area. If the force created by wind is represented by multiple vectors in a vector field, what assumption would have to be made about the third dimension?

Model and Apply

6. Complete the table for the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$. Then graph each vector.

(x, y)	$\langle -y, x \rangle$	(x, y)	$\langle -y, x \rangle$
(2, 0)		(−2, 1)	
(1, 2)		(−2, 0)	
(2, 1)		(−1, −2)	
(0, 2)		(0, −2)	
(−1, 2)		(1, −2)	
(−2, −1)		(2, −1)	

CHAPTER 9

Polar Coordinates and Complex Numbers

Then

In **Chapter 7**, you identified and graphed rectangular equations of conic sections.

Now

In **Chapter 9**, you will:

- Graph polar coordinates and equations.
- Convert between polar and rectangular coordinates and equations.
- Identify polar equations of conic sections.
- Convert complex numbers between polar and rectangular form.

Why? ▲

CONCERTS Polar equations can be used to model sound patterns to help determine stage arrangement, speaker and microphone placement, and volume and recording levels. Polar equations can also be used with lighting and camera angles when concerts are filmed.

PREREAD Use the Lesson Openers in Chapter 9 to make two or three predictions about what you will learn in this chapter.

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