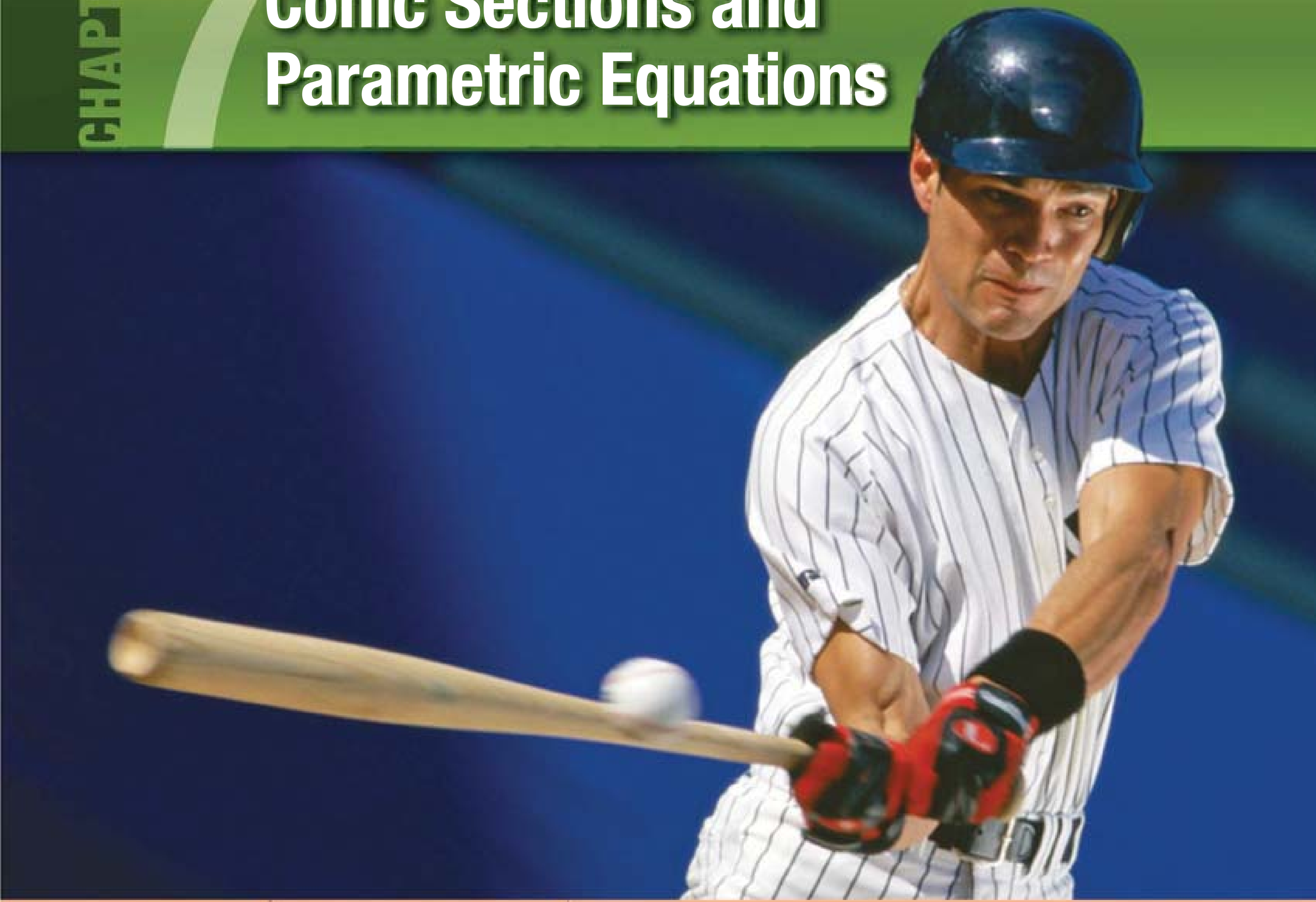
CHAPTER 7

Conic Sections and Parametric Equations



Then

In **Chapter 6**, you learned how to solve systems of linear equations using matrices.

Now

In **Chapter 7**, you will:

- Analyze, graph, and write equations of parabolas, circles, ellipses, and hyperbolas.
- Use equations to identify types of conic sections.
- Graph rotated conic sections.
- Solve problems related to the motion of projectiles.

Why?

BASEBALL When a baseball is hit, the path of the ball can be represented and traced by parametric equations.

PREREAD Scan the Study Guide and Review and use it to make two or three predictions about what you will learn in Chapter 7.

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Animation | Vocabulary | eGlossary | Personal Tutor | Graphing Calculator | Audio | Self-Check Practice | Worksheets

Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

For each function, find the axis of symmetry, the y-intercept, and the vertex. (Lesson 0-3)

1. $f(x) = x^2 - 2x - 12$ **2.** $f(x) = x^2 + 2x + 6$

3. $f(x) = 2x^2 + 4x - 8$ **4.** $f(x) = 2x^2 - 12x + 3$

5. $f(x) = 3x^2 - 12x - 4$ **6.** $f(x) = 4x^2 + 8x - 1$

7. BUSINESS The cost of producing x bicycles can be represented by $C(x) = 0.01x^2 - 0.5x + 550$. Find the axis of symmetry, the y-intercept, and the vertex of the function. (Lesson 0-3)

Find the discriminant of each quadratic function. (Lesson 0-3)

8. $f(x) = 2x^2 - 5x + 3$ **9.** $f(x) = 2x^2 + 6x - 9$

10. $f(x) = 3x^2 + 2x + 1$ **11.** $f(x) = 3x^2 - 8x - 3$

12. $f(x) = 4x^2 - 3x - 7$ **13.** $f(x) = 4x^2 - 2x + 11$

Find the equations of any vertical or horizontal asymptotes. (Lesson 2-5)

14. $f(x) = \frac{x-2}{x+4}$ **15.** $h(x) = \frac{x^2-4}{x+5}$

16. $f(x) = \frac{x(x-1)}{(x+2)(x-3)}$ **17.** $g(x) = \frac{x+3}{(x-1)(x+5)}$

18. $h(x) = \frac{2x^2-5x-12}{x^2+4x}$ **19.** $f(x) = \frac{2x^2-13x+6}{x-4}$

20. WILDLIFE The number of deer $D(x)$ after x years living on a wildlife preserve can be represented by $D(x) = \frac{12x+50}{0.02x+4}$. Determine the maximum number of deer that can live in the preserve. (Lesson 2-5)

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

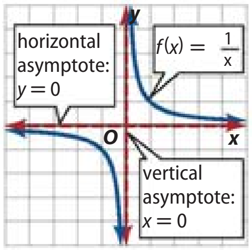
NewVocabulary

English		Español
conic section	p. 422	sección cónica
degenerate conic	p. 422	degenerado cónico
locus	p. 422	lugar geométrico
parabola	p. 422	parábola
focus	p. 422	foco
directrix	p. 422	directrix
axis of symmetry	p. 422	eje de simetría
vertex	p. 422	vértice
latus rectum	p. 430	recto de latus
ellipse	p. 432	elipse
foci	p. 432	focos
major axis	p. 432	eje principal
center	p. 432	centro
minor axis	p. 432	eje menor
vertices	p. 432	vértices
co-vertices	p. 432	co-vértices
eccentricity	p. 435	excentricidad
hyperbola	p. 442	hipérbola
transverse axis	p. 442	eje transversal
conjugate axis	p. 442	eje conjugado
parametric equation	p. 464	ecuación paramétrica
parameter	p. 464	parámetro
orientation	p. 464	orientación

ReviewVocabulary

transformations p. 48 transformaciones changes that affect the appearance of a parent function

asymptotes p. 46 asíntotas lines or curves that graphs approach



LESSON 7-1 Parabolas

Then

- You identified, analyzed, and graphed quadratic functions. (Lesson 1-5)

Now

1. Analyze and graph equations of parabolas.
2. Write equations of parabolas.

Why?

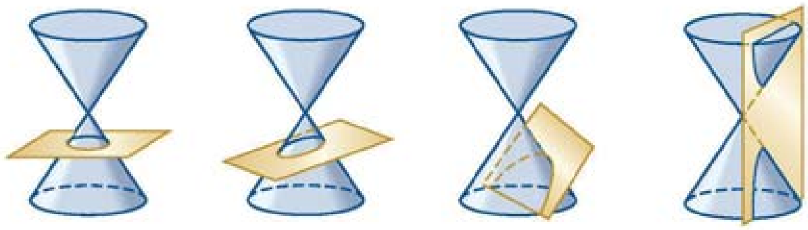
- Trough solar collectors use the properties of parabolas to focus radiation onto a receiver and generate solar power.

NewVocabulary

conic section
degenerate conic
locus
parabola
focus
directrix
axis of symmetry
vertex
latus rectum

1 Analyze and Graph Parabolas

Conic sections, or *conics*, are the figures formed when a plane intersects a double-napped right cone. A double-napped cone is two cones opposite each other and extending infinitely upward and downward. The four common conic sections that will be covered in this chapter are the *parabola*, the *ellipse*, the *circle*, and the *hyperbola*.



circle | ellipse | parabola | hyperbola

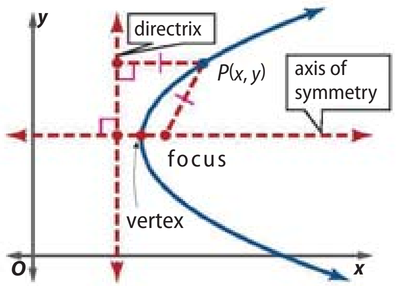
When the plane intersects the vertex of the cone, the figures formed are **degenerate conics**. A degenerate conic may be a point, a line, or two intersecting lines.

point (degenerate ellipse) | line (degenerate parabola) | intersecting lines (degenerate hyperbola)

The general form of the equations for conic sections is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C cannot all be zero. More specific algebraic forms for each type of conic will be addressed as they are introduced.

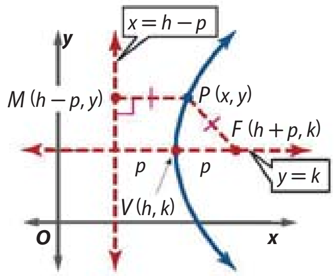
A **locus** is a set of all points that fulfill a geometric property. A **parabola** represents the locus of points in a plane that are equidistant from a fixed point, called the **focus**, and a specific line, called the **directrix**.

A parabola is symmetric about the line perpendicular to the directrix through the focus called the **axis of symmetry**. The **vertex** is the intersection of the parabola and the axis of symmetry.



Previously, you learned the quadratic function $f(x) = ax^2 + bx + c$, where $a \neq 0$, represents a parabola that opens either up or down. The definition of a parabola as a locus can be used to derive a general equation of a parabola that opens up, down, left, or right.

Let $P(x, y)$ be any point on the parabola with vertex $V(h, k)$ where $p = FV$, the distance from the vertex to the focus. By the definition of a parabola, the distance from any point on the parabola to the focus must equal the distance from that point to the directrix. So, if $FV = p$, then $VT = p$. From the definition of a parabola, you know that $PF = PM$. Because M lies on the directrix, the coordinates of M are $(h - p, y)$.



You can use the Distance Formula to determine the equation for the parabola.

$$PF = PM$$

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = \sqrt{[x - (h - p)]^2 + (y - y)^2} \quad \text{Distance Formula}$$

$$[x - (h + p)]^2 + (y - k)^2 = [x - (h - p)]^2 + 0^2 \quad \text{Square each side.}$$

$$x^2 - 2x(h + p) + (h + p)^2 + (y - k)^2 = x^2 - 2x(h - p) + (h - p)^2 \quad \text{Multiply.}$$

$$x^2 - 2xh - 2xp + h^2 + 2hp + p^2 + (y - k)^2 = x^2 - 2xh + 2xp + h^2 - 2hp + p^2 \quad \text{Multiply.}$$

$$(y - k)^2 = 4xp - 4hp \quad \text{Simplify.}$$

$$(y - k)^2 = 4p(x - h) \quad \text{Factor.}$$

ReadingMath

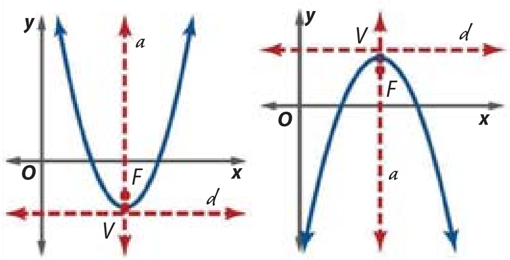
Concavity In this lesson, you will refer to parabolas as curves that open up, down, to the right, or to the left. In calculus, you will learn and use the term *concavity.* In this case, the curves are *concave* up, *concave* down, *concave* right, or *concave* left, respectively.

An equation for a parabola that opens horizontally is $(y - k)^2 = 4p(x - h)$. Similarly, for parabolas that open vertically, you can derive the equation $(x - h)^2 = 4p(y - k)$.

These represent the standard equations for parabolas, where $p \neq 0$. The values of the constants h, k, and p determine characteristics of parabolas such as the coordinates of the vertex and the direction of the parabola.

KeyConcept Standard Form of Equations for Parabolas

$(x - h)^2 = 4p(y - k)$



$p > 0$ $\qquad$ $p < 0$

Orientation: opens vertically

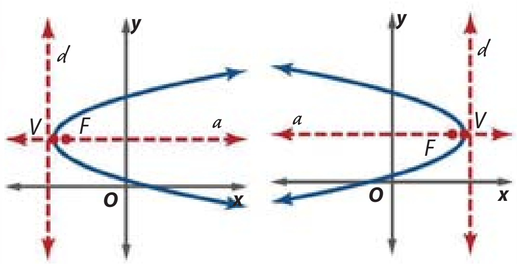
Vertex: (h, k)

Focus: $(h, k + p)$

Axis of Symmetry a: $x = h$

Directrix d: $y = k - p$

$(y - k)^2 = 4p(x - h)$



$p > 0$ $\qquad$ $p < 0$

Orientation: opens horizontally

Vertex: (h, k)

Focus: $(h + p, k)$

Axis of Symmetry a: $y = k$

Directrix d: $x = h - p$

You can use the standard form of the equation for a parabola to determine characteristics of the parabola such as the vertex, focus, and directrix.

Example 1 Determine Characteristics and Graph

For $(y + 5)^2 = -12(x - 2)$, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

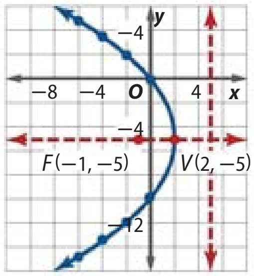
The equation is in standard form and the squared term is y, which means that the parabola opens horizontally. The equation is in the form $(y - k)^2 = 4p(x - h)$, so $h = 2$ and $k = -5$. Because $4p = -12$, $p = -3$ and the graph opens to the left. Use the values of h, k, and p to determine the characteristics of the parabola.

vertex: $(2, -5)$ (h, k) directrix: $x = 5$ $x = h - p$

focus: $(-1, -5)$ $(h + p, k)$ axis of symmetry: $y = -5$ $y = k$

Graph the vertex, focus, axis, and directrix of the parabola. Then make a table of values to graph the general shape of the curve.

x	y
0	−0.1, −9.9
−2	1.9, −11.9
−4	3.5, −13.5
−6	4.8, −14.8



GuidedPractice

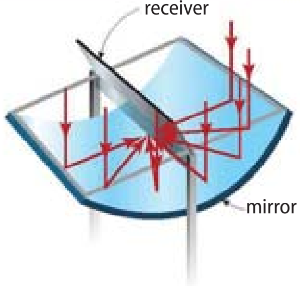
For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

1A. $8(y + 3) = (x - 4)^2$

1B. $2(x + 6) = (y + 1)^2$

Real-World Example 2 Characteristics of Parabolas

SOLAR ENERGY **A trough solar collector is a length of mirror in a parabolic shape that focuses radiation from the Sun onto a linear receiver located at the focus of the parabola. The cross section of a single trough can be modeled using $x^2 = 3.04y$, where x and y are measured in meters. Where is the linear receiver located in this cross section?**



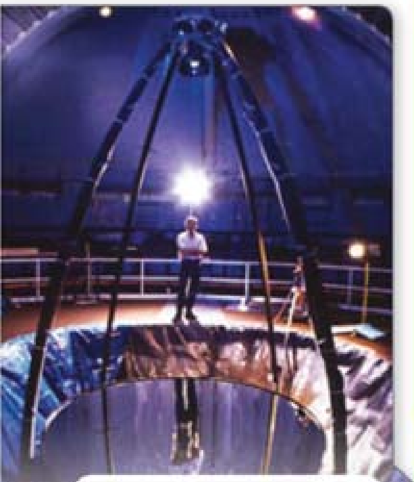
The linear receiver is located at the focus of the parabola. Because the x-term is squared and p is positive, the parabola opens up and the focus is located at $(h, k + p)$.

The equation is provided in standard form, and h and k are both zero. Because $4p = 3.04$, p is 0.76. So, the location of the focus is $(0, 0 + 0.76)$ or $(0, 0.76)$.

The location of the focus for the cross-section of the given parabola is $(0, 0.76)$. Therefore, the linear receiver is 0.76 meter above the vertex of the parabola.

GuidedPractice

2. **ASTRONOMY** Liquid-mirror telescopes consist of a thin layer of liquid metal in the shape of a parabola with a camera located at the focal point. Suppose a liquid mirror telescope can be modeled using the equation $x^2 = 44.8y - 268.8$ when $-5 \le x \le 5$. If x and y are measured in feet, what is the shortest distance between the surface of the liquid mirror and the camera?



Real-WorldLink

The 3.0 primary mirror of NASA's Orbital Debris Observatory is created by spinning a plate coated with a thin layer of mercury, which flows into the perfect shape for a telescope mirror.

Source: *Getty Images*

To determine the characteristics of a parabola, you may sometimes need to write an equation in standard form. In some cases, you can simply rearrange the equation, but other times it may be necessary to use mathematical skills such as completing the square.

JSC/NASA

Example 3 Write in Standard Form

Write $y = -\frac{1}{4}x^2 + 3x + 6$ in standard form. Identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

$y = -\frac{1}{4}x^2 + 3x + 6$	Original equation
$y = -\frac{1}{4}(x^2 - 12x) + 6$	Factor $-\frac{1}{4}$ from x-terms.
$y = -\frac{1}{4}(x^2 - 12x + 36 - 36) + 6$	Complete the square.
$y = -\frac{1}{4}(x^2 - 12x + 36) + 9 + 6$	$-\frac{1}{4}(-36) = 9$
$y = -\frac{1}{4}(x - 6)^2 + 15$	Factor.
$-4(y - 15) = (x - 6)^2$	Standard form of a parabola

Because the x-term is squared and $p = -1$, the graph opens down. Use the standard form of the equation to determine the characteristics of the parabola.

vertex: $(6, 15)$ (h, k)

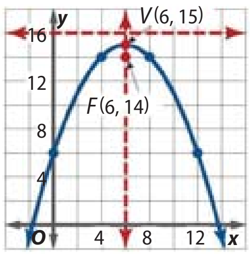
focus: $(6, 14)$ $(h, k + p)$

directrix: $y = 16$ $y = k - p$

axis of symmetry: $x = 6$ $x = h$

Graph the vertex, focus, axis, and directrix of the parabola. Then make a table of values to graph the curve. The curve should be symmetric about the axis of symmetry.

x	y
0	6
4	14
8	14
12	6



GuidedPractice

Write each equation in standard form. Identify the vertex, focus, axis of symmetry, and directrix. Then graph each parabola.

3A. $x^2 - 4y + 3 = 7$

3B. $3y^2 + 6y + 15 = 12x$

2 Equations of Parabolas

Specific characteristics can be used to determine the equation of a parabola.

> **StudyTip**
> **Orientation** If the vertex and focus share a common x-coordinate, then the parabola opens up or down. If the vertex and focus share a common y-coordinate, then the parabola opens to the right or left.

Example 4 Write Equations Given Characteristics

Write an equation for and graph a parabola with the given characteristics.

a. focus $(3, -4)$ and vertex $(1, -4)$

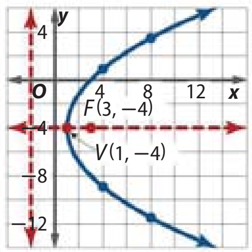
Because the focus and vertex share the same y-coordinate, the graph is horizontal. The focus is $(h + p, k)$, so the value of p is $3 - 1$ or 2. Because p is positive, the graph opens to the right.

Write the equation for the parabola in standard form using the values of h, p, and k.

$(y - k)^2 = 4p(x - h)$	Standard form
$[y - (-4)]^2 = 4(2)(x - 1)$	$k = -4$, $p = 2$, and $h = 1$
$(y + 4)^2 = 8(x - 1)$	Simplify.

The standard form of the equation is $(y + 4)^2 = 8(x - 1)$.

Graph the vertex and focus. Then graph the parabola.



b. vertex $(-2, 4)$, directrix $y = 1$

The directrix is a horizontal line, so the parabola opens vertically. Because the directrix lies below the vertex, the parabola opens up.

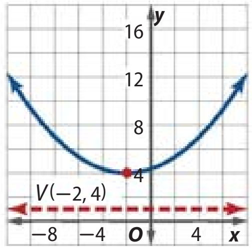
Use the equation of the directrix to find p.

$y = k - p$	Equation of directrix
$1 = 4 - p$	$y = 1$ and $k = 4$
$-3 = -p$	Subtract 4 from each side.
$3 = p$	Divide each side by -1.

Substitute the values for h, k, and p into the standard form equation for a parabola opening vertically.

$(x - h)^2 = 4p(y - k)$	Standard form
$[x - (-2)]^2 = 4(3)(y - 4)$	$h = -2$, $k = 4$, and $p = 3$
$(x + 2)^2 = 12(y - 4)$	Simplify.

Graph the parabola.



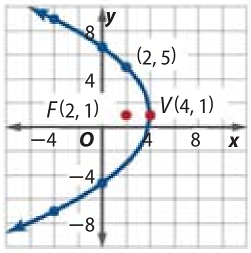
c. focus $(2, 1)$, opens left, contains $(2, 5)$

Because the parabola opens to the left, the vertex is $(2 - p, 1)$. Use the standard form of the equation of a horizontal parabola and the point $(2, 5)$ to find the equation.

$(y - k)^2 = 4p(x - h)$	Standard form
$(5 - 1)^2 = 4p[2 - (2 - p)]$	$h = 2 - p$, $k = 1$, $x = 2$, and $y = 5$
$16 = 4p(p)$	Simplify.
$4 = p^2$	Multiply; then divide each side by 4.
$\pm 2 = p$	Take the square root of each side.

Because the parabola opens to the left, the value of p must be negative. Therefore, $p = -2$. The vertex is $(4, 1)$, and the standard form of the equation is $(y - 1)^2 = -8(x - 4)$.

Graph the parabola.



GuidedPractice

Write an equation for and graph a parabola with the given characteristics.

4A. focus $(-6, 2)$, vertex $(-6, -1)$

4B. focus $(5, -2)$, vertex $(9, -2)$

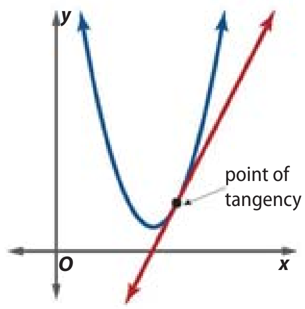
4C. focus $(-3, -4)$, opens down, contains $(5, -10)$

4D. focus $(-1, 5)$, opens right, contains $(8, -7)$

ReviewVocabulary

Tangent A line that is tangent to a circle intersects the circle in exactly one point. The point of intersection is called the *point of tangency.*

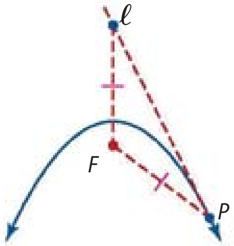
In calculus, you will often be asked to determine equations of lines that are tangent to curves. Equations of lines that are tangent to parabolas can be found without using calculus.



KeyConcept Line Tangent to a Parabola

A line ℓ that is tangent to a parabola at a point P forms an isosceles triangle such that:

- The segment from P to the focus forms one leg of the triangle.
- The segment along the axis of symmetry from the focus to another point on the tangent line forms the other leg.



Example 5 Find a Tangent Line at a Point

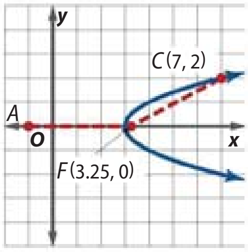
Write an equation for the line tangent to $x = y^2 + 3$ at $C(7, 2)$.

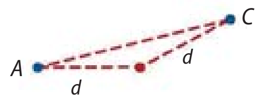
The graph opens horizontally. Determine the vertex and focus.

$x = y^2 + 3$ Original equation

$1(x - 3) = (y - 0)^2$ Write in standard form.

Because $4p = 1$, $p = 0.25$. The vertex is $(3, 0)$ and the focus is $(3.25, 0)$. As shown in the two figures, we need to determine d, the distance between the focus and the point of tangency, C.





This is one leg of the isosceles triangle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(7 - 3.25)^2 + (2 - 0)^2}$ $(x_1, y_1) = (3.25, 0)$ and $(x_2, y_2) = (7, 2)$

$= 4.25$ Simplify.

Use d to find A, the endpoint of the other leg of the isosceles triangle.
$A(3.25 - 4.25, 0)$ or $A(-1, 0)$

Points A and C both lie on the line tangent to the parabola. Find an equation of this line.

$m = \frac{2 - 0}{7 - (-1)}$ or $\frac{1}{4}$ Slope Formula

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 2 = \frac{1}{4}(x - 7)$ $m = \frac{1}{4}$, $y_1 = 2$, and $x_1 = 7$

$y - 2 = \frac{x}{4} - \frac{7}{4}$ Distributive Property

$y = \frac{x}{4} + \frac{1}{4}$ Add 2 to each side.

As shown in Figure 7.1.1, the equation for the line tangent to $x = y^2 + 3$ at $(7, 2)$ is $y = \frac{x}{4} + \frac{1}{4}$.

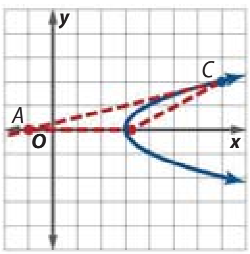


Figure 7.1.1

GuidedPractice

Write an equation for the line tangent to each parabola at each given point.

5A. $y = 4x^2 + 4$; $(-1, 8)$

5B. $x = 5 - \frac{y^2}{4}$; $(1, -4)$

StudyTip

Normals To Conics A *normal* to a conic at a point is the line perpendicular to the tangent line to the conic at that point. In Example 5, since the equation of the tangent line to the graph of $x = y^2 + 3$ at $(7, 2)$ is $y - 2 = \frac{1}{4}(x - 7)$, the equation of the normal line to the parabola at that same point is $y - 2 = -4(x - 7)$.

Exercises

= Step-by-Step Solutions begin on page R29.

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola. (Example 1)

1. $(x-3)^2 = 12(y-7)$

2. $(x+1)^2 = -12(y-6)$

3. $(y-4)^2 = 20(x+2)$

4. $-1(x+7) = (y+5)^2$

5. $(x+8)^2 = 8(y-3)$

6. $-40(x+4) = (y-9)^2$

7. $(y+5)^2 = 24(x-1)$

8. $2(y+12) = (x-6)^2$

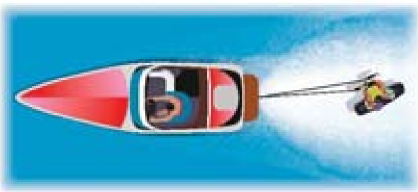
9. $-4(y+2) = (x+8)^2$

10. $10(x+11) = (y+3)^2$

11. SKATEBOARDING A group of high school students designing a half-pipe have decided that the ramps, or transitions, could be obtained by splitting a parabola in half. A parabolic cross section of the ramps can be modeled by $x^2 = 8(y-2)$, where the values of x and y are measured in feet. Where is the focus of the parabola in relation to the ground if the ground represents the directrix? (Example 2)

12. COMMUNICATION The cross section of a satellite television dish has a parabolic shape that focuses the satellite signals onto a receiver located at the focus of the parabola. The parabolic cross section can be modeled by $(x-6)^2 = 12(y-10)$, where the values of x and y are measured in inches. Where is the receiver located in relation to this particular cross section? (Example 2)

13. BOATING As a speed boat glides through the water, it creates a wake in the shape of a parabola. The vertex of this parabola meets with the stern of the boat. A swimmer on a wakeboard, attached by a piece of rope, is being pulled by the boat. When he is directly behind the boat, he is positioned at the focus of the parabola. The parabola formed by the wake can be modeled using $y^2 - 180x + 10y + 565 = 0$, where x and y are measured in feet. (Example 3)



a. Write the equation in standard form.

b. How long is the length of rope attaching the swimmer to the stern of the boat?

14. BASEBALL During Philadelphia Phillies baseball games, the team's mascot, The Phanatic, launches hot dogs into the stands. The launching device propels the hot dogs into the air at an initial velocity of 64 feet per second. A hot dog's distance y above ground after x seconds can be illustrated by $y = -16x^2 + 64x + 6$. (Example 3)

a. Write the equation in standard form.

b. What is the maximum height that a hot dog can reach?

Write each equation in standard form. Identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola. (Example 3)

15. $x^2 - 17 = 8y + 39$

16. $y^2 + 33 = -8x - 23$

17. $3x^2 + 72 = -72y$

18. $-12y + 10 = x^2 - 4x + 14$

19. $60x - 80 = 3y^2 + 100$

20. $-33 = x^2 - 12y - 6x$

21. $-72 = 2y^2 - 16y - 20x$

22. $y^2 + 21 = -20x - 6y - 68$

23. $x^2 - 18y + 12x = 126$

24. $-34 = 2x^2 + 20x + 8y$

25. LIGHTING Stadium lights at an athletic field need to reflect light at maximum intensity. The bulb should be placed at the focal point of the parabolic globe surrounding it. If the shape of the globe is given by $x^2 = 36y$, where x and y are in inches, how far from the vertex of the globe should the bulb be placed for maximum light? (Example 3)

Write an equation for and graph a parabola with the given focus F and vertex V. (Example 4)

26. $F(-9, -7), V(-9, -4)$

27. $F(2, -1), V(-4, -1)$

28. $F(-3, -2), V(1, -2)$

29. $F(-3, 4), V(-3, 2)$

30. $F(-2, -4), V(-2, -5)$

31. $F(-1, 4), V(7, 4)$

32. $F(14, -8), V(7, -8)$

33. $F(1, 3), V(1, 6)$

34. $F(-4, 9), V(-2, 9)$

35. $F(8, -3), V(8, -7)$

Write an equation for and graph each parabola with focus F and the given characteristics. (Example 4)

36. $F(3, 3)$; opens up; contains $(23, 18)$

37. $F(1, 2)$; opens down; contains $(7, 2)$

38. $F(11, 4)$; opens right; contains $(20, 16)$

39. $F(-4, 0)$; opens down; contains $(4, -15)$

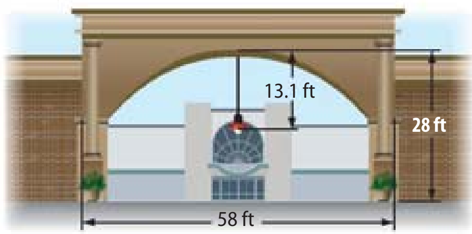
40. $F(1, 3)$; opens left; contains $(-14, 11)$

41. $F(-5, -9)$; opens right; contains $(10, -1)$

42. $F(-7, 6)$; opens left; contains $(-4, 10)$

43. $F(-5, -2)$; opens up; contains $(-13, -2)$

44. ARCHITECTURE The entrance to an open-air plaza has a parabolic arch above two columns. The light in the center is located at the focus of the parabola. (Example 4)



a. Write an equation that models the parabola.

b. Graph the equation.

Write an equation for the line tangent to each parabola at each given point. (Example 5)

45. $(x+7)^2 = -\frac{1}{2}(y-3)$, $(-5, -5)$

46. $y^2 = \frac{1}{5}(x-4)$, $(24, 2)$

47. $(x+6)^2 = 3(y-2)$, $(0, 14)$

48. $(x-3)^2 = y+4$, $(-1, 12)$

49 $-0.25(x-6)^2 = y-9$, $(10, 5)$

50. $-4x = (y+5)^2$, $(0, -5)$

Determine the orientation of each parabola.

51. directrix $y = 4$, $p = -2$

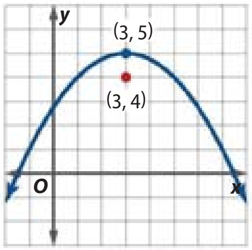
52. $y^2 = -8(x-6)$

53. vertex $(-5, 1)$, focus $(-5, 3)$

54. focus $(7, 10)$, directrix $x = 1$

Write an equation for each parabola.

55.



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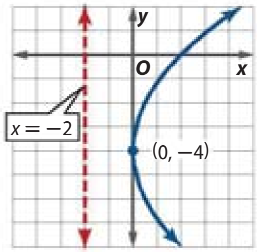

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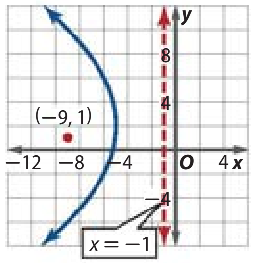

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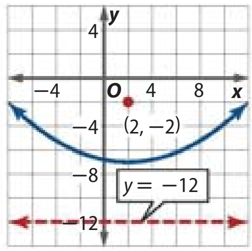
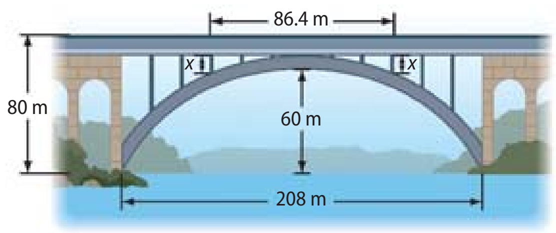


59. **BRIDGES** The arch of the railroad track bridge below is in the shape of a parabola. The two main support towers are 208 meters apart and 80 meters tall. The distance from the top of the parabola to the water below is 60 meters.



a. Write an equation that models the shape of the arch. Let the railroad track represent the x-axis.

b. Two vertical supports attached to the arch are equidistant from the center of the parabola as shown in the diagram. Find their lengths if they are 86.4 meters apart.

Write an equation for and graph a parabola with each set of characteristics.

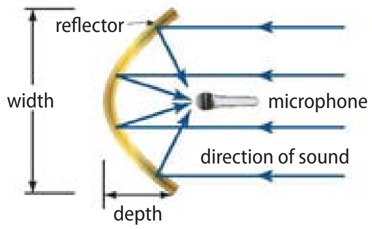
60. vertex $(1, 8)$, contains $(11, 13)$, opens vertically

61. vertex $(-6, 4)$, contains $(-10, 8)$, opens horizontally

62. opens vertically, passes through points $(-12, -14)$, $(0, -2)$, and $(6, -5)$

63. opens horizontally, passes through points $(-1, -1)$, $(5, 3)$, and $(15, 7)$

64. **SOUND** Parabolic reflectors with microphones located at the focus are used to capture sounds from a distance. The sound waves enter the reflector and are concentrated toward the microphone.



a. How far from the reflector should a microphone be placed if the reflector has a width of 3 feet and a depth of 1.25 feet?

b. Write an equation to model a different parabolic reflector that is 4 feet wide and 2 feet deep, if the vertex of the reflector is located at $(3, 5)$ and the parabola opens to the left.

c. Graph the equation. Specify the domain and range.

Determine the point of tangency for each equation and line.

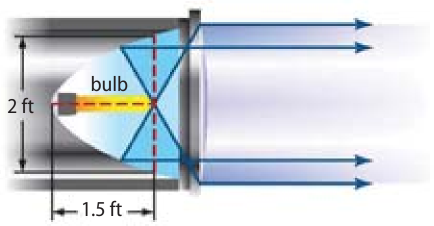
65. $(x+2)^2 = 2y$
$y = 4x$

66. $(y-8)^2 = 12x$
$y = x + 11$

67. $(y+3)^2 = -x + 4$
$y = -\frac{1}{4}x - 1$

68. $(x+5)^2 = -4(y+1)$
$y = 2x + 13$

69. **ILLUMINATION** In a searchlight, the bulb is placed at the focus of a parabolic mirror 1.5 feet from the vertex. This causes the light rays from the bulb to bounce off the mirror as parallel rays, thus providing a concentrated beam of light.

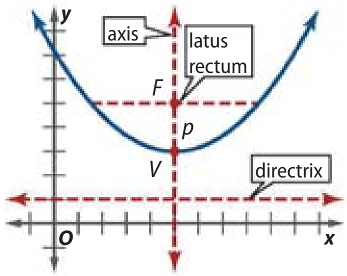


a. Write an equation for the parabola if the focal diameter of the bulb is 2 feet, as shown in the diagram.

b. Suppose the focal diameter is increased to 3 feet. If the depth of both searchlights is 3.5 feet, how much greater is the width of the opening of the larger light? Round to the nearest hundredth.

70. **PROOF** Use the standard form of the equation of a parabola to prove the general form of the equation.

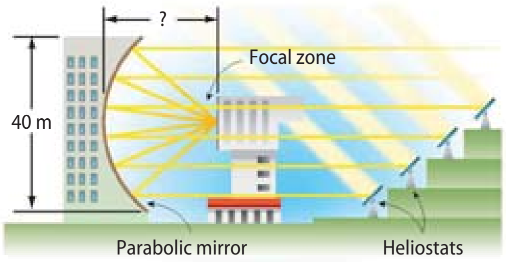
71. The **latus rectum** of a parabola is the line segment that passes through the focus, is perpendicular to the axis of the parabola, and has endpoints on the parabola. The length of the latus rectum is $|4p|$ units, where p is the distance from the vertex to the focus.



a. Write an equation for the parabola with vertex at $(-3, 2)$, axis $y = 2$, and latus rectum 8 units long.

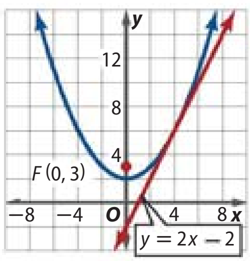
b. Prove that the endpoints of the latus rectum and point of intersection of the axis and directrix are the vertices of a right isosceles triangle.

72. **SOLAR ENERGY** A solar furnace in France's Eastern Pyrenees uses a parabolic mirror that is illuminated by sunlight reflected off a field of heliostats, which are devices that track and redirect sunlight. Experiments in solar research are performed in the *focal zone* part of a tower. If the parabolic mirror is 6.25 meters deep, how many meters in front of the parabola is the focal zone?



Write a possible equation for a parabola with focus *F* such that the line given is tangent to the parabola.

73.



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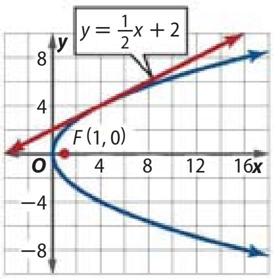

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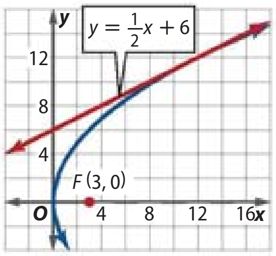

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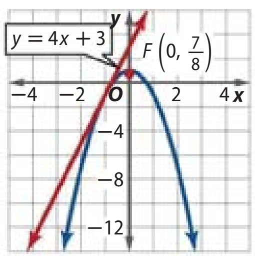


77. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate how the shape of a parabola changes as the position of the focus changes.

a. **GEOMETRIC** Find the distance between the vertex and the focus of each parabola.

i. $y^2 = 4(x - 2)$ **ii.** $y^2 = 8(x - 2)$ **iii.** $y^2 = 16(x - 2)$

b. **GRAPHICAL** Graph the parabolas in part **a** using a different color for each. Label each focus.

c. **VERBAL** Describe the relationship between a parabola's shape and the distance between its vertex and focus.

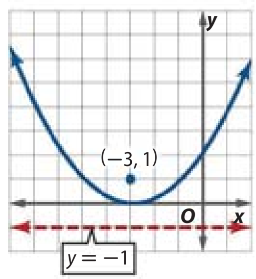
d. **ANALYTICAL** Write an equation for a parabola that has the same vertex as $(x + 1)^2 = 20(y + 7)$ but is narrower.

e. **ANALYTICAL** Make a conjecture about the graphs of $x^2 = -2(y + 1)$, $x^2 = -12(y + 1)$, and $x^2 = -5(y + 1)$. Check your conjecture by graphing the parabolas.

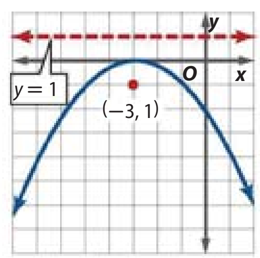
H.O.T. Problems Use Higher-Order Thinking Skills

78. **ERROR ANALYSIS** Abigail and Jaden are graphing $x^2 + 6x - 4y + 9 = 0$. Is either of them correct? Explain your reasoning.

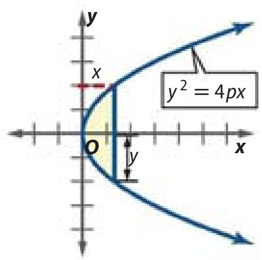
Abigail



Jaden



79 **CHALLENGE** The area of the *parabolic sector* shaded in the graph at the right is given by $A = \frac{4}{3}xy$. Find the equation of the parabola if the sector area is 2.4 square units, and the width of the sector is 3 units.



80. **REASONING** Which point on a parabola is closest to the focus? Explain your reasoning.

81. **REASONING** Without graphing, determine the quadrants in which the graph of $(y - 5)^2 = -8(x + 2)$ will have *no* points. Explain your reasoning.

82. **WRITING IN MATH** The *concavity* of a function's graph describes whether the graph opens upward (concave up) or downward (concave down). Explain how you can determine the concavity of a parabola given its focus and vertex.

83. **PREWRITE** Write a letter outlining and explaining the content you have learned in this lesson. Make an outline that addresses purpose, audience, a controlling idea, logical sequence, and time frame for completion.

Spiral Review

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints. (Lesson 6-5)

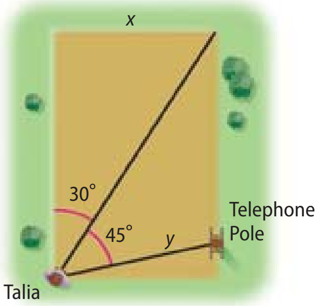
84. $x \leq 5$
$y \geq -2$
$y \leq x - 1$
$f(x, y) = x - 2y$

85. $y \geq -x + 2$
$2 \leq x \leq 7$
$y \leq \frac{1}{2}x + 5$
$f(x, y) = 8x + 3y$

86. $x \geq -3$
$y \geq 1$
$3x + y \leq 6$
$f(x, y) = 5x - 2y$

87. Find the partial fraction decomposition of $\frac{2y + 5}{y^2 + 3y + 2}$. (Lesson 6-4)

88. SURVEYING Talia is surveying a rectangular lot for a new office building. She measures the angle between one side of the lot and the line from her position to the opposite corner of the lot as 30°. She then measures the angle between that line and the line to a telephone pole on the edge of the lot as 45°. If Talia is 100 yards from the opposite corner of the lot, how far is she from the telephone pole? (Lesson 5-4)



Find the value of each expression using the given information. (Lesson 5-1)

89. cot θ and csc θ;
$\tan \theta = \frac{6}{7}$, $\sec \theta > 0$

90. cos θ and tan θ;
$\csc \theta = -2$, $\cos \theta < 0$

Locate the vertical asymptotes, and sketch the graph of each function. (Lesson 4-5)

91. $y = \tan x + 4$

92. $y = \sec x + 2$

93. $y = \csc x - \frac{3}{4}$

Evaluate each expression. (Lesson 3-2)

94. $\log_{16} 4$

95. $\log_4 16^x$

96. $\log_3 27^x$

Graph each function. (Lesson 2-2)

97. $f(x) = x^3 - x^2 - 4x + 4$

98. $g(x) = x^4 - 7x^2 + x + 5$

99. $h(x) = x^4 - 4x^2 + 2$

Skills Review for Standardized Tests

100. REVIEW What is the solution set for $3(4x + 1)^2 = 48$?

A $\left\{\frac{5}{4}, -\frac{3}{4}\right\}$

B $\left\{-\frac{5}{4}, \frac{3}{4}\right\}$

C $\left\{\frac{15}{4}, -\frac{17}{4}\right\}$

D $\left\{\frac{1}{3}, -\frac{4}{3}\right\}$

E $\left\{\frac{7}{4}, -\frac{9}{4}\right\}$

101. SAT/ACT If x is a positive number, then $\frac{x^{\frac{3}{2}} \cdot x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = ?$

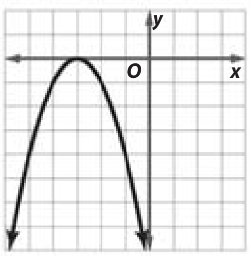
F $x^{-\frac{1}{4}}$

G $\sqrt{x^3}$

H $x^{\frac{3}{4}}$

J $\sqrt{x^5}$

102. Which is the parent function of the graph shown below?



A $y = x$

B $y = \sqrt{x}$

C $y = |x|$

D $y = x^2$

103. REVIEW What are the x-intercepts of the graph of $y = -2x^2 - 5x + 12$?

F $-\frac{3}{2}, 4$

G $-4, \frac{3}{2}$

H $-2, \frac{1}{2}$

J $-\frac{1}{2}, 2$

LESSON 7-2

Ellipses and Circles

Then

- You analyzed and graphed parabolas. (Lesson 7-1)

Now

- **1** Analyze and graph equations of ellipses and circles.
- **2** Use equations to identify ellipses and circles.

Why?

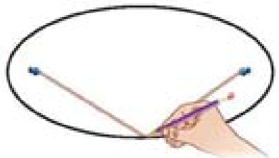
- Due to acceleration and inertia, the safest shape for a roller coaster loop can be approximated using an ellipse rather than a circle. The elliptical shape helps to minimize force on the riders' heads and necks.

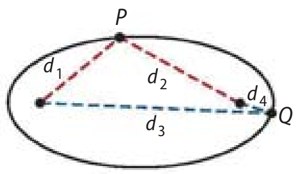
NewVocabulary

ellipse
foci
major axis
center
minor axis
vertices
co-vertices
eccentricity

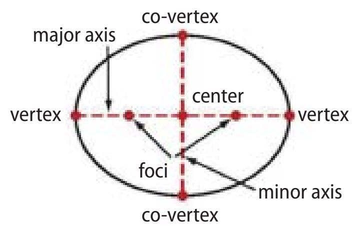
1 Analyze and Graph Ellipses and Circles

An **ellipse** is the locus of points in a plane such that the sum of the distances from two fixed points, called **foci**, is constant. To visualize this concept, consider a length of string tacked at the foci of an ellipse. You can draw an ellipse by using a pencil to trace a curve with the string pulled tight. For any two points on the ellipse, the sum of the lengths of the segments to each focus is constant. In other words, $d_1 + d_2 = d_3 + d_4$, and this sum is constant.



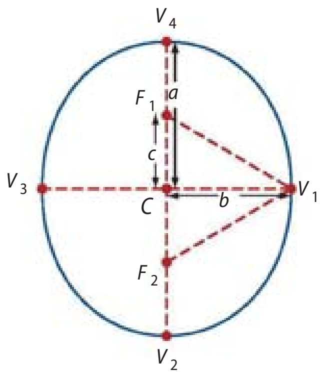


The segment that contains the foci of an ellipse and has endpoints on the ellipse is called the **major axis,** and the midpoint of the major axis is the **center**. The segment through the center with endpoints on the ellipse and perpendicular to the major axis is the **minor axis**. The two endpoints of the major axis are the **vertices,** and the endpoints of the minor axis are the **co-vertices**.



The center of the ellipse is the midpoint of both the major and minor axes. So, the segments from the center to each vertex are congruent, and the segments from the center to each co-vertex are congruent. The distance from each vertex to the center is a units, and the distance from the center to each co-vertex is b units. The distance from the center to each focus is c units.

Consider $\overline{V_1F_1}$ and $\overline{V_1F_2}$. Because $\triangle F_1V_1C \cong \triangle F_2V_1C$ by the Leg-Leg Theorem, $V_1F_1 = V_1F_2$. We can use the definition of an ellipse to find the lengths V_1F_1 and V_1F_2 in terms of the lengths given.

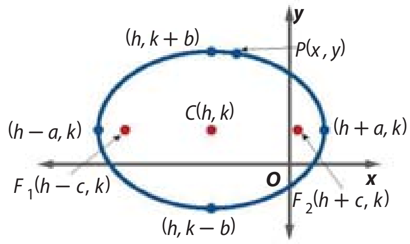


$V_1F_1 + V_1F_2 = V_2F_1 + V_2F_2$	Definition of an ellipse
$V_1F_1 + V_1F_2 = V_2F_1 + V_4F_1$	$V_2F_2 = V_4F_1$
$V_1F_1 + V_1F_2 = V_2V_4$	$V_2F_1 + V_4F_1 = V_2V_4$
$V_1F_1 + V_1F_2 = 2a$	$V_2V_4 = 2a$
$V_1F_1 + V_1F_1 = 2a$	$V_1F_1 = V_1F_2$
$2(V_1F_1) = 2a$	Simplify.
$V_1F_1 = a$	Divide.

Because $V_1F_1 = a$ and $\triangle F_1V_1C$ is a right triangle, $b^2 + c^2 = a^2$ by the Pythagorean Theorem.

Rommel/Masterfile

Let $P(x, y)$ be any point on the ellipse with center $C(h, k)$. The coordinates of the foci, vertices, and co-vertices are shown at the right. By the definition of an ellipse, the sum of distances from any point on the ellipse to the foci is constant. Thus, $PF_1 + PF_2 = 2a$.

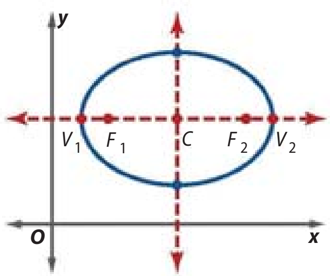


$$
\begin{aligned}
PF_1 + PF_2 &= 2a && \text{Definition of ellipse}\\
\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2} &= 2a && \text{Distance Formula}\\
\sqrt{(x-h+c)^2+(y-k)^2} &= 2a-\sqrt{(x-h-c)^2+(y-k)^2} && \text{Distributive and Subtraction Properties}\\
\sqrt{[(x-h)+c]^2+(y-k)^2} &= 2a-\sqrt{[(x-h)-c]^2+(y-k)^2} && \text{Associative Property}\\
[(x-h)+c]^2+(y-k)^2 &= 4a^2-4a\sqrt{[(x-h)-c]^2+(y-k)^2}+[(x-h)-c]^2+(y-k)^2 &&\\
(x-h)^2+2c(x-h)+c^2+(y-k)^2 &= 4a^2-4a\sqrt{[(x-h)-c]^2+(y-k)^2}+(x-h)^2-2c(x-h)+c^2+(y-k)^2 &&\\
4a\sqrt{[(x-h)-c]^2+(y-k)^2} &= 4a^2-4c(x-h) && \text{Subtraction and Addition Properties}\\
a\sqrt{[(x-h)-c]^2+(y-k)^2} &= a^2-c(x-h) && \text{Divide each side by 4.}\\
a^2[(x-h)^2-2c(x-h)+c^2+(y-k)^2] &= a^4-2a^2c(x-h)+c^2(x-h)^2 && \text{Square each side.}\\
a^2(x-h)^2-2a^2c(x-h)+a^2c^2+a^2(y-k)^2 &= a^4-2a^2c(x-h)+c^2(x-h)^2 && \text{Distributive Property}\\
a^2(x-h)^2-c^2(x-h)^2+a^2(y-k)^2 &= a^4-a^2c^2 && \text{Subtraction Property}\\
(a^2-c^2)(x-h)^2+a^2(y-k)^2 &= a^2(a^2-c^2) && \text{Factor.}\\
b^2(x-h)^2+a^2(y-k)^2 &= a^2b^2 && a^2-c^2=b^2\\
\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} &= 1 && \text{Divide each side by } a^2b^2.
\end{aligned}
$$

The standard form for an ellipse centered at (h, k), where $a > b$, is given below.

KeyConcept Standard Forms of Equations for Ellipses

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$



Orientation: horizontal major axis

Center: (h, k)

Foci: $(h \pm c, k)$

Vertices: $(h \pm a, k)$

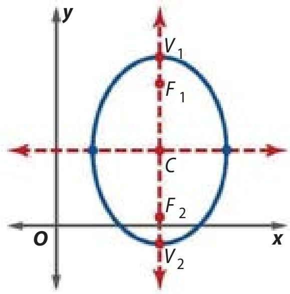
Co-vertices: $(h, k \pm b)$

Major axis: $y = k$

Minor axis: $x = h$

a, b, c relationship: $c^2 = a^2 - b^2$ or $c = \sqrt{a^2 - b^2}$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$



Orientation: vertical major axis

Center: (h, k)

Foci: $(h, k \pm c)$

Vertices: $(h, k \pm a)$

Co-vertices: $(h \pm b, k)$

Major axis: $x = h$

Minor axis: $y = k$

a, b, c relationship: $c^2 = a^2 - b^2$ or $c = \sqrt{a^2 - b^2}$

Example 1 Graph Ellipses

Graph the ellipse given by each equation.

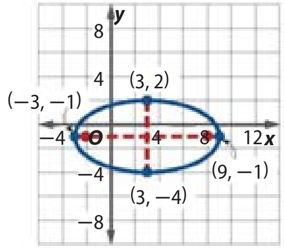
a. $\dfrac{(x-3)^2}{36}+\dfrac{(y+1)^2}{9}=1$

The equation is in standard form with $h = 3$, $k = -1$, $a = \sqrt{36}$ or 6, $b = \sqrt{9}$ or 3, and $c = \sqrt{36-9}$ or $3\sqrt{3}$. Use these values to determine the characteristics of the ellipse.

orientation:	horizontal	When the equation is in standard form, the x^2-term contains a^2
center:	$(3, -1)$	(h, k)
foci:	$(3 \pm 3\sqrt{3}, -1)$	$(h \pm c, k)$
vertices:	$(-3, -1)$ and $(9, -1)$	$(h \pm a, k)$
co-vertices:	$(3, -4)$ and $(3, 2)$	$(h, k \pm b)$
major axis:	$y = -1$	$y = k$
minor axis:	$x = 3$	$x = h$

Graph the center, vertices, foci, and axes. Then make a table of values to sketch the ellipse.

x	y
0	1.60, −3.60
6	1.60, −3.60



b. $4x^2 + y^2 - 24x + 4y + 24 = 0$

First, write the equation in standard form.

$4x^2 + y^2 - 24x + 4y + 24 = 0$ — Original equation

$(4x^2 - 24x) + (y^2 + 4y) = -24$ — Isolate and group like terms.

$4(x^2 - 6x) + (y^2 + 4y) = -24$ — Factor.

$4(x^2 - 6x + 9) + (y^2 + 4y + 4) = -24 + 4(9) + 4$ — Complete the squares.

$4(x-3)^2 + (y+2)^2 = 16$ — Factor and simplify.

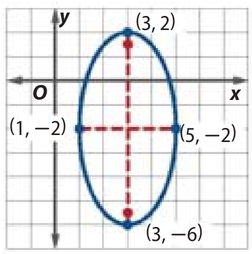
$\dfrac{(x-3)^2}{4}+\dfrac{(y+2)^2}{16}=1$ — Divide each side by 16.

The equation is in standard form with $h = 3$, $k = -2$, $a = \sqrt{16}$ or 4, $b = \sqrt{4}$ or 2, and $c = \sqrt{16-4}$ or $2\sqrt{3}$. Use these values to determine the characteristics of the ellipse.

orientation:	vertical	When the equation is in standard form, the y^2-term contains a^2.
center:	$(3, -2)$	(h, k)
foci:	$(3, -2 \pm 2\sqrt{3})$	$(h, k \pm c)$
vertices:	$(3, -6)$ and $(3, 2)$	$(h, k \pm a)$
co-vertices:	$(5, -2)$ and $(1, -2)$	$(h \pm b, k)$
major axis:	$x = 3$	$x = h$
minor axis:	$y = -2$	$y = k$

Graph the center, vertices, foci, and axes. Then make a table of values to sketch the ellipse.

x	y
2	1.46, −5.46
4	1.46, −5.46



Study Tip

Orientation If the *y*-coordinate is the same for both vertices of an ellipse, then the major axis is horizontal. If the *x*-coordinate is the same for both vertices of an ellipse, then the major axis is vertical.

Guided Practice

1A. $\dfrac{(x-6)^2}{9}+\dfrac{(y+3)^2}{16}=1$

1B. $x^2 + 4y^2 + 4x - 40y + 103 = 0$

Example 2 Write Equations Given Characteristics

Write an equation for an ellipse with each set of characteristics.

a. major axis from (−6, 2) to (−6, −8); minor axis from (−3, −3) to (−9, −3)

Use the major and minor axes to determine a and b.

Half the length of major axis

$$a = \frac{2-(-8)}{2} \text{ or } 5$$

Half the length of minor axis

$$b = \frac{-3-(-9)}{2} \text{ or } 3$$

The center of the ellipse is at the midpoint of the major axis.

$$(h, k) = \left(\frac{-6+(-6)}{2}, \frac{2+(-8)}{2}\right) \quad \text{Midpoint Formula}$$

$$= (-6, -3) \quad \text{Simplify.}$$

The x-coordinates are the same for both endpoints of the major axis, so the major axis is vertical and the value of a belongs with the y^2-term. An equation for the ellipse is $\frac{(y+3)^2}{25} + \frac{(x+6)^2}{9} = 1$. The graph of the ellipse is shown in Figure 7.2.1.

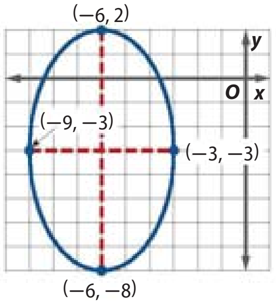


Figure 7.2.1

b. vertices at (−4, 4) and (6, 4); foci at (−2, 4) and (4, 4)

The length of the major axis, $2a$, is the distance between the vertices.

$$2a = \sqrt{(-4-6)^2 + (4-4)^2} \quad \text{Distance Formula}$$

$$a = 5 \quad \text{Solve for } a.$$

$2c$ represents the distance between the foci.

$$2c = \sqrt{(-2-4)^2 + (4-4)^2} \quad \text{Distance Formula}$$

$$c = 3 \quad \text{Solve for } c.$$

Find the value of b.

$$c^2 = a^2 - b^2 \quad \text{Equation relating } a, b, \text{ and } c$$

$$3^2 = 5^2 - b^2 \quad a = 5 \text{ and } c = 3$$

$$b = 4 \quad \text{Solve for } b.$$

The vertices are equidistant from the center.

$$(h, k) = \left(\frac{-4+6}{2}, \frac{4+4}{2}\right) \quad \text{Midpoint Formula}$$

$$= (1, 4) \quad \text{Simplify.}$$

The y-coordinates are the same for both endpoints of the major axis, so the major axis is horizontal and the value of a belongs with the x^2-term. An equation for the ellipse is $\frac{(x-1)^2}{25} + \frac{(y-4)^2}{16} = 1$. The graph of the ellipse is shown in Figure 7.2.2.

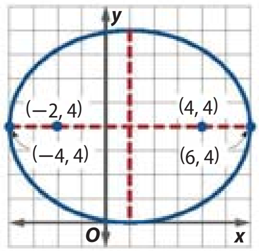


Figure 7.2.2

GuidedPractice

2A. foci at (19, 3) and (−7, 3); length of major axis equals 30

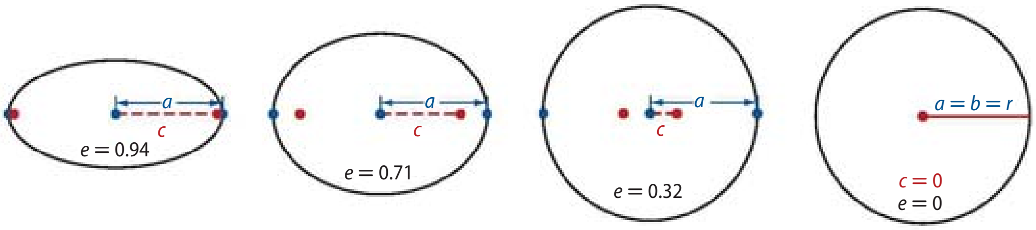
2B. vertices at (−2, −4) and (−2, 8); length of minor axis equals 10

The **eccentricity** of an ellipse is the ratio of c to a. This value will always be between 0 and 1 and will determine how "circular" or "stretched" the ellipse will be.

KeyConcept Eccentricity

For any ellipse, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, where $c^2 = a^2 - b^2$, the eccentricity $e = \frac{c}{a}$.

The value c represents the distance between one of the foci and the center of the ellipse. As the foci are moved closer together, c and e both approach 0. When the eccentricity reaches 0, the ellipse is a circle and both a and b are equal to the radius of the circle.



Example 3 Determine the Eccentricity of an Ellipse

Determine the eccentricity of the ellipse given by $\frac{(x-6)^2}{100} + \frac{(y+1)^2}{9} = 1$.

First, determine the value of c.

$c^2 = a^2 - b^2$ — Equation relating a, b, and c

$c^2 = 100 - 9$ — $a^2 = 100$ and $b^2 = 9$

$c = \sqrt{91}$ — Solve for c.

Use the values of c and a to find the eccentricity.

$e = \frac{c}{a}$ — Eccentricity equation

$e = \frac{\sqrt{91}}{10}$ or about 0.95 — $a = 10$ and $c = \sqrt{91}$

The eccentricity of the ellipse is about 0.95, so the ellipse will appear stretched, as shown in Figure 7.2.3.

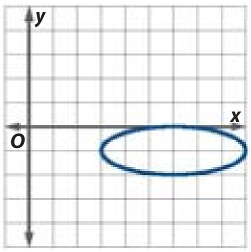


Figure 7.2.3

GuidedPractice

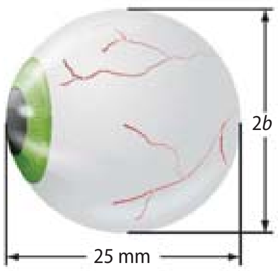
Determine the eccentricity of the ellipse given by each equation.

3A. $\frac{x^2}{18} + \frac{(y+8)^2}{48} = 1$

3B. $\frac{(x-4)^2}{19} + \frac{(y+7)^2}{17} = 1$

Real-World Example 4 Use Eccentricity

OPTICS **The shape of an eye can be modeled by a prolate, or three-dimensional, ellipse. The eccentricity of the center cross-section for an eye with normal vision is about 0.28. If a normal eye is about 25 millimeters deep, what is the approximate height of the eye?**



Use the eccentricity to determine the value of c.

$e = \frac{c}{a}$ — Definition of eccentricity

$0.28 = \frac{c}{12.5}$ — $e = 0.28$ and $a = 12.5$

$c = 3.5$ — Solve for c.

Use the values of c and a to determine b.

$c^2 = a^2 - b^2$ — Equation relating a, b, and c

$3.5^2 = 12.5^2 - b^2$ — $c = 3.5$ and $a = 12.5$

$b = 12$ — Solve for b.

Because the value of b is 12, the height of the eye is $2b$ or 24 millimeters.

GuidedPractice

4. The eccentricity of a nearsighted eye is 0.39. If the depth of the eye is 25 millimeters, what is the height of the eye?

Real-WorldCareer

Ophthalmic Technician Ophthalmic technicians work with ophthalmologists to care for patients with eye disease or injury. They perform exams such as visual status and assist in surgical settings. They must complete a one-year training program in addition to a high school diploma or GED.

2 Identify Conic Sections

The equation of a circle can be derived using the eccentricity of an ellipse.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Equation of an ellipse with center at (0, 0)

$\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$ $a = b$ when $e = 0$

$x^2 + y^2 = a^2$ Multiply each side by a^2.

$x^2 + y^2 = r^2$ a is the radius of the circle.

KeyConcept Standard Form of Equations for Circles

The standard form of an equation for a circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

If you are given the equation for a conic section, you can determine what type of conic is represented using the characteristics of the equation.

Example 5 Determine Types of Conics

Write each equation in standard form. Identify the related conic.

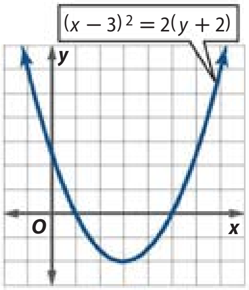


Figure 7.2.4

a. $x^2 - 6x - 2y + 5 = 0$

$x^2 - 6x - 2y + 5 = 0$	Original equation
$(x^2 - 6x) - 2y = -5$	Isolate and group like terms.
$(x^2 - 6x + 9) - 2y = -5 + 9$	Complete the square.
$(x - 3)^2 - 2y = 4$	Factor and simplify.
$(x - 3)^2 = 2y + 4$	Add $2y$ to each side.
$(x - 3)^2 = 2(y + 2)$	Factor.

Because only one term is squared, the graph is a parabola with vertex $(3, -2)$, as in Figure 7.2.4.

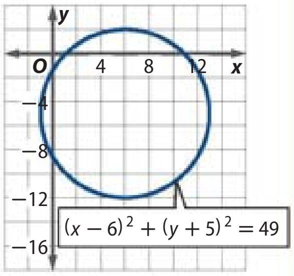


Figure 7.2.5

b. $x^2 + y^2 - 12x + 10y + 12 = 0$

$x^2 + y^2 - 12x + 10y + 12 = 0$	Original equation
$(x^2 - 12x) + (y^2 + 10y) = -12$	Isolate and group like terms.
$(x^2 - 12x + 36) + (y^2 + 10y + 25) = -12 + 36 + 25$	Complete the squares.
$(x - 6)^2 + (y + 5)^2 = 49$	Factor and simplify.

Because the equation is of the form $(x - h)^2 + (y - k)^2 = r^2$, the graph is a circle with center $(6, -5)$ and radius 7, as in Figure 7.2.5.

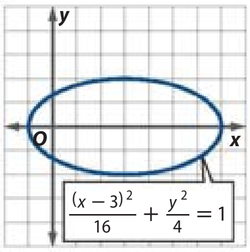


Figure 7.2.6

c. $x^2 + 4y^2 - 6x - 7 = 0$

$x^2 + 4y^2 - 6x - 7 = 0$	Original equation
$(x^2 - 6x) + 4y^2 = 7$	Isolate and group like terms.
$(x^2 - 6x + 9) + 4y^2 = 7 + 9$	Complete the square.
$(x - 3)^2 + 4y^2 = 16$	Factor and simplify.
$\dfrac{(x - 3)^2}{16} + \dfrac{y^2}{4} = 1$	Divide each side by 16.

Because the equation is of the form $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$, the graph is an ellipse with center $(3, 0)$, as in Figure 7.2.6.

GuidedPractice

5A. $y^2 - 3x + 6y + 12 = 0$

5B. $4x^2 + 4y^2 - 24x + 32y + 36 = 0$

5C. $4x^2 + 3y^2 + 36y + 60 = 0$

Exercises

= Step-by-Step Solutions begin on page R29.

Graph the ellipse given by each equation. (Example 1)

1. $\frac{(x+2)^2}{9} + \frac{y^2}{49} = 1$
2. $\frac{(x+4)^2}{9} + \frac{(y+3)^2}{4} = 1$
3. $x^2 + 9y^2 - 14x + 36y + 49 = 0$
4. $4x^2 + y^2 - 64x - 12y + 276 = 0$
5. $9x^2 + y^2 + 126x + 2y + 433 = 0$
6. $x^2 + 25y^2 - 12x - 100y + 111 = 0$

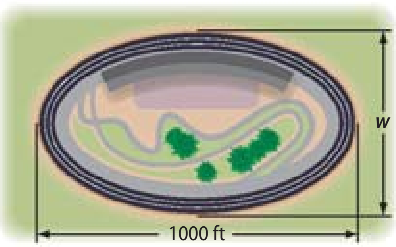
Write an equation for the ellipse with each set of characteristics. (Example 2)

7. vertices $(-7, -3)$, $(13, -3)$; foci $(-5, -3)$, $(11, -3)$
8. vertices $(4, 3)$, $(4, -9)$; length of minor axis is 8
9. vertices $(7, 2)$, $(-3, 2)$; foci $(6, 2)$, $(-2, 2)$
10. major axis $(-13, 2)$ to $(1, 2)$; minor axis $(-6, 4)$ to $(-6, 0)$
11. foci $(-6, 9)$, $(-6, -3)$; length of major axis is 20
12. co-vertices $(-13, 7)$, $(-3, 7)$; length of major axis is 16
13. foci $(-10, 8)$, $(14, 8)$; length of major axis is 30

Determine the eccentricity of the ellipse given by each equation. (Example 3)

14. $\frac{(x+5)^2}{72} + \frac{(y-3)^2}{54} = 1$
15. $\frac{(x+6)^2}{40} + \frac{(y-2)^2}{12} = 1$
16. $\frac{(x-8)^2}{14} + \frac{(y+3)^2}{57} = 1$
17. $\frac{(x+8)^2}{27} + \frac{(y-7)^2}{33} = 1$
18. $\frac{(x-1)^2}{12} + \frac{(y+2)^2}{9} = 1$
19. $\frac{(x-11)^2}{17} + \frac{(y+15)^2}{23} = 1$
20. $\frac{x^2}{38} + \frac{(y-12)^2}{13} = 1$
21. $\frac{(x+9)^2}{10} + \frac{(y+11)^2}{8} = 1$

22. **RACING** The design of an elliptical racetrack with an eccentricity of 0.75 is shown. (Example 4)



a. What is the maximum width w of the track?

b. Write an equation for the ellipse if the origin x is located at the center of the racetrack.

23. **CARPENTRY** A carpenter has been hired to construct a sign for a pet grooming business. The plans for the sign call for an elliptical shape with an eccentricity of 0.60 and a length of 36 inches. (Example 4)

a. What is the maximum height of the sign?

b. Write an equation for the ellipse if the origin is located at the center of the sign.

Write each equation in standard form. Identify the related conic. (Example 5)

24. $x^2 + y^2 + 6x - 4y - 3 = 0$
25. $4x^2 + 8y^2 - 8x + 48y + 44 = 0$
26. $x^2 - 8x - 8y - 40 = 0$
27. $y^2 - 12x + 18y + 153 = 0$
28. $x^2 + y^2 - 8x - 6y - 39 = 0$
29. $3x^2 + y^2 - 42x + 4y + 142 = 0$
30. $5x^2 + 2y^2 + 30x - 16y + 27 = 0$
31. $2x^2 + 7y^2 + 24x + 84y + 310 = 0$

32. **HISTORY** The United States Capitol has a room with an elliptical ceiling. This type of room is called a *whispering gallery* because sound that is projected from one focus of an ellipse reflects off the ceiling and back to the other focus. The room in the Capitol is 96 feet in length, 45 feet wide, and has a ceiling that is 23 feet high.

a. Write an equation modeling the shape of the room. Assume that it is centered at the origin and that the major axis is horizontal.

b. Find the location of the two foci.

c. How far from one focus would one have to stand to be able to hear the sound reflecting from the other focus?

Write an equation for a circle that satisfies each set of conditions. Then graph the circle.

33. center at $(3, 0)$, radius 2
34. center at $(-1, 7)$, diameter 6
35. center at $(-4, -3)$, tangent to $y = 3$
36. center at $(2, 0)$, endpoints of diameter at $(-5, 0)$ and $(9, 0)$

37. **FORMULA** Derive the general form of the equation for an ellipse with a vertical major axis centered at the origin.

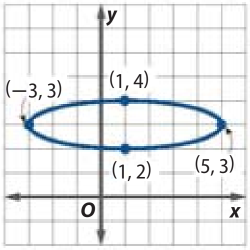
38. **MEDICAL TECHNOLOGY** Indoor Positioning Systems (IPS) use ultrasound waves to detect tags that are linked to digital files containing information regarding a person or item being monitored. Hospitals often use IPS to detect the location of moveable equipment and patients.

a. If the tracking system receiver must be centrally located for optimal functioning, where should a receiver be situated in a hospital complex that is 800 meters by 942 meters?

b. Write an equation that models the sonar range of the IPS.

Write an equation for each ellipse.

39.



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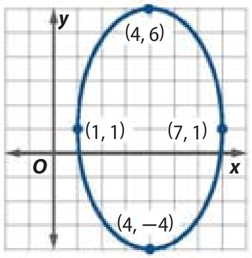
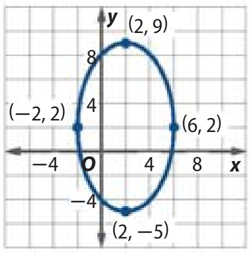
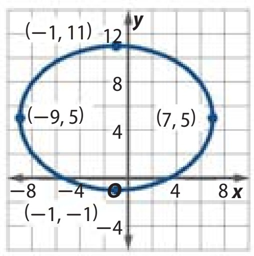


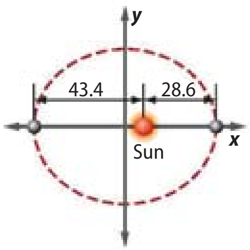
41.



42.



43. **PLANETARY MOTION** Each of the planets in the solar system move around the Sun in an elliptical orbit, where the Sun is one focus of the ellipse. Mercury is 43.4 million miles from the Sun at its farthest point and 28.6 million miles at its closest, as shown below. The diameter of the Sun is 870,000 miles.



a. Find the length of the minor axis.

b. Find the eccentricity of the elliptical orbit.

Find the center, foci, and vertices of each ellipse.

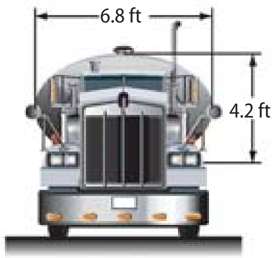
44. $\frac{(x+5)^2}{16} + \frac{y^2}{7} = 1$

45. $\frac{x^2}{100} + \frac{(y+6)^2}{25} = 1$

46. $9y^2 - 18y + 25x^2 + 100x - 116 = 0$

47. $65x^2 + 16y^2 + 130x - 975 = 0$

48. **TRUCKS** Elliptical tanker trucks like the one shown are often used to transport liquids because they are more stable than circular tanks and the movement of the fluid is minimized.



a. Draw and label the elliptical cross-section of the tank on a coordinate plane.

b. Write an equation to represent the elliptical shape of the tank.

c. Find the eccentricity of the ellipse.

Write the standard form of the equation for each ellipse.

49. The vertices are at $(-10, 0)$ and $(10, 0)$, and the eccentricity e is $\frac{3}{5}$.

50. The co-vertices are at $(0, 1)$ and $(6, 1)$, and the eccentricity e is $\frac{4}{5}$.

51. The center is at $(2, -4)$, one focus is at $(2, -4 + 2\sqrt{5})$, and the eccentricity e is $\frac{\sqrt{5}}{3}$.

52. **ROLLER COASTERS** The shape of a roller coaster loop in an amusement park can be modeled by

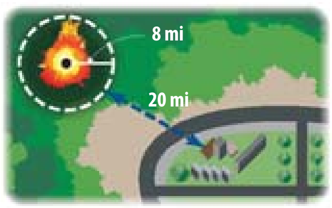
$$\frac{y^2}{3306.25} + \frac{x^2}{2025} = 1.$$

a. What is the width of the loop along the horizontal axis?

b. Determine the height of the roller coaster from the ground when it reaches the top of the loop, if the lower rail is 20 feet from ground level.

c. Find the eccentricity of the ellipse.

53. **FOREST FIRES** The radius of a forest fire is expanding at a rate of 4 miles per day. The current state of the fire is shown below, where a city is located 20 miles southeast of the fire.

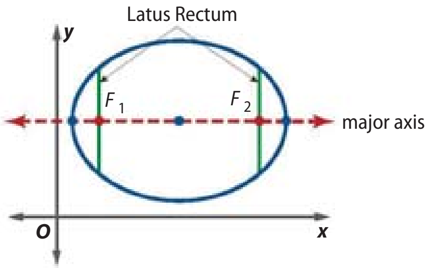


a. Write the equation of the circle at the current time and the equation of the circle at the time the fire reaches the city.

b. Graph both circles.

c. If the fire continues to spread at the same rate, how many days will it take to reach the city?

54. The *latus rectum* of an ellipse is a line segment that passes through a focus, is perpendicular to the major axis of the ellipse, and has endpoints on the ellipse. The length of each latus rectum is $\frac{2b^2}{a}$ units, where a is half the length of the major axis and b is half the length of the minor axis.



Write the equation of a horizontal ellipse with center at (3, 2), major axis is 16 units long, and latus rectum 12 units long.

Find the coordinates of points where a line intersects a circle.

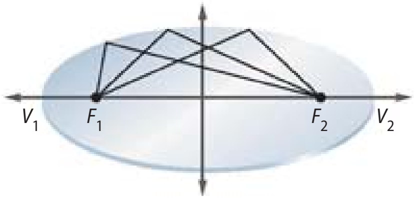
55. $y = x - 8$, $(x - 7)^2 + (y + 5)^2 = 16$

56. $y = x + 9$, $(x - 3)^2 + (y + 5)^2 = 169$

57. $y = -x + 1$, $(x - 5)^2 + (y - 2)^2 = 50$

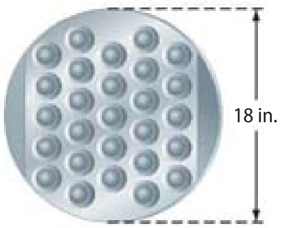
58. $y = -\frac{1}{3}x - 3$, $(x + 3)^2 + (y - 3)^2 = 25$

59. REFLECTION *Silvering* is the process of coating glass with a reflective substance. The interior of an ellipse can be silvered to produce a mirror with rays that originate at the ellipse's focus and then reflect to the other focus as shown.



If the segment V_1F_1 is 2 cm long and the eccentricity of the mirror is 0.5, find the equation of the ellipse in standard form.

60. CHEMISTRY Distillation columns are used to separate chemical substances based on the differences in their rates of evaporation. The columns may contain plates with bubble caps or small circular openings.



a. Write an equation for the plate shown, assuming that the center is at $(-4, -1)$.

b. What is the surface area of the plate not covered by bubble caps if each cap is 2 inches in diameter?

61. GEOMETRY The graphs of $x - 5y = -3$, $2x + 3y = 7$, and $4x - 7y = 27$ contain the sides of a triangle. Write the equation of a circle that circumscribes the triangle.

Write the standard form of the equation of a circle that passes through each set of points. Then identify the center and radius of the circle.

62. (2, 3), (8, 3), (5, 6)

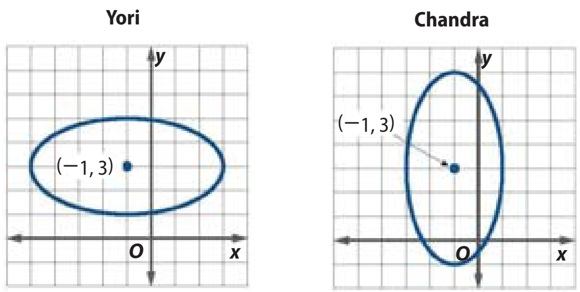
63. (1, −11), (−3, −7), (5, −7)

64. (0, 9), (0, 3), (−3, 6)

65. (7, 4), (−1, 12), (−9, 4)

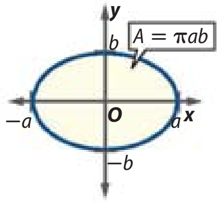
H.O.T. Problems Use Higher-Order Thinking Skills

66. ERROR ANALYSIS Yori and Chandra are graphing an ellipse that has a center at (−1, 3), a major axis of length 8, and a minor axis of length 4. Is either of them correct? Explain your reasoning.



67. REASONING Determine whether an ellipse represented by $\frac{x^2}{p} + \frac{y^2}{p + r} = 1$, where $r > 0$, will have the same foci as the ellipse represented by $\frac{x^2}{p + r} + \frac{y^2}{p} = 1$. Explain your reasoning.

CHALLENGE The area A of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $A = \pi ab$. Write an equation of an ellipse with each of the following characteristics.



68. $b + a = 12$, $A = 35\pi$

69 $a - b = 5$, $A = 24\pi$

70. WRITING IN MATH Explain how to find the foci and vertices of an ellipse if you are given the standard form of the equation.

71. REASONING Is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ symmetric with respect to the origin? Explain your reasoning.

72. OPEN ENDED If the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, where $h > 0$ and $k < 0$, what is the domain of the circle? Verify your answer with an example, both algebraically and graphically.

73. WRITING IN MATH Explain why an ellipse becomes circular as the value of b approaches the value of a.

Spiral Review

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola. (Lesson 7-1)

74. $y = 3x^2 - 24x + 50$

75. $y = -2x^2 + 5x - 10$

76. $x = 5y^2 - 10y + 9$

77. MANUFACTURING A toy company is introducing two new dolls to its customers: My First Baby, which talks, laughs, and cries, and My Real Baby, which uses a bottle and crawls. In one hour, the company can produce 8 First Babies or 20 Real Babies. Because of the demand, the company must produce at least twice as many First Babies as Real Babies. The company spends no more than 48 hours per week making these two dolls. Find the number and type of dolls that should be produced to maximize the profit. (Lesson 6-5)

Profit per Doll ($)	
First Baby	Real Baby
3.00	7.50

Verify each identity. (Lesson 5-4)

78. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos\theta$

79. $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = \sin\theta$

80. $\sin(3\pi - x) = \sin x$

Find all solutions to each equation in the interval $(0, 2\pi)$. (Lesson 5-3)

81. $\sin\theta = \cos\theta$

82. $\sin\theta = 1 + \cos\theta$

83. $2\sin^2 x + 3\sin x + 1 = 0$

Solve each inequality. (Lesson 2-6)

84. $x^2 - 5x - 24 > 0$

85. $x^2 + 2x - 35 \leq 0$

86. $-2y^2 + 7y + 4 < 0$

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring. (Lesson 2-2)

87. $f(x) = 3x^4 + 18x^3 + 24x^2$

88. $f(x) = 8x^6 + 48x^5 + 40x^4$

89. $f(x) = 5x^5 - 15x^4 - 50x^3$

Simplify. (Lesson 0-2)

90. $(2 + 4i) + (-1 + 5i)$

91. $(-2 - i)^2$

92. $\frac{i}{1 + 2i}$

Skills Review for Standardized Tests

93. SAT/ACT Point B lies 10 units from point A, which is the center of a circle of radius 6. If a tangent line is drawn from B to the circle, what is the distance from B to the point of tangency?

A 6
B 8
C 10
D $2\sqrt{34}$
E $2\sqrt{41}$

94. REVIEW What is the standard form of the equation of the conic given below?

$$2x^2 + 4y^2 - 8x + 24y + 32 = 0$$

F $\frac{(x-4)^2}{3} + \frac{(y+3)^2}{11} = 1$

G $\frac{(x-2)^2}{6} + \frac{(y+3)^2}{3} = 1$

H $\frac{(x+2)^2}{5} + \frac{(y+3)^2}{4} = 1$

J $\frac{(x-4)^2}{11} + \frac{(y+3)^2}{3} = 1$

95. Ruben is making an elliptical target for throwing darts. He wants the target to be 27 inches wide and 15 inches high. Which equation should Ruben use to draw the target?

A $\frac{x^2}{7.5} + \frac{y^2}{13.5} = 1$

B $\frac{x^2}{56.25} + \frac{y^2}{182.25} = 1$

C $\frac{x^2}{182.25} + \frac{y^2}{56.25} = 1$

D $\frac{x^2}{13.5} + \frac{y^2}{7.5} = 1$

96. REVIEW If $m = \frac{1}{x}$, $n = 7m$, $p = \frac{1}{n}$, $q = 14p$, and $r = \frac{1}{\frac{1}{2}q}$, find x.

F r
G q
H p
J $\frac{1}{r}$

LESSON 7-3

Hyperbolas

Then

- You analyzed and graphed ellipses and circles. (Lesson 7-2)

Now

1. Analyze and graph equations of hyperbolas.
2. Use equations to identify types of conic sections.

Why?

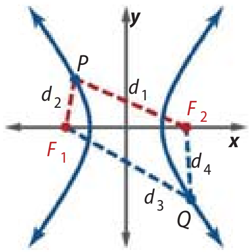
- Lightning detection systems use multiple sensors to digitize lightning strike waveforms and record details of the strike using extremely accurate GPS timing signals. Two sensors detect a signal at slightly different times and generate a point on a hyperbola where the distance from each sensor is proportional to the difference in the time of arrival. The sensors make it possible to transmit the exact location of a lightning strike in real time.

NewVocabulary

hyperbola
transverse axis
conjugate axis

1 Analyze and Graph Hyperbolas

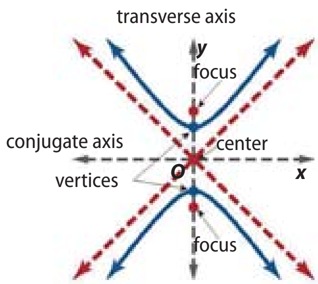
While an ellipse is the locus of all points in a plane such that the *sum* of the distances from two foci is constant, a **hyperbola** is the locus of all points in a plane such that the *absolute value of the differences* of the distances from two foci is constant.



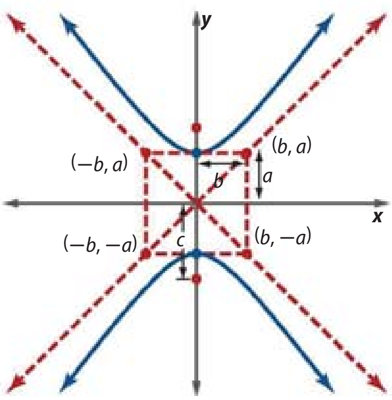
$$|d_1 - d_2| = |d_3 - d_4|$$

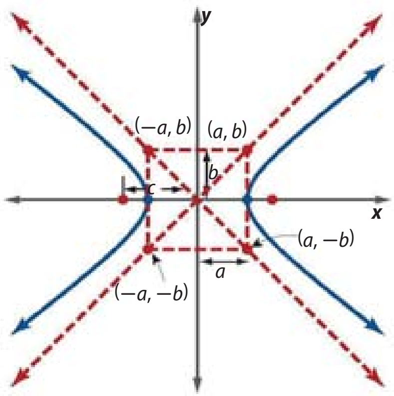
The graph of a hyperbola consists of two disconnected branches that approach two asymptotes. The midpoint of the segment with endpoints at the foci is the center. The vertices are at the intersection of this segment and each branch of the curve.

Like an ellipse, a hyperbola has two axes of symmetry. The **transverse axis** has a length of $2a$ units and connects the vertices. The **conjugate axis** is perpendicular to the transverse, passes through the center, and has a length of $2b$ units.



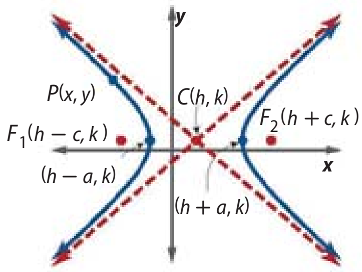
The relationship among the values of a, b, and c is different for a hyperbola than it is for an ellipse. For a hyperbola, the relationship is $c^2 = a^2 + b^2$. In addition, for any point on the hyperbola, the absolute value of the difference between the distances from the point to the foci is $2a$.





Comstock/JupiterImages

As with other conic sections, the definition of a hyperbola can be used to derive its equation. Let $P(x, y)$ be any point on the hyperbola with center $C(h, k)$. The coordinates of the foci and vertices are shown at the right. By the definition of a hyperbola, the absolute value of the difference of distances from any point on the hyperbola to the foci is constant. Thus, $|PF_1 - PF_2| = 2a$. Therefore, either $PF_1 - PF_2 = 2a$ or $PF_2 - PF_1 = 2a$. For the proof below, we will assume $PF_1 - PF_2 = 2a$.

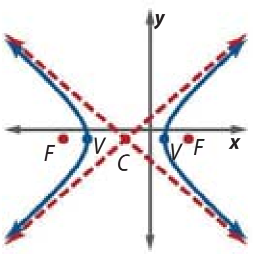


$PF_1 - PF_2 = 2a$	Definition of hyperbola
$\sqrt{[x-(h-c)]^2+(y-k)^2} - \sqrt{[x-(h+c)]^2+(y-k)^2} = 2a$	Distance Formula
$\sqrt{(x-h+c)^2+(y-k)^2} = 2a + \sqrt{(x-h-c)^2+(y-k)^2}$	Distributive and Subtraction Properties
$\sqrt{[(x-h)+c]^2+(y-k)^2} = 2a + \sqrt{[(x-h)-c]^2+(y-k)^2}$	Associative Property
$[(x-h)+c]^2+(y-k)^2 = 4a^2 + 4a\sqrt{[(x-h)-c]^2+(y-k)^2} + [(x-h)-c]^2+(y-k)^2$	
$(x-h)^2+2c(x-h)+c^2+(y-k)^2 = 4a^2 + 4a\sqrt{[(x-h)-c]^2+(y-k)^2} + (x-h)^2-2c(x-h)+c^2+(y-k)^2$	
$-4a\sqrt{[(x-h)-c]^2+(y-k)^2} = 4a^2 - 4c(x-h)$	Subtraction Property
$a\sqrt{[(x-h)-c]^2+(y-k)^2} = -a^2 + c(x-h)$	Divide each side by −4.
$a^2[(x-h)^2-2c(x-h)+c^2+(y-k)^2] = a^4 - 2a^2c(x-h) + c^2(x-h)^2$	Square each side.
$a^2(x-h)^2 - 2a^2c(x-h) + a^2c^2 + a^2(y-k)^2 = a^4 - 2a^2c(x-h) + c^2(x-h)^2$	Distributive Property
$a^2(x-h)^2 - c^2(x-h)^2 + a^2(y-k)^2 = a^4 - a^2c^2$	Addition and Subtraction Properties
$(a^2-c^2)(x-h)^2 + a^2(y-k)^2 = a^2(a^2-c^2)$	Distributive Property
$-b^2(x-h)^2 + a^2(y-k)^2 = a^2(-b^2)$	$a^2 - c^2 = -b^2$
$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	Divide each side by $a^2(-b^2)$.

The general equation for a hyperbola centered at (h, k) is given below.

KeyConcept Standard Forms of Equations for Hyperbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



Orientation: horizontal transverse axis

Center: (h, k)

Vertices: $(h \pm a, k)$

Foci: $(h \pm c, k)$

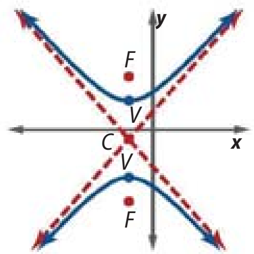
Transverse axis: $y = k$

Conjugate axis: $x = h$

Asymptotes: $y - k = \pm\frac{b}{a}(x - h)$

***a*, *b*, *c* relationship:** $c^2 = a^2 + b^2$ or $c = \sqrt{a^2 + b^2}$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



Orientation: vertical transverse axis

Center: (h, k)

Vertices: $(h, k \pm a)$

Foci: $(h, k \pm c)$

Transverse axis: $x = h$

Conjugate axis: $y = k$

Asymptotes: $y - k = \pm\frac{a}{b}(x - h)$

***a*, *b*, *c* relationship:** $c^2 = a^2 + b^2$ or $c = \sqrt{a^2 + b^2}$

Example 1 Graph Hyperbolas in Standard Form

Graph the hyperbola given by each equation.

a. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

The equation is in standard form with h and k both equal to zero. Because $a^2 = 9$ and $b^2 = 25$, $a = 3$ and $b = 5$. Use the values of a and b to find c.

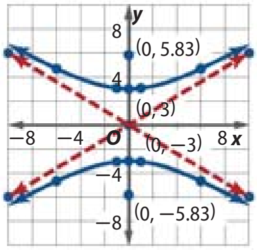
$c^2 = a^2 + b^2$	Equation relating a, b, and c for a hyperbola
$c^2 = 3^2 + 5^2$	$a = 3$ and $b = 5$
$c = \sqrt{34}$ or about 5.83	Solve for c.

Use these values for h, k, a, b, and c to determine the characteristics of the hyperbola.

orientation:	vertical	When the equation is in standard form, the x^2-term is subtracted.
center:	(0, 0)	(h, k)
vertices:	(0, 3) and (0, −3)	$(h, k \pm a)$
foci:	$(0, \sqrt{34})$ and $(0, -\sqrt{34})$	$(h, k \pm c)$
asymptotes:	$y = \frac{3}{5}x$ and $y = -\frac{3}{5}x$	$y - k = \pm\frac{a}{b}(x - h)$

Graph the center, vertices, foci, and asymptotes. Then make a table of values to sketch the hyperbola.

x	y
−6	−4.69, 4.69
−1	−3.06, 3.06
1	−3.06, 3.06
6	−4.69, 4.69



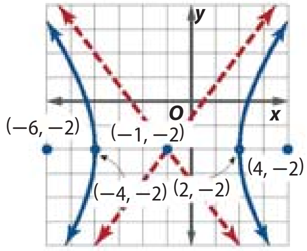
b. $\frac{(x+1)^2}{9} - \frac{(y+2)^2}{16} = 1$

The equation is in standard form with $h = -1$, $k = -2$, $a = \sqrt{9}$ or 3, $b = \sqrt{16}$ or 4, and $c = \sqrt{9 + 16}$ or 5. Use these values to determine the characteristics of the hyperbola.

orientation:	horizontal	When the equation is in standard form, the y^2-term is subtracted.
center:	(−1, −2)	(h, k)
vertices:	(2, −2) and (−4, −2)	$(h \pm a, k)$
foci:	(4, −2) and (−6, −2)	$(h \pm c, k)$
asymptotes:	$y + 2 = \frac{4}{3}(x + 1)$ and $y + 2 = -\frac{4}{3}(x + 1)$, or $y = \frac{4}{3}x - \frac{2}{3}$ and $y = -\frac{4}{3}x - \frac{10}{3}$	$y - k = \pm\frac{b}{a}(x - h)$

Graph the center, vertices, foci, and asymptotes. Then make a table of values to sketch the hyperbola.

x	y
−6	−7.33, 3.33
−5	−5.53, 1.53
3	−5.53, 1.53
4	−7.33, 3.33



Guided Practice

1A. $\frac{x^2}{4} - \frac{y^2}{1} = 1$

1B. $\frac{y^2}{5} - \frac{x^2}{3} = 1$

Math History Link

Hypatia (c. 370 A.D.–415 A.D.) Hypatia was a mathematician, scientist, and philosopher who worked as a professor at a university in Alexandria, Egypt. Hypatia edited the book *On the Conics of Apollonius*, which developed the ideas of hyperbolas, parabolas, and ellipses.

Source: Agnes Scott College

If you know the equation for a hyperbola in standard form, you can use the characteristics to graph the curve. If you are given the equation in another form, you will need to write the equation in standard form to determine the characteristics.

Example 2 Graph a Hyperbola

Graph the hyperbola given by $25x^2 - 16y^2 + 100x + 96y = 444$.

First, write the equation in standard form.

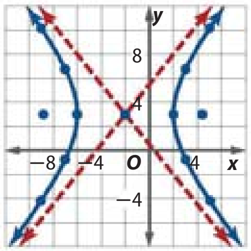
$25x^2 - 16y^2 + 100x - 96 = 444$	Original equation
$(25x^2 + 100x) - (16y^2 + 96y) = 444$	Group like terms.
$25(x^2 + 4x) - 16(y^2 - 6y) = 444$	Factor.
$25(x^2 + 4x + 4) - 16(y^2 - 6y + 9) = 444 + 25(4) - 16(9)$	Complete the squares.
$25(x + 2)^2 - 16(y - 3)^2 = 400$	Factor and simplify.
$\frac{(x+2)^2}{16} - \frac{(y-3)^2}{25} = 1$	Divide each side by 400.

The equation is now in standard form with $h = -2$, $k = 3$, $a = \sqrt{16}$ or 4, $b = \sqrt{25}$ or 5, and $c = \sqrt{16 + 25}$, which is $\sqrt{41}$ or about 6.4. Use these values to determine the characteristics of the hyperbola.

orientation:	horizontal	When the equation is in standard form, the y^2-term is subtracted.
center:	$(-2, 3)$	(h, k)
vertices:	$(-6, 3)$ and $(2, 3)$	$(h \pm a, k)$
foci:	$(-8.4, 3)$ and $(4.4, 3)$	$(h \pm c, k)$
asymptotes:	$y - 3 = \frac{5}{4}(x + 2)$ and $y - 3 = -\frac{5}{4}(x + 2)$, or $y = \frac{5}{4}x + \frac{11}{2}$ and $y = -\frac{5}{4}x + \frac{1}{2}$	$y - k = \pm\frac{b}{a}(x - h)$

Graph the center, vertices, foci, and asymptotes. Then, make a table of values to sketch the hyperbola.

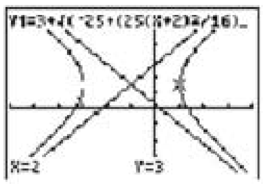
x	y
−9	−4.18, 10.18
−7	−0.75, 6.75
3	−0.75, 6.75
5	−4.18, 10.18



CHECK Solve the equation for y to obtain two functions of x,

$$y = 3 + \sqrt{-25 + \frac{25(x+2)^2}{16}} \text{ and } 3 - \sqrt{-25 + \frac{25(x+2)^2}{16}}.$$

Graph the equations in the same window, along with the equations of the asymptote and compare with your graph, by testing a few points. ✔



$[-12, 8]$ scl: 1 by $[-8, 12]$ scl: 1

GuidedPractice

Graph the hyperbola given by each equation.

2A. $\frac{(y+4)^2}{64} - \frac{(x+1)^2}{81} = 1$

2B. $2x^2 - 3y^2 - 12x = 36$

When graphing a hyperbola remember that the graph will approach the asymptotes as it moves away from the vertices. Plot near the vertices to improve the accuracy of your graph.

StudyTip

Standard Form When converting from general form to standard form, always remember that the difference of the two algebraic terms must be equal to 1. When you divide by the number on the right side of the equation, only perfect square trinomials should remain in the numerators of the subtracted fractions.

You can determine the equation for a hyperbola if you are given characteristics that provide sufficient information.

Example 3 Write Equations Given Characteristics

Write an equation for the hyperbola with the given characteristics.

a. vertices $(-3, -6)$, $(-3, 2)$; foci $(-3, -7)$, $(-3, 3)$

Because the x-coordinates of the vertices are the same, the transverse axis is vertical. Find the center and the values of a, b, and c.

center: $(-3, -2)$	Midpoint of segment between foci
$a = 4$	Distance from each vertex to center
$c = 5$	Distance from each focus to center
$b = 3$	$c^2 = a^2 + b^2$

Because the transverse axis is vertical, the a^2-term goes with the y^2-term. An equation for the hyperbola is $\frac{(y+2)^2}{16} - \frac{(x+3)^2}{9} = 1$. The graph of the hyperbola is shown in Figure 7.3.1.

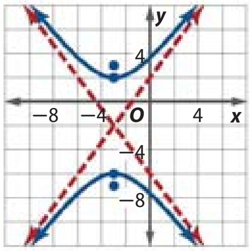


Figure 7.3.1

b. vertices $(-3, 0)$, $(-9, 0)$; asymptotes $y = 2x - 12$, $y = -2x + 12$

Because the y-coordinates of the vertices are the same, the transverse axis is horizontal.

center: $(-6, 0)$	Midpoint of segment between vertices
$a = 3$	Distance from each vertex to center

The slopes of the asymptotes are $\pm\frac{b}{a}$. Use the positive slope to find b.

$\frac{b}{a} = 2$	Positive slope of asymptote
$\frac{b}{3} = 2$	$a = 3$
$b = 6$	Solve for b.

Because the transverse axis is horizontal, the a^2-term goes with the x^2-term. An equation for the hyperbola is $\frac{(x+6)^2}{9} - \frac{y^2}{36} = 1$. The graph of the hyperbola is shown in Figure 7.3.2.

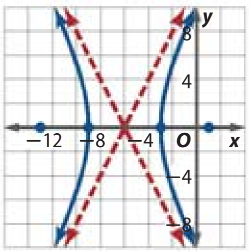


Figure 7.3.2

GuidedPractice

3A. vertices $(3, 2)$, $(3, 6)$; conjugate axis length 10 units

3B. foci $(2, -2)$, $(12, -2)$; asymptotes $y = \frac{3}{4}x - \frac{29}{4}$, $y = -\frac{3}{4}x + \frac{13}{4}$

Another characteristic that can be used to describe a hyperbola is the eccentricity. The formula for eccentricity is the same for all conics, $e = \frac{c}{a}$. Recall that for an ellipse, the eccentricity is greater than 0 and less than 1. For a hyperbola, the eccentricity will always be greater than 1.

Example 4 Find the Eccentricity of a Hyperbola

Determine the eccentricity of the hyperbola given by $\frac{(y-4)^2}{48} - \frac{(x+5)^2}{36} = 1$.

Find c and then determine the eccentricity.

$c^2 = a^2 + b^2$	Equation relating a, b, and c
$c^2 = 48 + 36$	$a^2 = 48$ and $b^2 = 36$
$c = \sqrt{84}$	Solve for c.

$e = \frac{c}{a}$	Eccentricity equation
$= \frac{\sqrt{84}}{\sqrt{48}}$	$c = \sqrt{84}$ and $a = \sqrt{48}$
≈ 1.32	Simplify.

The eccentricity of the hyperbola is about 1.32.

Guided Practice

Determine the eccentricity of the hyperbola given by each equation.

4A. $\frac{(x+8)^2}{64} - \frac{(y-4)^2}{80} = 1$

4B. $\frac{(y-2)^2}{15} - \frac{(x+9)^2}{75} = 1$

2 Identify Conic Sections

You can determine the type of conic when the equation for the conic is in general form, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The discriminant, or $B^2 - 4AC$, can be used to identify the conic.

Key Concept Classify Conics Using the Discriminant

The graph of a second degree equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

- a circle if $B^2 - 4AC < 0$; $B = 0$ and $A = C$.
- an ellipse if $B^2 - 4AC < 0$; either $B \neq 0$ or $A \neq C$.
- a parabola if $B^2 - 4AC = 0$.
- a hyperbola if $B^2 - 4AC > 0$.

When $B = 0$, the conic will be either vertical or horizontal. When $B \neq 0$, the conic will be neither vertical nor horizontal.

Example 5 Identify Conic Sections

Use the discriminant to identify each conic section.

a. $\mathbf{4x^2 + 3y^2 - 2x + 5y - 60 = 0}$

A is 4, B is 0, and C is 3.

Find the discriminant.

$B^2 - 4AC = 0^2 - 4(4)(3)$ or -48

The discriminant is less than 0, so the conic must be either a circle or an ellipse. Because $A \neq C$, the conic is an ellipse.

b. $\mathbf{2y^2 + 6x - 3y + 4xy + 2x^2 - 88 = 0}$

A is 2, B is 4, and C is 2.

Find the discriminant.

$B^2 - 4AC = 4^2 - 4(2)(2)$ or 0

The discriminant is 0, so the conic is a parabola.

c. $\mathbf{18x - 12y^2 + 4xy + 10x^2 - 6y + 24 = 0}$

A is 10, B is 4, and C is -12.

Find the discriminant.

$B^2 - 4AC = 4^2 - 4(10)(-12)$ or 496

The discriminant is greater than 0, so the conic is a hyperbola.

Study Tip

Identifying Conics When a conic has been rotated as in Example 5b, its equation cannot be written in standard form. In this case, only the discriminant can be used to determine the type of conic without graphing. You will learn more about rotated conics in the next lesson.

Guided Practice

5A. $3x^2 + 4x - 2y + 3y^2 + 6xy + 64 = 0$

5B. $6x^2 + 2xy - 15x = 3y^2 + 5y + 18$

5C. $4xy + 8x - 3y = 2x^2 + 8y^2$

Researchers can determine the location of a lightning strike on the hyperbolic path formed with the detection sensors located at the foci.

Real-WorldLink

A lightning rod provides a low-resistance path to ground for electrical currents from lightning strikes.

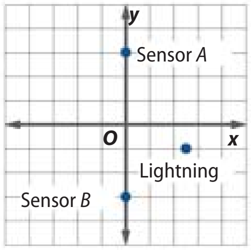
Source: *How Stuff Works*

Real-World Example 6 Apply Hyperbolas

METEOROLOGY Two lightning detection sensors are located 6 kilometers apart, where sensor *A* is due north of sensor *B*. As a bolt of lightning strikes, researchers determine the lightning strike occurred east of both sensors and 1.5 kilometers farther from sensor *A* than sensor *B*.

a. Find the equation for the hyperbola on which the lightning strike is located.

First, place the two sensors on a coordinate grid so that the origin is the midpoint of the segment between sensor *A* and sensor *B*. The lightning is east of the sensors and closer to sensor *B*, so it should be in the 4^{th} quadrant.



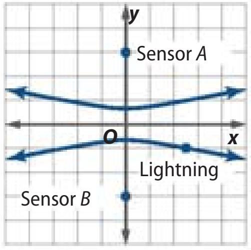
The two sensors are located at the foci of the hyperbola, so c is 3. Recall that the absolute value of the difference of the distances from any point on a hyperbola to the foci is $2a$. Because the lightning strike is 1.5 kilometers farther from sensor A than sensor B, $2a = 1.5$ and a is 0.75. Use these values of a and c to find b^2.

$c^2 = a^2 + b^2$ — Equation relating a, b, and c

$3^2 = 0.75^2 + b^2$ — $c = 3$ and $a = 0.75$

$8.4375 = b^2$ — Solve for b^2.

The transverse axis is vertical and the center of the hyperbola is located at the origin, so the equation will be of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Substituting the values of a^2 and b^2, the equation for the hyperbola is $\frac{y^2}{0.5625} - \frac{x^2}{8.4375} = 1$.



The lightning strike occurred along the hyperbola $\frac{y^2}{0.5625} - \frac{x^2}{8.4375} = 1$.

b. Find the coordinates of the lightning strike if it occurred 2.5 kilometers east of the sensors.

Because the lightning strike occurred 2.5 kilometers east of the sensors, $x = 2.5$. The lightning was closer to sensor B than sensor A, so it lies on the lower branch. Substitute the value of x into the equation and solve for y.

$\frac{y^2}{0.5625} - \frac{x^2}{8.4375} = 1$ — Original equation

$\frac{y^2}{0.5625} - \frac{2.5^2}{8.4375} = 1$ — $x = 2.5$

$y \approx -0.99$ — Solve.

The value of y is about -0.99, so the location of the lightning strike is at $(2.5, -0.99)$.

GuidedPractice

6. **METEOROLOGY** Sensor A is located 30 miles due west of sensor B. A lightning strike occurs 9 miles farther from sensor A than sensor B.
   - **A.** Find the equation for the hyperbola on which the lightning strike occurred.
   - **B.** Find the coordinates of the location of the lightning strike if it occurred 8 miles north of the sensors.

Exercises

● = Step-by-Step Solutions begin on page R29.

Graph the hyperbola given by each equation. (Example 1)

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

2. $\frac{y^2}{4} - \frac{x^2}{17} = 1$

3. $\frac{x^2}{49} - \frac{y^2}{30} = 1$

4. $\frac{y^2}{34} - \frac{x^2}{14} = 1$

5. $\frac{x^2}{9} - \frac{y^2}{21} = 1$

6. $\frac{x^2}{36} - \frac{y^2}{4} = 1$

7. $\frac{y^2}{81} - \frac{x^2}{8} = 1$

8. $\frac{y^2}{25} - \frac{x^2}{14} = 1$

9. $3x^2 - 2y^2 = 12$

10. $3y^2 - 5x^2 = 15$

11. **LIGHTING** The light projected on a wall by a table lamp can be represented by a hyperbola. The light from a certain table lamp can be modeled by $\frac{y^2}{225} - \frac{x^2}{81} = 1$. Graph the hyperbola. (Example 1)

Graph the hyperbola given by each equation. (Example 2)

12. $\frac{(x+5)^2}{9} - \frac{(y+4)^2}{48} = 1$

13. $\frac{(y-7)^2}{4} - \frac{x^2}{33} = 1$

14. $\frac{(x-2)^2}{25} - \frac{(y-6)^2}{60} = 1$

15. $\frac{(x-5)^2}{49} - \frac{(y-1)^2}{17} = 1$

16. $\frac{(y-3)^2}{16} - \frac{(x-4)^2}{42} = 1$

17. $\frac{(x+6)^2}{64} - \frac{(y+5)^2}{58} = 1$

18. $x^2 - 4y^2 - 6x - 8y = 27$

19. $-x^2 + 3y^2 - 4x + 6y = 28$

20. $13x^2 - 2y^2 + 208x + 16y = -748$

21. $-5x^2 + 2y^2 - 70x - 8y = 287$

22. **EARTHQUAKES** Shortly after a seismograph detects an earthquake, a second seismograph positioned due north of the first detects the earthquake. The epicenter of the earthquake lies on a branch of the hyperbola represented by $\frac{(y-30)^2}{900} - \frac{(x-60)^2}{1600} = 1$, where the seismographs are located at the foci. Graph the hyperbola. (Example 2)

Write an equation for the hyperbola with the given characteristics. (Example 3)

23. foci $(-1, 9)$, $(-1, -7)$; conjugate axis length of 14 units

24. vertices $(7, 5)$, $(-5, 5)$; foci $(11, 5)$, $(-9, 5)$

25. foci $(9, -1)$, $(-3, -1)$; conjugate axis length of 6 units

26. vertices $(-1, 9)$, $(-1, 3)$; asymptotes $y = \pm\frac{3}{7}x + \frac{45}{7}$

27. vertices $(-3, -12)$, $(-3, -4)$; foci $(-3, -15)$, $(-3, -1)$

28. foci $(9, 7)$, $(-17, 7)$; asymptotes $y = \pm\frac{5}{12}x + \frac{104}{12}$

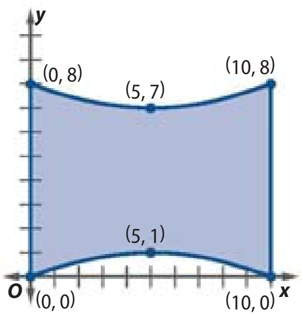
29. center $(-7, 2)$; asymptotes $y = \pm\frac{7}{5}x + \frac{59}{5}$, transverse axis length of 10 units

30. center $(0, -5)$; asymptotes $y = \pm\frac{\sqrt{19}}{6}x - 5$, conjugate axis length of 12 units

31. vertices $(0, -3)$, $(-4, -3)$; conjugate axis length of 12 units

32. vertices $(2, 10)$, $(2, -2)$; conjugate axis length of 16 units

33. **ARCHITECTURE** The graph below shows the outline of a floor plan for an office building.



a. Write an equation that could model the curved sides of the building.

b. Each unit on the coordinate plane represents 15 feet. What is the narrowest width of the building? (Example 3)

Determine the eccentricity of the hyperbola given by each equation. (Example 4)

34. $\frac{(y-1)^2}{10} - \frac{(x-6)^2}{13} = 1$

35. $\frac{(x+4)^2}{24} - \frac{(y+1)^2}{15} = 1$

36. $\frac{(x-3)^2}{38} - \frac{(y-2)^2}{5} = 1$

37. $\frac{(y+2)^2}{32} - \frac{(x+5)^2}{25} = 1$

38. $\frac{(y-4)^2}{23} - \frac{(x+11)^2}{72} = 1$

39. $\frac{(x-1)^2}{16} - \frac{(y+4)^2}{29} = 1$

Determine the eccentricity of the hyperbola given by each equation. (Example 4)

40. $11x^2 - 2y^2 - 110x + 24y = -181$

41. $-4x^2 + 3y^2 + 72x - 18y = 321$

42. $3x^2 - 2y^2 + 12x - 12y = 42$

43. $-x^2 + 7y^2 + 24x + 70y = -24$

Use the discriminant to identify each conic section. (Example 5)

44. $14y + y^2 = 4x - 97$

45. $18x - 3x^2 + 4 = -8y^2 + 32y$

46. $14 + 4y + 2x^2 = -12x - y^2$

47. $12y - 76 - x^2 = 16x$

48. $2x + 8y + x^2 + y^2 = 8$

49. $5y^2 - 6x + 3x^2 - 50y = -3x^2 - 113$

50. $x^2 + y^2 + 8x - 6y + 9 = 0$

51. $-56y + 5x^2 = 211 + 4y^2 + 10x$

52. $-8x + 16 = 8y + 24 - x^2$

53. $x^2 - 4x = -y^2 + 12y - 31$

54. **PHYSICS** A hyperbola occurs naturally when two nearly identical glass plates in contact on one edge and separated by about 5 millimeters at the other edge are dipped in a thick liquid. The liquid will rise by capillarity to form a hyperbola caused by the surface tension. Find a model for the hyperbola if the conjugate axis is 50 centimeters and the transverse axis is 30 centimeters.

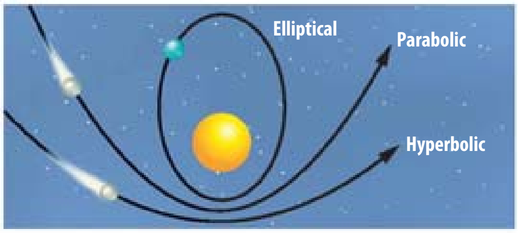
55 **AVIATION** The Federal Aviation Administration performs flight trials to test new technology in aircraft. When one of the test aircraft collected its data, it was 18 kilometers farther from Airport B than Airport A. The two airports are 72 kilometers apart along the same highway, with Airport B due south of Airport A. (Example 6)

a. Write an equation for the hyperbola centered at the origin on which the aircraft was located when the data were collected.

b. Graph the equation, indicating on which branch of the hyperbola the plane was located.

c. When the data were collected, the plane was 40 miles from the highway. Find the coordinates of the plane.

56. **ASTRONOMY** While each of the planets in our solar system move around the Sun in elliptical orbits, comets may have elliptical, parabolic, or hyperbolic orbits where the center of the sun is a focus. (Example 5)



The paths of three comets are modeled below, where the values of x and y are measured in gigameters. Use the discriminant to identify each conic.

a. $3x^2 - 18x - 580850 = 4.84y^2 - 38.72y$

b. $-360x - 8y = -y^2 - 1096$

c. $-24.88y + x^2 = 6x - 3.11y^2 + 412341$

Derive the general form of the equation for a hyperbola with each of the following characteristics.

57. vertical transverse axis centered at the origin

58. horizontal transverse axis centered at the origin

Solve each system of equations. Round to the nearest tenth if necessary.

59. $2y = x - 10$ and $\dfrac{(x-3)^2}{16} - \dfrac{(y+2)^2}{84} = 1$

60. $y = -\dfrac{1}{4}x + 3$ and $\dfrac{x^2}{36} - \dfrac{(y-4)^2}{4} = 1$

61. $y = 2x$ and $\dfrac{(y+2)^2}{64} - \dfrac{(x+5)^2}{49} = 1$

62. $3x - y = 9$ and $\dfrac{(x-5)^2}{36} + \dfrac{y^2}{16} = 1$

63. $\dfrac{y^2}{36} + \dfrac{x^2}{25} = 1$ and $\dfrac{y^2}{36} - \dfrac{x^2}{25} = 1$

64. $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$ and $\dfrac{(x+1)^2}{49} + \dfrac{(y+2)^2}{4} = 1$

65. **FIREWORKS** A fireworks grand finale is heard by Carson and Emmett, who are 3 miles apart talking on their cell phones. Emmett hears the finale about 1 second before Carson. Assume that sound travels at 1100 feet per second.

a. Write an equation for the hyperbola on which the fireworks were located. Place the locations of Carson and Emmett on the x-axis, with Carson on the left and the midpoint between them at the origin.

b. Describe the branch of the hyperbola on which the fireworks display was located.

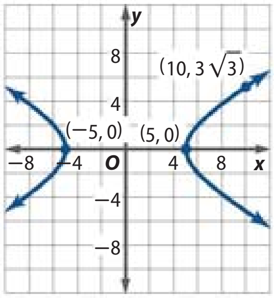
66. **ARCHITECTURE** The Kobe Port Tower is a *hyperboloid* structure in Kobe, Japan. This means that the shape is generated by rotating a hyperbola around its conjugate axis. Suppose the hyperbola used to generate the hyperboloid modeling the shape of the tower has an eccentricity of 19.

a. If the tower is 8 meters wide at its narrowest point, determine an equation of the hyperbola used to generate the hyperboloid.

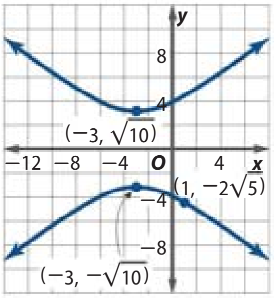
b. If the top of the tower is 32 meters above the center of the hyperbola and the base is 76 meters below the center, what is the radius of the top and the radius of the base of the tower?

Write an equation for each hyperbola.

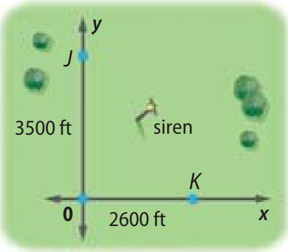
67.



68.



69. SOUND When a tornado siren goes off, three people are located at J, K, and O, as shown on the graph below.



The person at J hears the siren 2 seconds before the person at O. The person at K hears the siren 1 second before the person at O. Find each possible location of the tornado siren. Assume that sound travels at 1100 feet per second. (*Hint*: Each location of the siren is a point of intersection between a hyperbola that has foci at O and J and a hyperbola that has foci at O and K.)

Write an equation for the hyperbola with the given characteristics.

70. The center is at (5, 1), a vertex is at (5, 9), and an equation of an asymptote is $3y = 4x - 17$.

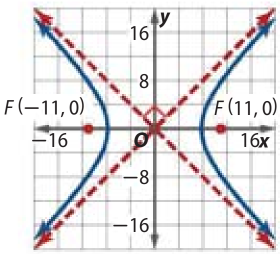
71. The hyperbola has its center at (−4, 3) and a vertex at (1, 3). The equation of one of its asymptotes is $7x + 5y = -13$.

72. The foci are at $(0, 2\sqrt{6})$ and $(0, -2\sqrt{6})$. The eccentricity is $\frac{2\sqrt{6}}{3}$.

73 The eccentricity of the hyperbola is $\frac{7}{6}$ and the foci are at (−1, −2) and (13, −2).

74. The hyperbola has foci at (−1, 9) and (−1, −7) and the slopes of the asymptotes are $\pm\frac{\sqrt{15}}{7}$.

75. For an *equilateral hyperbola*, $a = b$ when the equation of the hyperbola is written in standard form. The asymptotes of an equilateral hyperbola are perpendicular. Write an equation for the equilateral hyperbola below.



76. MULTIPLE REPRESENTATIONS In this problem, you will explore a special type of hyperbola called a *conjugate hyperbola*. This occurs when the conjugate axis of one hyperbola is the transverse axis of another.

a. GRAPHICAL Sketch the graphs of $\frac{x^2}{36} - \frac{y^2}{64} = 1$ and $\frac{y^2}{64} - \frac{x^2}{36} = 1$ on the same coordinate plane.

b. ANALYTICAL Compare the foci, vertices, and asymptotes of the graphs.

c. ANALYTICAL Write an equation for the conjugate hyperbola for $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

d. GRAPHICAL Sketch the graphs of the new conjugate hyperbolas.

e. VERBAL Make a conjecture about the similarities of conjugate hyperbolas.

H.O.T. Problems Use Higher-Order Thinking Skills

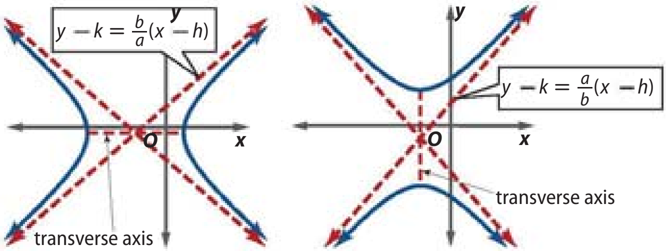
77. OPEN ENDED Write an equation for a hyperbola where the distance between the foci is twice the length of the transverse axis.

78. REASONING Consider $rx^2 = -sy^2 - t$. Describe the type of conic section that is formed for each of the following. Explain your reasoning.

a. $rs = 0$ **b.** $rs > 0$

c. $r = s$ **d.** $rs < 0$

79. WRITING IN MATH Explain why the equation for the asymptotes of a hyperbola changes from $\pm\frac{b}{a}$ to $\pm\frac{a}{b}$ depending on the location of the transverse axis.



80. REASONING Suppose you are given two of the following characteristics: vertices, foci, transverse axis, conjugate axis, or asymptotes. Is it *sometimes*, *always*, or *never* possible to write the equation for the hyperbola?

81. CHALLENGE A hyperbola has foci at $F_1(0, 9)$ and $F_2(0, -9)$ and contains point P. The distance between P and F_1 is 6 units greater than the distance between P and F_2. Write the equation of the hyperbola in standard form.

82. PROOF An equilateral hyperbola is formed when $a = b$ in the standard form of the equation for a hyperbola. Prove that the eccentricity of every equilateral hyperbola is $\sqrt{2}$.

83. WRITING IN MATH Describe the steps for finding the equation of a hyperbola if the foci and length of the transverse axis are given.

Spiral Review

Graph the ellipse given by each equation. (Lesson 7-2)

84. $(x-8)^2 + \frac{(y-2)^2}{81} = 1$

85. $\frac{x^2}{64} + \frac{(y+5)^2}{49} = 1$

86. $\frac{(x-2)^2}{16} + \frac{(y+5)^2}{36} = 1$

87. PROJECTILE MOTION The height of a baseball hit by a batter with an initial speed of 80 feet per second can be modeled by $h = -16t^2 + 80t + 5$, where t is the time in seconds. (Lesson 7-1)

a. How high above the ground is the vertex located?

b. If an outfielder's catching height is the same as the initial height of the ball, about how long after the ball is hit will the player catch the ball?

Write each system of equations as a matrix equation, $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix to solve the system. (Lesson 6-2)

88. $3x_1 + 11x_2 - 9x_3 = 25$
$-8x_1 + 5x_2 + x_3 = -31$
$x_1 - 9x_2 + 4x_3 = 13$

89. $x_1 - 7x_2 + 8x_3 = -3$
$6x_1 + 5x_2 - 2x_3 = 2$
$3x_1 - 4x_2 + 9x_3 = 26$

90. $2x_1 - 5x_2 + x_3 = 28$
$3x_1 + 4x_2 + 5x_3 = 17$
$7x_1 - 2x_2 + 3x_3 = 33$

Solve each equation for all values of θ. (Lesson 5-3)

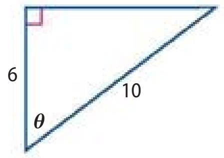
91. $\tan\theta = \sec\theta - 1$

92. $\sin\theta + \cos\theta = 0$

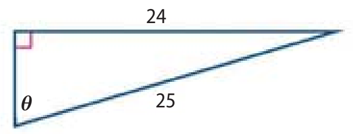
93. $\csc\theta - \cot\theta = 0$

Find the exact values of the six trigonometric functions of θ. (Lesson 4-1)

94.



95.



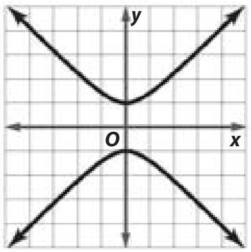
Use the given zero to find all complex zeros of each function. Then write the linear factorization of the function. (Lesson 2-4)

96. $f(x) = 2x^5 - 11x^4 + 69x^3 + 135x^2 - 675x;\ 3 - 6i$

97. $f(x) = 2x^5 - 9x^4 + 146x^3 + 618x^2 + 752x + 291;\ 4 + 9i$

Skills Review for Standardized Tests

98. REVIEW What is the equation of the graph?



A $y = x^2 + 1$

B $y - x = 1$

C $y^2 - x^2 = 1$

D $x^2 + y^2 = 1$

E $xy = 1$

99. REVIEW The graph of $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{5}\right)^2 = 1$ is a hyperbola. Which set of equations represents the asymptotes of the hyperbola's graph?

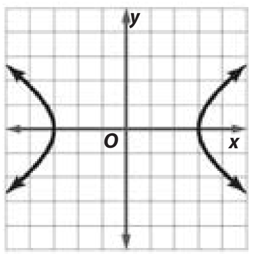
F $y = \frac{4}{5}x,\ y = -\frac{4}{5}x$

G $y = \frac{1}{4}x,\ y = -\frac{1}{4}x$

H $y = \frac{5}{4}x,\ y = -\frac{5}{4}x$

J $y = \frac{1}{5}x,\ y = -\frac{1}{5}x$

100. The foci of the graph are at $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$. Which equation does the graph represent?



A $\frac{x^2}{9} - \frac{y^2}{4} = 1$

B $\frac{x^2}{3} - \frac{y^2}{2} = 1$

C $\frac{x^2}{3} - \frac{y^2}{\sqrt{13}} = 1$

D $\frac{x^2}{9} - \frac{y^2}{13} = 1$

101. SAT/ACT If $z = \frac{3y}{x^3}$, then what is the effect on the value of z when y is multiplied by 4 and x is doubled?

F z is unchanged.

G z is halved.

H z is doubled.

J z is multiplied by 4.

CHAPTER 7

Mid-Chapter Quiz

Lessons 7-1 through 7-3

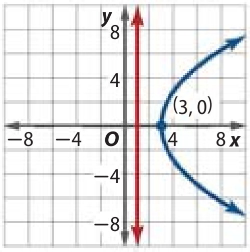
Write an equation for and graph a parabola with the given focus F and vertex V. (Lesson 7-1)

1. $F(1, 5)$, $V(1, 3)$

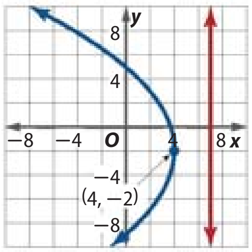
2. $F(5, -7)$, $V(1, -7)$

3. MULTIPLE CHOICE In each of the following, a parabola and its directrix are shown. In which parabola is the focus farthest from the vertex? (Lesson 7-1)

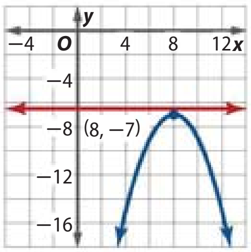
A



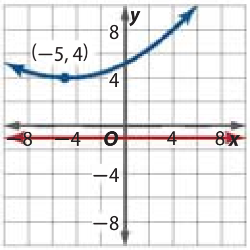
C



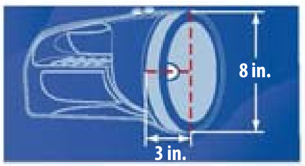
B



D



4. DESIGN The cross-section of the mirror in the flashlight design below is a parabola. (Lesson 7-1)



a. Write an equation that models the parabola.

b. Graph the equation.

Graph the ellipse given by each equation. (Lesson 7-2)

5. $\dfrac{(x+4)^2}{81} + \dfrac{(y+2)^2}{16} = 1$

6. $\dfrac{(x-3)^2}{4} + \dfrac{(y-6)^2}{36} = 1$

Write an equation for the ellipse with each set of characteristics. (Lesson 7-2)

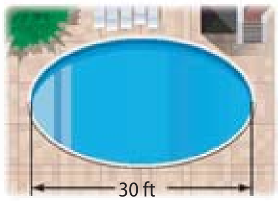
7. vertices $(9, -3)$, $(-3, -3)$; foci $(7, -3)$, $(-1, -3)$

8. foci $(3, 1)$, $(3, 7)$; length of minor axis equals 8

9. major axis $(1, -1)$ to $(1, -13)$;
minor axis $(-2, -7)$ to $(4, -7)$

10. vertices $(8, 5)$, $(8, -9)$;
length of minor axis equals 6

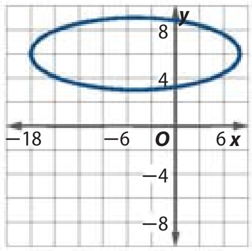
11. SWIMMING The shape of a swimming pool is designed as an ellipse with a length of 30 feet and an eccentricity of 0.68. (Lesson 7-2)



a. What is the width of the pool?

b. Write an equation for the ellipse if the point of origin is the center of the pool.

12. MULTIPLE CHOICE Which of the following is a possible eccentricity for the graph? (Lesson 7-2)



A 0

B $\frac{1}{4}$

C 1

D $\frac{9}{5}$

Graph the hyperbola given by each equation. (Lesson 7-3)

13. $\dfrac{x^2}{81} - \dfrac{(y+7)^2}{81} = 1$

14. $\dfrac{(y-3)^2}{4} - \dfrac{(x-3)^2}{16} = 1$

Write an equation for the hyperbola with the given characteristics. (Lesson 7-3)

15. vertices $(0, 5)$, $(0, -5)$; conjugate axis length of 6

16. foci $(10, 0)$, $(-6, 0)$; transverse axis length of 4

17. vertices $(-11, 0)$, $(11, 0)$; foci $(-14, 0)$, $(14, 0)$

18. foci $(5, 7)$, $(5, -9)$; transverse axis length of 10

Use the discriminant to identify each conic section. (Lesson 7-3)

19. $x^2 + 4y^2 - 2x - 24y + 34 = 0$

20. $4x^2 - 25y^2 - 24x - 64 = 0$

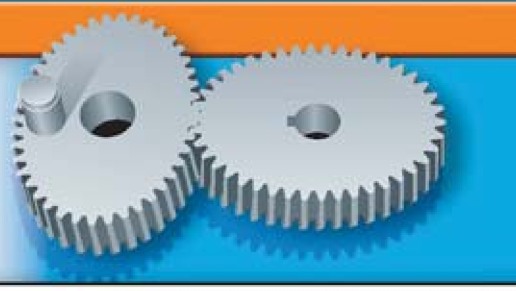
21. $2x^2 - y + 5 = 0$

22. $25x^2 + 25y^2 - 100x - 100y + 196 = 0$

LESSON 7-4

Rotations of Conic Sections

Then	Now	Why?
You identified and graphed conic sections. (Lessons 7-1 through 7-3)	1 Find rotation of axes to write equations of rotated conic sections. 2 Graph rotated conic sections.	Elliptical gears are paired by rotating them about their foci. The driver gear turns at a constant speed, and the driven gear changes its speed continuously during each revolution.



1 Rotations of Conic Sections

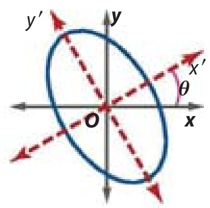
In the previous lesson, you learned that when a conic section is vertical or horizontal with its axes parallel to the x- and y-axis, $B = 0$ in its general equation. The equation of such a conic does not contain an xy-term.

$Ax^2 + Cy^2 + Dx + Ey + F = 0$ Axes of conic are parallel to coordinate axes.

In this lesson, you will examine conics with axes that are rotated and no longer parallel to the coordinate axes. In the general equation for such rotated conics, $B \neq 0$, so there is an xy-term.

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ Axes of conic are rotated from coordinate axes.

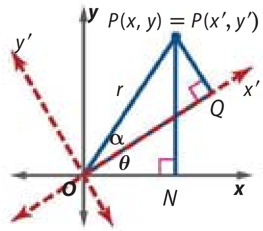
If the xy-term were eliminated, the equation of the rotated conic could be written in standard form by completing the square. To eliminate this term, we rotate the coordinate axes until they are parallel to the axes of the conic.



When the coordinate axes are rotated through an angle θ as shown, the origin remains fixed and new axes x' and y' are formed. The equation of the conic in the new $x'y'$-plane has the following general form.

$A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ Equation in $x'y'$-plane

Trigonometry can be used to develop formulas relating a point $P(x, y)$ in the xy-plane and $P(x', y')$ in the $x'y'$ plane.



Consider the figure at the right. Notice that in right triangle PNO, $OP = r$, $ON = x$, $PN = y$, and $m\angle NOP = \alpha + \theta$. Using $\triangle PNO$, you can establish the following relationships.

$x = r\cos(\alpha + \theta)$ Cosine ratio

$\quad = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$ Cosine Sum Identity

$y = r\sin(\alpha + \theta)$ Sine ratio

$\quad = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$ Sine Sum Identity

Using right triangle POQ, in which $OP = r$, $OQ = x'$, $PQ = y'$, and $m\angle QOP = \alpha$, you can establish the relationships $x' = r\cos\alpha$ and $y' = r\sin\alpha$. Substituting these values into the previous equations, you obtain the following.

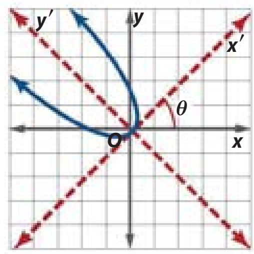
$x = x'\cos\theta - y'\sin\theta$ $y = y'\cos\theta + x'\sin\theta$

KeyConcept Rotation of Axes of Conics

An equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in the xy-plane can be rewritten as $A(x')^2 + C(y')^2 + Dx' + Ey' + F = 0$ in the rotated $x'y'$-plane.

The equation in the $x'y'$-plane can be found using the following equations, where θ is the angle of rotation.

$x = x'\cos\theta - y'\sin\theta$ $y = x'\sin\theta + y'\cos\theta$



Example 1 Write an Equation in the $x'y'$-Plane

Use $\theta = \frac{\pi}{4}$ to write $6x^2 + 6xy + 9y^2 = 53$ in the $x'y'$-plane. Then identify the conic.

Find the equations for x and y.

$x = x'\cos\theta - y'\sin\theta$	Rotation equations for x and y	$y = x'\sin\theta + y'\cos\theta$
$= \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$= \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$

Substitute into the original equation.

$$6x^2 \quad + \quad 6xy \quad + \quad 9y^2 = 53$$

$$6\left(\frac{\sqrt{2}x' - \sqrt{2}y'}{2}\right)^2 + 6\left(\frac{\sqrt{2}x' - \sqrt{2}y'}{2}\right)\left(\frac{\sqrt{2}x' + \sqrt{2}y'}{2}\right) + 9\left(\frac{\sqrt{2}x' + \sqrt{2}y'}{2}\right)^2 = 53$$

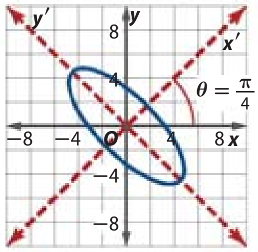
$$\frac{6[2(x')^2 - 4x'y' + 2(y')^2]}{4} + \frac{6[2(x')^2 - 2(y')^2]}{4} + \frac{9[2(x')^2 + 4x'y' + 2(y')^2]}{4} = 53$$

$$3(x')^2 - 6x'y' + 3(y')^2 + 3(x')^2 - 3(y')^2 + \frac{9}{2}(x')^2 + 9x'y' + \frac{9}{2}(y')^2 - 53 = 0$$

$$6(x')^2 - 12x'y' + 6(y')^2 + 6(x')^2 - 6(y')^2 + 9(x')^2 + 18x'y' + 9(y')^2 - 106 = 0$$

$$21(x')^2 + 6x'y' + 9(y')^2 - 106 = 0$$

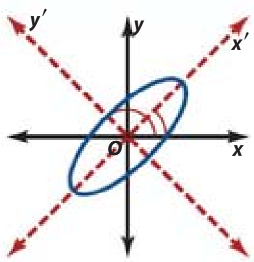
The equation in the $x'y'$-plane is $21(x')^2 + 6x'y' + 9(y')^2 - 106 = 0$. For this equation, $B^2 - 4AC = 6^2 - 4(21)(9)$ or -720. Since $-720 < 0$, the conic is an ellipse as shown.



GuidedPractice

1. Use $\theta = \frac{\pi}{6}$ to write $7x^2 + 4\sqrt{3}xy + 3y^2 - 60 = 0$ in the $x'y'$-plane. Then identify the conic.

> **StudyTip**
>
> **Angle of Rotation** The angle of rotation θ is an acute angle due to the fact that either the x'-axis or the y'-axis will be in the first quadrant. For example, while the plane in the figure below could be rotated 123°, a 33° rotation is all that is needed to align the axes.



When the angle of rotation θ is chosen appropriately, the $x'y'$-term is eliminated from the general form equation, and the axes of the conic will be parallel to the axes of the $x'y'$-plane.

After substituting $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ into the general form of a conic, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the coefficient of the $x'y'$-term is $B\cos 2\theta + (C - A)\sin 2\theta$. By setting this equal to 0, the $x'y'$-term can be eliminated.

$B\cos 2\theta + (C - A)\sin 2\theta = 0$	Coefficient of $x'y'$-term
$B\cos 2\theta = -(C - A)\sin 2\theta$	Subtract $(C - A)\sin 2\theta$ from each side.
$B\cos 2\theta = (A - C)\sin 2\theta$	Distributive Property
$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B}$	Divide each side by $B\sin 2\theta$.
$\cot 2\theta = \frac{A - C}{B}$	$\frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$

KeyConcept Angle of Rotation Used to Eliminate xy-Term

An angle of rotation θ such that $\cot 2\theta = \frac{A - C}{B}$, $B \neq 0$, $0 < \theta < \frac{\pi}{2}$, will eliminate the xy-term from the equation of the conic section in the rotated $x'y'$-coordinate system.

Example 2 Write an Equation in Standard Form

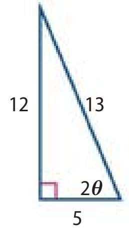
Using a suitable angle of rotation for the conic with equation $8x^2 + 12xy + 3y^2 = 4$, write the equation in standard form.

The conic is a hyperbola because $B^2 - 4AC > 0$. Find θ.

$$\cot 2\theta = \frac{A - C}{B} \qquad \text{Rotation of the axes}$$

$$= \frac{5}{12} \qquad A = 8,\ B = 12, \text{ and } C = 3$$

The figure illustrates a triangle for which $\cot 2\theta = \frac{5}{12}$. From this, $\sin 2\theta = \frac{12}{13}$ and $\cos 2\theta = \frac{5}{13}$.



Use the half-angle identities to determine $\sin \theta$ and $\cos \theta$.

Half-Angle Identities

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \qquad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$

$\cos 2\theta = \frac{5}{13}$

$$= \sqrt{\frac{1 - \frac{5}{13}}{2}} \qquad = \sqrt{\frac{1 + \frac{5}{13}}{2}}$$

Simplify.

$$= \frac{2\sqrt{13}}{13} \qquad = \frac{3\sqrt{13}}{13}$$

StudyTip

x′y′ **Term** When you correctly substitute values of x' and y' in for x and y, the coefficient of the $x'y'$ term will become zero. If the coefficient of this term is not zero, then an error has occurred.

Next, find the equations for x and y.

Rotation equations for x and y

$$x = x' \cos \theta - y' \sin \theta \qquad y = x' \sin \theta + y' \cos \theta$$

$\sin \theta = \frac{2\sqrt{13}}{13}$ and $\cos \theta = \frac{3\sqrt{13}}{13}$

$$= \frac{3\sqrt{13}}{13}x' - \frac{2\sqrt{13}}{13}y' \qquad = \frac{2\sqrt{13}}{13}x' + \frac{3\sqrt{13}}{13}y'$$

Simplify.

$$= \frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13} \qquad = \frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}$$

Substitute these values into the original equation.

$$8x^2 + 12xy + 3y^2 = 4$$

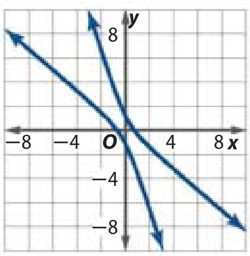
$$8\left(\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13}\right)^2 + 12\,\frac{3\sqrt{13}x' - 2\sqrt{13}y'}{13} \cdot \frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13} + 3\left(\frac{2\sqrt{13}x' + 3\sqrt{13}y'}{13}\right)^2 = 4$$

$$\frac{72(x')^2 - 96x'y' + 32(y')^2}{13} + \frac{72(x')^2 + 60x'y' - 72(y')^2}{13} + \frac{12(x')^2 + 36x'y' + 27(y')^2}{13} = 4$$

$$\frac{156(x')^2 - 13(y')^2}{13} = 4$$

$$3(x')^2 - \frac{(y')^2}{4} = 1$$

The standard form of the equation in the $x'y'$-plane is $\frac{(x')^2}{\frac{1}{3}} - \frac{(y')^2}{4} = 1$. The graph of this hyperbola is shown.



GuidedPractice

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form.

2A. $2x^2 - 12xy + 18y^2 - 4y = 2$

2B. $20x^2 + 20xy + 5y^2 - 12x - 36y - 200 = 0$

Two other formulas relating x' and y' to x and y can be used to find an equation in the xy-plane for a rotated conic.

KeyConcept Rotation of Axes of Conics

When an equation of a conic section is rewritten in the $x'y'$-plane by rotating the coordinate axes through θ, the equation in the xy-plane can be found using

$$x' = x\cos\theta + y\sin\theta, \text{ and } y' = y\cos\theta - x\sin\theta.$$

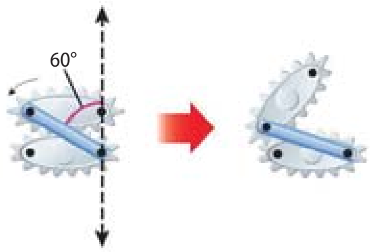
Real-WorldLink

In a system of gears where both gears spin, such as a bicycle, the speed of the gears in relation to each other is related to their size. If the diameter of one of the gears is $\frac{1}{2}$ of the diameter of the second gear, the first gear will rotate twice as fast as the second gear.

Source: *How Stuff Works*

Example 3 Write an Equation in the *xy*-Plane

PHYSICS Elliptical gears can be used to generate variable output speeds. After a 60° rotation, the equation for the rotated gear in the $x'y'$-plane is $\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1$. Write an equation for the ellipse formed by the rotated gear in the xy-plane.



Use the rotation formulas for x' and y' to find the equation of the rotated conic in the xy-plane.

$x' = x\cos\theta + y\sin\theta$	Rotation equations for x' and y'	$y' = y\cos\theta - x\sin\theta$
$= x\cos 60° + y\sin 60°$	$\theta = 60°$	$= y\cos 60° - x\sin 60°$
$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$\sin 60° = \frac{1}{2}$ and $\cos 60° = \frac{\sqrt{3}}{2}$	$= \frac{1}{2}y - \frac{\sqrt{3}}{2}x$

Substitute these values into the original equation.

$$\frac{(x')^2}{36} + \frac{(y')^2}{18} = 1 \quad \text{Original equation}$$

$$(x')^2 + 2(y')^2 = 36 \quad \text{Multiply each side by 36.}$$

$$\left(\frac{x + \sqrt{3}y}{2}\right)^2 + 2\left(\frac{y - \sqrt{3}x}{2}\right)^2 = 36 \quad \text{Substitute.}$$

$$\frac{x^2 + 2\sqrt{3}xy + 3y^2}{4} + \frac{2y^2 - 4\sqrt{3}xy + 6x^2}{4} = 36 \quad \text{Simplify.}$$

$$\frac{7x^2 - 2\sqrt{3}xy + 5y^2}{4} = 36 \quad \text{Combine like terms.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 = 144 \quad \text{Multiply each side by 4.}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0 \quad \text{Subtract 144 from each side.}$$

The equation of the rotated ellipse in the xy-plane is $7x^2 - 2\sqrt{3}xy + 5y^2 - 144 = 0$.

GuidedPractice

3. If the equation for the gear after a 30° rotation in the $x'y'$-plane is $(x')^2 + 4(y')^2 - 40 = 0$, find the equation for the gear in the xy-plane.

2 Graph Rotated Conics

When the equations of rotated conics are given for the $x'y'$-plane, they can be graphed by finding points on the graph of the conic and then converting these points to the xy-plane.

PhotoStock-Israel/Alamy

Example 4 Graph a Conic Using Rotations

Graph $(x'-2)^2 = 4(y'-3)$ if it has been rotated 30° from its position in the xy-plane.

The equation represents a parabola, and it is in standard form. Use the vertex (2, 3) and axis of symmetry $x' = 2$ in the $x'y'$-plane to determine the vertex and axis of symmetry for the parabola in the xy-plane.

Find the equations for x and y for $\theta = 30°$.

$x = x'\cos\theta - y'\sin\theta$ — Rotation equations for x and y — $y = x'\sin\theta + y'\cos\theta$

$= \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$ — $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$ — $= \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

Use the equations to convert the $x'y'$-coordinates of the vertex into xy-coordinates.

$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$ — Conversion equation — $y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

$= \frac{\sqrt{3}}{2}(2) - \frac{1}{2}(3)$ — $x' = 2$ and $y' = 3$ — $= \frac{1}{2}(2) + \frac{\sqrt{3}}{2}(3)$

$= \sqrt{3} - \frac{3}{2}$ or about 0.23 — Multiply. — $= 1 + \frac{3\sqrt{3}}{2}$ or about 3.60

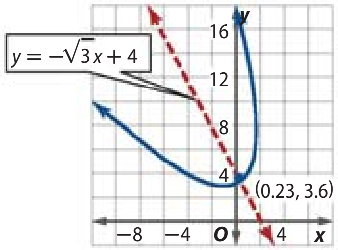
Find the equation for the axis of symmetry.

$x' = x\cos\theta + y\sin\theta$ — Conversion equation

$2 = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$ — $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$

$y = -\sqrt{3}x + 4$ — Solve for y.

The new vertex and axis of symmetry can be used to sketch the graph of the parabola in the xy-plane.



Study Tip

Graphing Convert other points on the conic from $x'y'$-coordinates to xy-coordinates. Then make a table of these values to complete the sketch of the conic.

Guided Practice

Graph each equation at the indicated angle.

4A. $\frac{(x')^2}{9} - \frac{(y')^2}{32} = 1$; 60°

4B. $\frac{(x')^2}{16} + \frac{(y')^2}{25} = 1$; 30°

One method of graphing conic sections with an xy-term is to solve the equation for y and graph with a calculator. Write the equation in quadratic form and then use the Quadratic Formula.

Example 5 Graph a Conic in Standard Form

Use a graphing calculator to graph the conic given by $4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0$.

$4y^2 + 8xy - 60y + 2x^2 - 40x + 155 = 0$ — Original equation

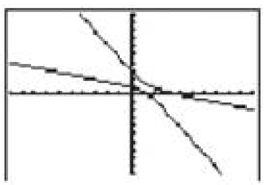
$4y^2 + (8x - 60)y + (2x^2 - 40x + 155) = 0$ — Quadratic form

$y = \frac{-(8x-60) \pm \sqrt{(8x-60)^2 - 4(4)(2x^2-40x+155)}}{2(4)}$ — $a = 4$, $b = 8x - 60$, and $c = 2x^2 - 40x + 155$

$= \frac{-8x + 60 \pm \sqrt{32x^2 - 320x + 1120}}{8}$ — Multiply and combine like terms.

$= \frac{-8x + 60 \pm 4\sqrt{2x^2 - 20x + 70}}{8}$ — Factor out $\sqrt{16}$.

$= \frac{-2x + 15 \pm \sqrt{2x^2 - 20x + 70}}{2}$ — Divide each term by 4.



[−40, 40] scl: 4 by [−40, 40] scl: 4

Graphing both of these equations on the same screen yields the hyperbola shown.

Study Tip

Arranging Terms Arrange the terms in descending powers of y in order to convert the equation to quadratic form.

Guided Practice

5. Use a graphing calculator to graph the conic given by $4x^2 - 6xy + 2y^2 - 60x - 20y + 275 = 0$.

Exercises

= Step-by-Step Solutions begin on page R29.

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic. (Example 1)

1. $x^2 - y^2 = 9,\ \theta = \frac{\pi}{3}$
2. $xy = -8,\ \theta = 45°$
3. $x^2 - 8y = 0,\ \theta = \frac{\pi}{2}$
4. $2x^2 + 2y^2 = 8,\ \theta = \frac{\pi}{6}$
5. $y^2 + 8x = 0,\ \theta = 30°$
6. $4x^2 + 9y^2 = 36,\ \theta = 30°$
7. $x^2 - 5x + y^2 = 3,\ \theta = 45°$
8. $49x^2 - 16y^2 = 784,\ \theta = \frac{\pi}{4}$
9. $4x^2 - 25y^2 = 64,\ \theta = 90°$
10. $6x^2 + 5y^2 = 30,\ \theta = 30°$

Using a suitable angle of rotation for the conic with each given equation, write the equation in standard form. (Example 1)

11. $xy = -4$
12. $x^2 - xy + y^2 = 2$
13. $145x^2 + 120xy + 180y^2 = 900$
14. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$
15. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$
16. $x^2 - 3y^2 - 8x + 30y = 60$
17. $8x^2 + 12xy + 3y^2 + 4 = 6$
18. $73x^2 + 72xy + 52y^2 + 25x + 50y - 75 = 0$

Write an equation for each conic in the xy-plane for the given equation in $x'y'$ form and the given value of θ. (Example 3)

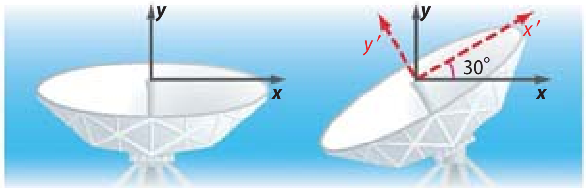
19. $(x')^2 + 3(y')^2 = 8,\ \theta = \frac{\pi}{4}$
20. $\frac{(x')^2}{25} - \frac{(y')^2}{225} = 1,\ \theta = \frac{\pi}{4}$
21. $\frac{(x')^2}{9} - \frac{(y')^2}{36} = 1,\ \theta = \frac{\pi}{3}$
22. $(x')^2 = 8y',\ \theta = 45°$
23. $\frac{(x')^2}{7} + \frac{(y')^2}{28} = 1,\ \theta = \frac{\pi}{6}$
24. $4x' = (y')^2,\ \theta = 30°$
25. $\frac{(x')^2}{64} - \frac{(y')^2}{16} = 1,\ \theta = 45°$
26. $(x')^2 = 5y',\ \theta = \frac{\pi}{3}$
27. $\frac{(x')^2}{4} - \frac{(y')^2}{9} = 1,\ \theta = 30°$
28. $\frac{(x')^2}{3} + \frac{(y')^2}{4} = 1,\ \theta = 60°$

29. **ASTRONOMY** Suppose $144(x')^2 + 64(y')^2 = 576$ models the shape in the $x'y'$-plane of a reflecting mirror in a telescope. (Example 4)
    a. If the mirror has been rotated 30°, determine the equation of the mirror in the xy-plane.
    b. Graph the equation.

Graph each equation at the indicated angle.

30. $\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1;\ 60°$
31. $\frac{(x')^2}{25} - \frac{(y')^2}{36} = 1;\ 45°$
32. $(x')^2 + 6x' - y' = -9;\ 30°$
33. $8(x')^2 + 6(y')^2 = 24;\ 30°$
34. $\frac{(x')^2}{4} - \frac{(y')^2}{16} = 1;\ 45°$
35. $y' = 3(x')^2 - 2x' + 5;\ 60°$

36. **COMMUNICATION** A satellite dish tracks a satellite directly overhead. Suppose $y = \frac{1}{6}x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 30°. (Example 4)
    a. Write an equation that models the new orientation of the dish.
    b. Use a graphing calculator to graph both equations on the same screen. Sketch this graph on your paper.



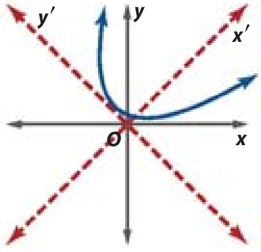
GRAPHING CALCULATOR **Graph the conic given by each equation.** (Example 5)

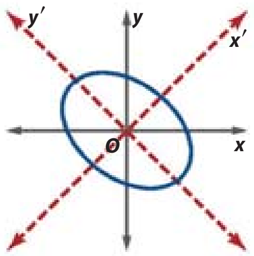
37. $x^2 - 2xy + y^2 - 5x - 5y = 0$
38. $2x^2 + 9xy + 14y^2 = 5$
39. $8x^2 + 5xy - 4y^2 = -2$
40. $2x^2 + 4\sqrt{3}xy + 6y^2 + 3x = y$
41. $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y = -12$
42. $9x^2 + 4xy + 6y^2 = 20$
43. $x^2 + 10\sqrt{3}xy + 11y^2 - 64 = 0$
44. $x^2 + y^2 - 4 = 0$
45. $x^2 - 2\sqrt{3}xy - y^2 + 18 = 0$
46. $2x^2 + 9xy + 14y^2 - 5 = 0$

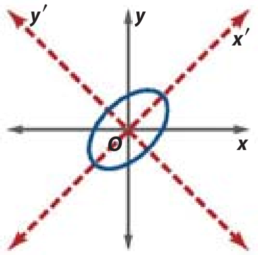
The graph of each equation is a degenerate case. Describe the graph.

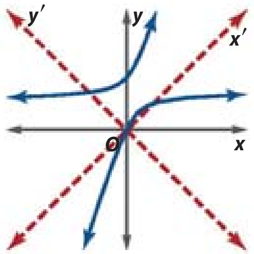
47. $y^2 - 16x^2 = 0$
48. $(x + 4)^2 - (x - 1)^2 = y + 8$
49. $(x + 3)^2 + y^2 + 6y + 9 - 6(x + y) = 18$

Match the graph of each conic with its equation.

50. 

51. 

52. 

53. 

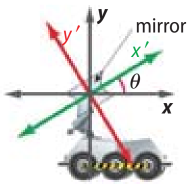
a. $x^2 - xy + y^2 = 2$

b. $145x^2 + 120xy + 180y^2 - 900 = 0$

c. $2x^2 - 72xy + 23y^2 + 100x - 50y = 0$

d. $16x^2 - 24xy + 9y^2 - 5x - 90y + 25 = 0$

54. **ROBOTICS** A hyperbolic mirror used in robotic systems is attached to the robot so that it is facing to the right. After it is rotated, the shape of its new position is represented by $51.75x^2 + 184.5\sqrt{3}xy - 132.75y^2 = 32{,}400$.



a. Solve the equation for y.

b. Use a graphing calculator to graph the equation.

c. Determine the angle θ through which the mirror has been rotated. Round to the nearest degree.

55. **INVARIANTS** When a rotation transforms an equation from the xy-plane to the $x'y'$-plane, the new equation is equivalent to the original equation. Some values are invariant under the rotation, meaning their values do not change when the axes are rotated. Use reasoning to explain how $A + C = A' + C'$ is a rotation invariant.

GRAPHING CALCULATOR **Graph each pair of equations and find any points of intersection. If the graphs have no points of intersection, write *no solution*.**

56. $x^2 + 2xy + y^2 - 8x - y = 0$
$8x^2 + 3xy - 5y^2 = 15$

57. $9x^2 + 4xy + 5y^2 - 40 = 0$
$x^2 - xy - 2y^2 - x - y + 2 = 0$

58. $x^2 + \sqrt{3}xy - 3 = 0$
$16x^2 - 20xy + 9y^2 = 40$

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate angles of rotation that produce the original graphs.

a. TABULAR For each equation in the table, identify the conic and find the minimum angle of rotation needed to transform the equation so that the rotated graph coincides with its original graph.

Equation	Conic	Minimum Angle of Rotation
$x^2 - 5x + 3 - y = 0$		
$6x^2 + 10y^2 = 15$		
$2xy = 9$		

b. VERBAL Describe the relationship between the lines of symmetry of the conics and the minimum angles of rotation needed to produce the original graphs.

c. ANALYTICAL A noncircular ellipse is rotated 50° about the origin. It is then rotated again so that the original graph is produced. What is the second angle of rotation?

H.O.T. Problems Use Higher-Order Thinking Skills

60. **ERROR ANALYSIS** Leon and Dario are describing the graph of $x^2 + 4xy + 6y^2 + 3x - 4y = 75$. Leon says that it is an ellipse. Dario thinks it is a parabola. Is either of them correct? Explain your reasoning.

61. **CHALLENGE** Show that a circle with the equation $x^2 + y^2 = r^2$ remains unchanged under any rotation θ.

62. **REASONING** *True* or *false*: Every angle of rotation θ can be described as an acute angle. Explain.

63. **PROOF** Prove $x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$. (*Hint:* Solve the system $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ by multiplying one equation by $\sin\theta$ and the other by $\cos\theta$.)

64. **REASONING** The angle of rotation θ can also be defined as $\tan 2\theta = \frac{B}{A - C}$, when $A \neq C$, or $\theta = \frac{\pi}{4}$, when $A = C$. Why does defining the angle of rotation in terms of cotangent not require an extra condition with an additional value for θ?

65. **WRITING IN MATH** The discriminant can be used to classify a conic $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ in the $x'y'$-plane. Explain why the values of A' and C' determine the type of conic. Describe the parameters necessary for an ellipse, a circle, a parabola, and a hyperbola.

66. **REASONING** *True* or *false*: Whenever the discriminant of an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is equal to zero, the graph of the equation is a parabola. Explain.

Spiral Review

Graph the hyperbola given by each equation. (Lesson 7-3)

67. $\frac{x^2}{9} - \frac{y^2}{64} = 1$

68. $\frac{y^2}{25} - \frac{x^2}{49} = 1$

69. $\frac{(x-3)^2}{64} - \frac{(y-7)^2}{25} = 1$

Determine the eccentricity of the ellipse given by each equation. (Lesson 7-2)

70. $\frac{(x+17)^2}{39} + \frac{(y+7)^2}{30} = 1$

71. $\frac{(x-6)^2}{12} + \frac{(y+4)^2}{15} = 1$

72. $\frac{(x-10)^2}{29} + \frac{(y+2)^2}{24} = 1$

73. INVESTING Randall has a total of $5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. Randall calculates that his interest earnings for the year will be $227.50. (Lesson 6-3)

a. Write a system of equations for the amount of money in each investment.

b. Use Cramer's Rule to determine how much money is in Randall's savings account and in the certificate of deposit.

74. OPTICS The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface that is R feet from a source of light with intensity I candelas is $E = \frac{I\cos\theta}{R^2}$, where θ is the measure of the angle between the direction of the light and a line perpendicular to the surface being illuminated.

Verify that $E = \frac{I\cot\theta}{R^2\csc\theta}$ is an equivalent formula. (Lesson 5-2)

Solve each equation. (Lesson 3-4)

75. $\log_4 8n + \log_4 (n-1) = 2$

76. $\log_9 9p + \log_9 (p+8) = 2$

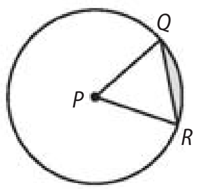
Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Lesson 2-3)

77. $f(x) = x^4 - x^3 - 16x^2 + 4x + 48$; $(x-4)$, $(x-2)$

78. $f(x) = 2x^4 + 9x^3 - 23x^2 - 81x + 45$; $(x+5)$, $(x+3)$

Skills Review for Standardized Tests

79. SAT/ACT P is the center of the circle and $PQ = QR$. If $\triangle PQR$ has an area of $9\sqrt{3}$ square units, what is the area of the shaded region in square units?



A $24\pi - 9\sqrt{3}$

B $9\pi - 9\sqrt{3}$

C $18\pi - 9\sqrt{3}$

D $6\pi - 9\sqrt{3}$

E $12\pi - 9\sqrt{3}$

80. REVIEW Which is NOT the equation of a parabola?

F $y = 2x^2 + 4x - 9$

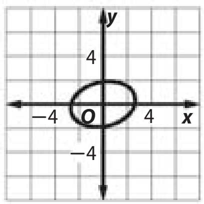
G $3x + 2y^2 + y + 1 = 0$

H $x^2 + 2y^2 + 8y = 8$

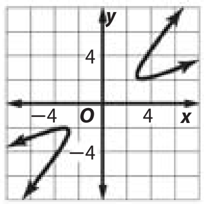
J $x = \frac{1}{2}(y-1)^2 + 5$

81. Which is the graph of the conic given by the equation $4x^2 - 2xy + 8y^2 - 7 = 0$?

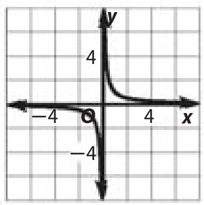
A



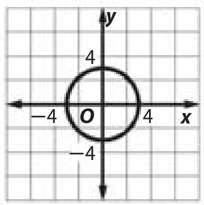
C



B



D



82. REVIEW How many solutions does the system $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$ and $(x-3)^2 + y^2 = 9$ have?

F 0

G 1

H 2

J 4

EXTEND 7-4

Graphing Technology Lab
Systems of Nonlinear Equations and Inequalities

Objective

- Use a graphing calculator to approximate solutions to systems of nonlinear equations and inequalities.

Graphs of conic sections represent a nonlinear system. Solutions of systems of nonlinear equations can be found algebraically. However, approximations can be found by using your graphing calculator. Graphing calculators can only graph functions. To graph a conic section that is not a function, solve the equation for *y*.

Activity 1 Nonlinear System

Solve the system by graphing.

$x^2 + y^2 = 13$

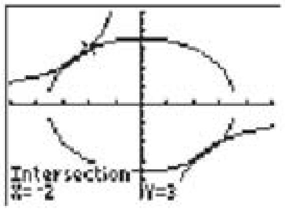
$xy + 6 = 0$

Step 1 Solve each equation for *y*.

$$y = \sqrt{13 - x^2} \text{ and } y = -\sqrt{13 - x^2} \qquad y = -\frac{6}{x}$$

Step 2 Graph the equations in the appropriate window.

Step 3 Use the intersect feature from the CALC menu to find the four points of intersection.



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The solutions are $(-3, 2)$, $(-2, 3)$, $(2, -3)$, and $(3, -2)$.

Exercises

Solve each system of equations by graphing. Round to the nearest tenth.

1. $xy = 2$
$x^2 - y^2 = 3$

2. $49 = y^2 + x^2$
$x = 1$

3. $x = 2 + y$
$x^2 + y^2 = 100$

4. $25 - 4x^2 = y^2$
$2x + y + 1 = 0$

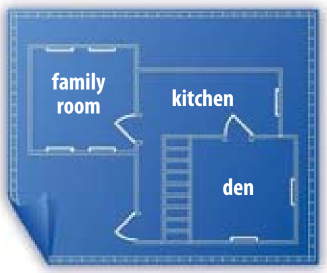
5. $y^2 = 9 - 3x^2$
$x^2 = 10 - 2y^2$

6. $y = -1 - x$
$4 + x = (y - 1)^2$

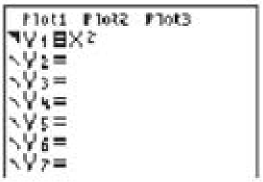
7. CHALLENGE A house contains two square rooms, the family room and the den. The total area of the two rooms is 468 square feet, and the den is 180 square feet smaller than the family room.

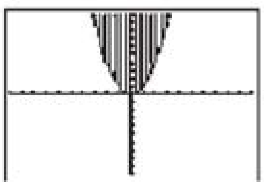
a. Write a system of second-degree equations that models this situation.

b. Graph the system found in part **a**, and estimate the length of each room.



Systems of nonlinear inequalities can also be solved using a graphing calculator. Recall from Chapter 1 that inequalities can be graphed by using the *greater than* and *less than* commands from the **Y=** menu. An inequality symbol is found by scrolling to the left of the equal sign and pressing ENTER until the shaded triangles are flashing. The triangle above represents *greater than* and the triangle below represents *less than.* The graph of $y \geq x^2$ is shown below.





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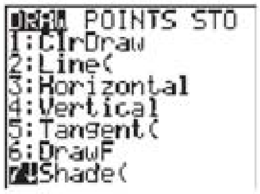
Inequalities with conic sections that are not functions, such as ellipses, circles, and some hyperbolas, can be graphed by using the Shade(command from the DRAW menu. The restrictive information required is Shade(*lowerfunc, upperfunc, Xleft, Xright, 3, 4*).



This command draws the lower function *lowerfunc* and the upper function *upperfunc* in terms of x. It then shades the area that is above *lowerfunc* and below *upperfunc* between the left and right boundaries *Xleft* and *Xright*. The final two entries *3* and *4* specify the type of shading and can remain constant.

TechnologyTip

Clear Screen To clear any drawings from the calculator screen, select ClrDraw from the DRAW menu.

Activity 2 Nonlinear System of Inequalities

Solve the system of inequalities by graphing.

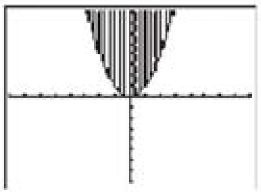
$x^2 + y^2 \leq 36$

$y - x^2 > 0$

Step 1 Solve each inequality for y.

$y \leq \sqrt{36 - x^2}$ and $y \geq -\sqrt{36 - x^2}$ $\qquad$ $y > x^2$

Step 2 Graph $y > x^2$, and shade the correct region. Make each inequality symbol by scrolling to the left of the equal sign and selecting ENTER until the shaded triangles are flashing.



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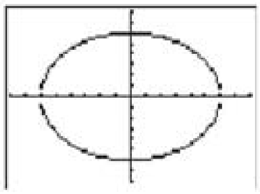
StudyTip

Left and Right Boundaries If the left and right boundaries are not apparent, enter window values that exceed both boundaries. For example, if the boundaries should be $x = -5$ and $x = 5$, entering −10 and 10 will still produce the correct graph.

Step 3 To graph $x^2 + y^2 \leq 36$, the lower boundary is $y = -\sqrt{36 - x^2}$ and the upper boundary is $y = \sqrt{36 - x^2}$. The two halves of the circle meet at $x = -6$ and $x = 6$ as shown.



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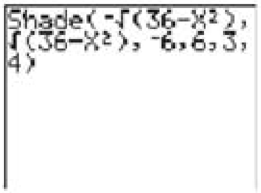
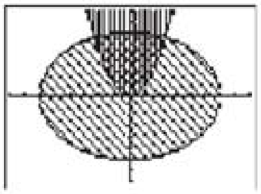
Step 4 From the DRAW menu, select 7: Shade. Enter Shade($-\sqrt{36 - x^2}$, $\sqrt{36 - x}$, −6, 6, 3, 4).



The solution of the system is represented by the double-shaded area.



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Exercises

Solve each system of inequalities by graphing.

8. $2y^2 \leq 32 - 2x^2$
$x + 4 \geq y^2$

9. $y + 5 \geq x^2$
$9y^2 \leq 36 + x^2$

10. $x^2 + 4y^2 \leq 32$
$4x^2 + y^2 \leq 32$

LESSON 7-5

Parametric Equations

Then	Now	Why?
You modeled motion using quadratic functions. (Lesson 1-5)	1 Graph parametric equations. 2 Solve problems related to the motion of projectiles.	You have used quadratic functions to model the paths of projectiles such as a tennis ball. Parametric equations can also be used to model and evaluate the trajectory and range of projectiles.

NewVocabulary
parametric equation
parameter
orientation
parametric curve

1 Graph Parametric Equations

So far in this text, you have represented the graph of a curve in the xy-plane using a single equation involving two variables, x and y. In this lesson you represent some of these same graphs using two equations by introducing a third variable.

Consider the graphs below, each of which models different aspects of what happens when a certain object is thrown into the air. Figure 7.5.1 shows the vertical distance the object travels as a function of time, while Figure 7.5.2 shows the object's horizontal distance as a function of time. Figure 7.5.3 shows the object's vertical distance as a function of its horizontal distance.

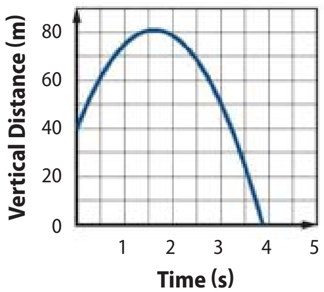


Figure 7.5.1

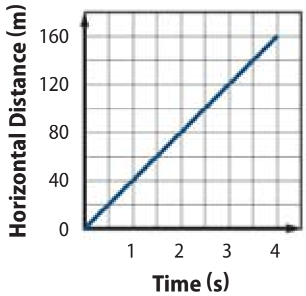


Figure 7.5.2

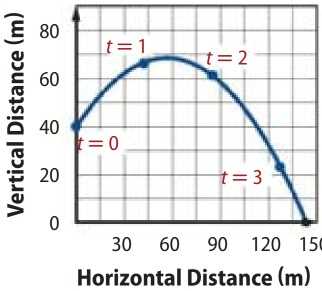


Figure 7.5.3

Each of these graphs and their equations tells part of what is happening in this situation, but not the whole story. To express the position of the object, both horizontally and vertically, as a function of time we can use **parametric equations**. The equations below both represent the graph shown in Figure 7.5.3.

Rectangular Equation

$y = -\frac{2}{225}x^2 + x + 40$

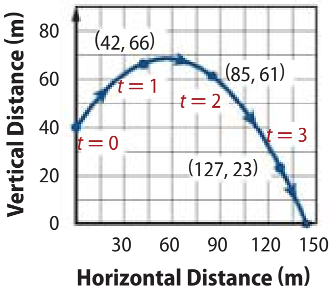
Parametric Equations

$x = 30\sqrt{2}t$ Horizontal component

$y = -16t^2 + 30\sqrt{2}t + 40$ Vertical component

From the parametric equations, we can now determine where the object was at a given time by evaluating the horizontal and vertical components for t. For example, when $t = 0$, the object was at (0, 40). The variable t is called a **parameter**.

The graph shown is plotted over the time interval $0 \leq t \leq 4$. Plotting points in the order of increasing values of t traces the curve in a specific direction called the **orientation** of the curve. This orientation is indicated by arrows on the curve as shown.



KeyConcept Parametric Equations

If f and g are continuous functions of t on the interval I, then the set of ordered pairs $(f(t), g(t))$ represent a **parametric curve**. The equations

$$x = f(t) \text{ and } y = g(t)$$

are parametric equations for this curve, t is the parameter, and I is the parameter interval.

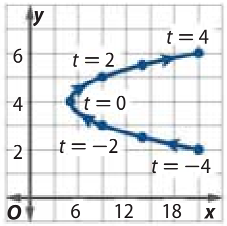
Example 1 Sketch Curves with Parametric Equations

Sketch the curve given by each pair of parametric equations over the given interval.

a. $x = t^2 + 5$ **and** $y = \frac{t}{2} + 4$; $-4 \leq t \leq 4$

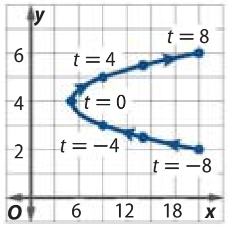
Make a table of values for $-4 \leq t \leq 4$. Then, plot the (x, y) coordinates for each t-value and connect the points to form a smooth curve. The arrows in the graph indicate the orientation of the curve as t moves from -4 to 4.

t	x	y	t	x	y
−4	21	2	1	6	4.5
−3	14	2.5	2	9	5
−2	9	3	3	14	5.5
−1	6	3.5	4	21	6
0	5	4			



b. $x = \frac{t^2}{4} + 5$ **and** $y = \frac{t}{4} + 4$; $-8 \leq t \leq 8$

t	x	y	t	x	y
−8	21	2	2	6	4.5
−6	14	2.5	4	9	5
−4	9	3	6	14	5.5
−2	6	3.5	8	21	6
0	5	4			



Guided Practice

1A. $x = 3t$ and $y = \sqrt{t} + 6$; $0 \leq t \leq 8$

1B. $x = t^2$ and $y = 2t + 3$; $-10 \leq t \leq 10$

> **Study Tip**
> **Plane Curves** Parametric equations can be used to represent curves that are not functions, as shown in Example 1.

Notice that the two different sets of parametric equations in Example 1 trace out the same curve. The graphs differ in their *speeds* or how rapidly each curve is traced out. If t represents time in seconds, then the curve in part **b** is traced in 16 seconds, while the curve in part **a** is traced out in 8 seconds.

Another way to determine the curve represented by a set of parametric equations is to write the set of equations in rectangular form. This can be done using substitution to eliminate the parameter.

Example 2 Write in Rectangular Form

Write $x = -3t$ and $y = t^2 + 2$ in rectangular form.

To eliminate the parameter t, solve $x = -3t$ for t. This yields $t = -\frac{1}{3}x$. Then substitute this value for t in the equation for y.

$$y = t^2 + 2 \quad \text{Equation for } y$$

$$= \left(-\frac{1}{3}x\right)^2 + 2 \quad \text{Substitute } -\frac{1}{3}x \text{ for } t.$$

$$= \frac{1}{9}x^2 + 2 \quad \text{Simplify.}$$

This set of parametric equations yields the parabola $y = \frac{1}{9}x^2 + 2$.

Guided Practice

2. Write $x = t^2 - 5$ and $y = 4t$ in rectangular form.

> **Study Tip**
> **Eliminating a Parameter** When you are eliminating a parameter to convert to rectangular form, you can solve either of the parametric equations first.

In Example 2, notice that a parameter interval for t was not specified. When not specified, the implied parameter interval is all values for t which produce real number values for x and y.

Sometimes the domain must be restricted after converting from parametric to rectangular form.

Example 3 Rectangular Form with Domain Restrictions

Write $x = \frac{1}{\sqrt{t}}$ and $y = \frac{t+1}{t}$ in rectangular form. Then graph the equation. State any restrictions on the domain.

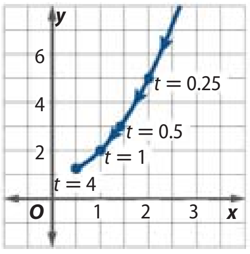
To eliminate t, square each side of $x = \frac{1}{\sqrt{t}}$. This yields $x^2 = \frac{1}{t}$, so $t = \frac{1}{x^2}$. Substitute this value for t in parametric equation for y.

$$y = \frac{t+1}{t} \qquad \text{Parametric equation for } y$$

$$= \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}} \qquad \text{Substitute } \frac{1}{x^2} \text{ for } t.$$

$$= \frac{\frac{x^2+1}{x^2}}{\frac{1}{x^2}} \qquad \text{Simplify the numerator.}$$

$$= x^2 + 1 \qquad \text{Simplify.}$$



While the rectangular equation is $y = x^2 + 1$, the curve is only defined for $t > 0$. From the parametric equation $x = \frac{1}{\sqrt{t}}$, the only possible values for x are values greater than zero. As shown in the graph, the domain of the rectangular equation needs to be restricted to $x > 0$.

GuidedPractice

3. Write $x = \sqrt{t+4}$ and $y = \frac{1}{t}$ in rectangular form. Graph the equation. State any restrictions on the domain.

The parameter in a parametric equation can also be an angle, θ.

Example 4 Rectangular Form with θ as Parameter

TechnologyTip

Parameters When graphing parametric equations on a calculator, θ and t are interchangeable.

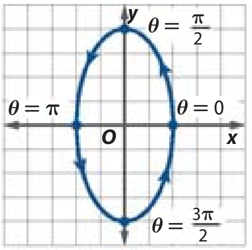
Write $x = 2\cos\theta$ and $y = 4\sin\theta$ in rectangular form. Then graph the equation.

To eliminate the angular parameter θ, first solve the equations for $\cos\theta$ and $\sin\theta$ to obtain $\cos\theta = \frac{x}{2}$ and $\sin\theta = \frac{y}{4}$. Then use the Pythagorean Identity to eliminate the parameter θ.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Pythagorean Identity}$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \cos\theta = \frac{x}{2} \text{ and } \sin\theta = \frac{y}{4}$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1 \qquad \text{Simplify.}$$



You should recognize this equation as that of an ellipse centered at the origin with vertices at (0, 4) and (0, −4) and covertices at (2, 0) and (−2, 0) as shown. As θ varies from 0 to 2π, the ellipse is traced out counterclockwise.

GuidedPractice

4. Write $x = 3\sin\theta$ and $y = 8\cos\theta$ in rectangular form. Then sketch the graph.

As you saw in Example 1, parametric representations of rectangular graphs are not unique. By varying the definition for the parameter, you can obtain parametric equations that produce graphs that vary only in speed and/or orientation.

StudyTip

Parametric Form The easiest method of converting an equation from rectangular to parametric form is to use $x = t$. When this is done, the other parametric equation is the original equation with t replacing x.

Example 5 Write Parametric Equations from Graphs

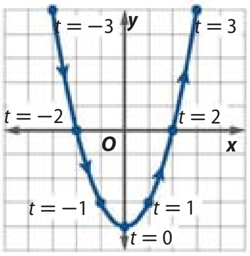
Use each parameter to write the parametric equations that can represent $y = x^2 - 4$. Then graph the equation, indicating the speed and orientation.

a. $t = x$

$y = x^2 - 4$ — Original equation

$\;\;= t^2 - 4$ — Substitute for x in original equation.

The parametric equations are $x = t$ and $y = t^2 - 4$. The associated speed and orientation are indicated on the graph.



b. $t = 4x + 1$

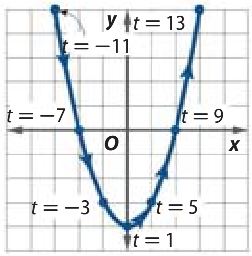
$x = \dfrac{t-1}{4}$ — Solve for x.

$y = \left(\dfrac{t-1}{4}\right)^2 - 4$ — Substitute for x in original equation.

$\;\;= \dfrac{t^2}{16} - \dfrac{t}{8} - \dfrac{63}{16}$ — Simplify.

$x = \dfrac{t-1}{4}$ and $y = \dfrac{t^2}{16} - \dfrac{t}{8} - \dfrac{63}{16}$ are the parametric equations.

Notice that the speed is much slower than part **a**.



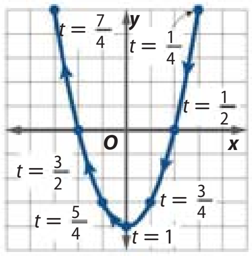
c. $t = 1 - \dfrac{x}{4}$

$4 - 4t = x$ — Solve for x.

$y = (4 - 4t)^2 - 4$ — Substitute for x in original equation.

$\;\;= 16t^2 - 32t + 12$ — Simplify.

The parametric equations are $x = 4 - 4t$ and $y = 16t^2 - 32t + 12$. Notice that the speed is much faster than part **a**. The orientation is also reversed, as indicated by the arrows.



GuidedPractice

Use each parameter to determine the parametric equations that can represent $x = 6 - y^2$. Then graph the equation, indicating the speed and orientation.

5A. $t = x + 1$ **5B.** $t = 3x$ **5C.** $t = 4 - 2x$

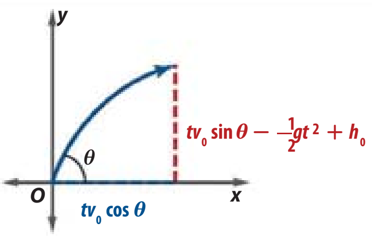
2 Projectile Motion

Parametric equations are often used to simulate projectile motion. The path of a projectile launched at an angle other than 90° with the horizontal can be modeled by the following parametric equations.

KeyConcept Projectile Motion

For an object launched at an angle θ with the horizontal at an initial velocity v_0, where g is the gravitational constant, t is time, and h_0 is the initial height:

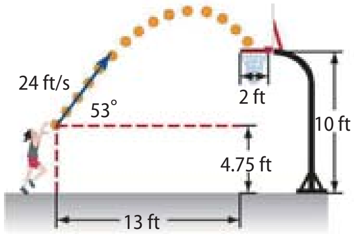
Horizontal Distance	$x = tv_0 \cos\theta$
Vertical Position	$y = tv_0 \sin\theta - \frac{1}{2}gt^2 + h_0$



Real-World Example 6 Projectile Motion

BASKETBALL Kaylee is practicing free throws for an upcoming basketball game. She releases the ball with an initial velocity of 24 feet per second at an angle of 53° with the horizontal. The horizontal distance from the free throw line to the front rim of the basket is 13 feet. The vertical distance from the floor to the rim is 10 feet. The front of the rim is 2 feet from the backboard. She releases the shot 4.75 feet from the ground. Does Kaylee make the basket?

Make a diagram of the situation.

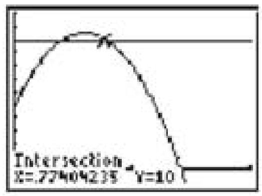


To determine whether she makes the shot, you need the horizontal distance that the ball has traveled when the height of the ball is 10 feet. First, write a parametric equation for the vertical position of the ball.

$y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$ — Parametric equation for vertical position

$= t(24) \sin 53 - \frac{1}{2}(32)t^2 + 4.75$ — $v_0 = 24$, $\theta = 53°$, $g = 32$, and $h_0 = 4.75$

Graph the equation for the vertical position and the line $y = 10$. The curve will intersect the line in two places. The second intersection represents the ball as it is moving down toward the basket. Use 5: intersect on the CALC menu to find the second point of intersection with $y = 10$. The value is about 0.77 second.



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Determine the horizontal position of the ball at 0.77 second.

$x = tv_0 \cos \theta$ — Parametric equation for horizontal position

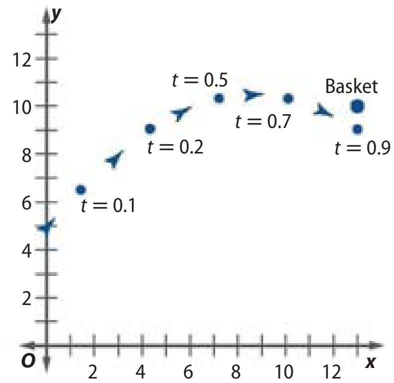
$= 0.77(24) \cos 53$ — $v_0 = 24$, $\theta = 53°$, and $t \approx 0.77$

≈ 11.1 — Use a calculator.

Because the horizontal position is less than 13 feet when the ball reaches 10 feet for the second time, the shot is short of the basket. Kaylee does not make the free throw.

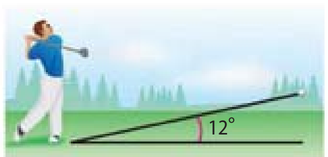
CHECK You can confirm the results of your calculation by graphing the parametric equations and determining the path of the ball in relation to the basket.

t	x	y	t	x	y
0	0	4.75	0.5	7.22	10.33
0.1	1.44	6.51	0.6	8.67	10.49
0.2	2.89	7.94	0.7	10.11	10.32
0.3	4.33	9.06	0.8	11.55	9.84
0.4	5.78	9.86	0.9	13.00	9.04



Guided Practice

6. GOLF Evan drives a golf ball with an initial velocity of 56 meters per second at an angle of 12° down a flat driving range. How far away will the golf ball land?



Study Tip

Gravity At the surface of Earth, the acceleration due to gravity is 9.8 meters per second squared or 32 feet per second squared. When solving problems, be sure to use the appropriate value for gravity based on the units of the velocity and position.

Real-World Link

In April 2007, Morgan Pressel became the youngest woman ever to win a major LPGA championship.

Source: *LPGA*

Exercises

= Step-by-Step Solutions begin on page R29.

Sketch the curve given by each pair of parametric equations over the given interval. (Example 1)

1. $x = t^2 + 3$ and $y = \frac{t}{4} - 5$; $-5 \le t \le 5$
2. $x = \frac{t^2}{2}$ and $y = -4t$; $-4 \le t \le 4$
3. $x = -\frac{5t}{2} + 4$ and $y = t^2 - 8$; $-6 \le t \le 6$
4. $x = 3t + 6$ and $y = \sqrt{t} + 1$; $0 \le t \le 9$
5. $x = 2t - 1$ and $y = -\frac{t^2}{2} + 7$; $-4 \le t \le 4$
6. $x = -2t^2$ and $y = \frac{t}{3} - 6$; $-6 \le t \le 6$
7. $x = \frac{t}{2}$ and $y = -\sqrt{t} + 5$; $0 \le t \le 8$
8. $x = t^2 - 4$ and $y = 3t - 8$; $-5 \le t \le 5$

Write each pair of parametric equations in rectangular form. Then graph the equation and state any restrictions on the domain. (Examples 2 and 3)

9. $x = 2t - 5$, $y = t^2 + 4$
10. $x = 3t + 9$, $y = t^2 - 7$
11. $x = t^2 - 2$, $y = 5t$
12. $x = t^2 + 1$, $y = -4t + 3$
13. $x = -t - 4$, $y = 3t^2$
14. $x = 5t - 1$, $y = 2t^2 + 8$
15. $x = 4t^2$, $y = \frac{6t}{5} + 9$
16. $x = \frac{t}{3} + 2$, $y = \frac{t^2}{6} - 7$

17. **MOVIE STUNTS** During the filming of a movie, a stunt double leaps off the side of a building. The pulley system connected to the stunt double allows for a vertical fall modeled by $y = -16t^2 + 15t + 100$, and a horizontal movement modeled by $x = 4t$, where x and y are measured in feet and t is measured in seconds. Write and graph an equation in rectangular form to model the stunt double's fall for $0 \le t \le 3$. (Example 3)

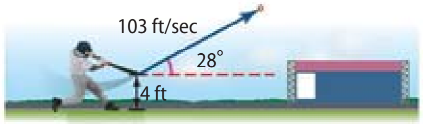
Write each pair of parametric equations in rectangular form. Then graph the equation. (Example 4)

18. $x = 3 \cos \theta$ and $y = 5 \sin \theta$
19. $x = 7 \sin \theta$ and $y = 2 \cos \theta$
20. $x = 6 \cos \theta$ and $y = 4 \sin \theta$
21. $x = 3 \cos \theta$ and $y = 3 \sin \theta$
22. $x = 8 \sin \theta$ and $y = \cos \theta$
23. $x = 5 \cos \theta$ and $y = 6 \sin \theta$
24. $x = 10 \sin \theta$ and $y = 9 \cos \theta$
25. $x = \sin \theta$ and $y = 7 \cos \theta$

Use each parameter to write the parametric equations that can represent each equation. Then graph the equations, indicating the speed and orientation. (Example 5)

26. $t = 3x - 2$; $y = x^2 + 9$
27. $t = 8x$; $y^2 = 9 - x^2$
28. $t = 2 - \frac{x}{3}$; $y = \frac{x^2}{12}$
29. $t = \frac{x}{5} + 4$; $y = 10 - x^2$
30. $t = 4x + 7$; $y = \frac{x^2 - 1}{2}$
31. $t = \frac{1 - x}{2}$; $y = \frac{3 - x^2}{4}$

32. **BASEBALL** A baseball player hits the ball at a 28° angle with an initial speed of 103 feet per second. The bat is 4 feet from the ground at the time of impact. Assuming that the ball is not caught, determine the distance traveled by the ball. (Example 6)



33. **FOOTBALL** Delmar attempts a 43-yard field goal. He kicks the ball at a 41° angle with an initial speed of 70 feet per second. The goal post is 15 feet high. Is the kick long enough to make the field goal? (Example 6)

Write each pair of parametric equations in rectangular form. Then state the restriction on the domain.

34. $x = \sqrt{t} + 4$; $y = 4t + 3$
35. $x = \log t$; $y = t + 3$
36. $x = \sqrt{t - 7}$; $y = -3t - 8$
37. $x = \log (t - 4)$; $y = t$
38. $x = \frac{1}{\sqrt{t + 3}}$; $y = t$
39. $x = \frac{1}{\log (t + 2)}$; $y = 2t - 4$

40. **TENNIS** Jill hits a tennis ball 55 centimeters above the ground at an angle of 15° with the horizontal. The ball has an initial speed of 18 meters per second.
    a. Use a graphing calculator to graph the path of the tennis ball using parametric equations.
    b. How long does the ball stay in the air before hitting the ground?
    c. If Jill is 10 meters from the net and the net is 1.5 meters above the ground, will the tennis ball clear the net? If so, by how many meters? If not, by how many meters is the ball short?

Write a set of parametric equations for the line or line segment with the given characteristics.

41. line with a slope of 3 that passes through (4, 7)
42. line with a slope of −0.5 that passes through (3, −2)
43. line segment with endpoints (−2, −6) and (2, 10)
44. line segment with endpoints (7, 13) and (13, 11)

Match each set of parametric equations with its graph.

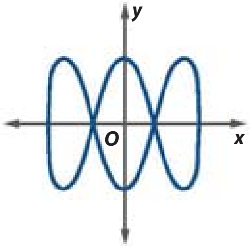
45. $x = \cos 2t$, $y = \sin 4t$

46. $x = \cos 3t$, $y = \sin t$

47. $x = \cos t$, $y = \sin 3t$

48. $x = \cos 4t$, $y = \sin 3t$

a.



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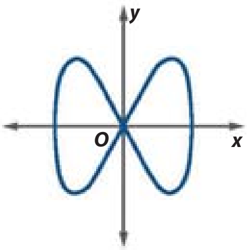
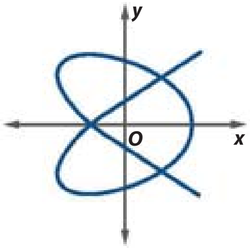
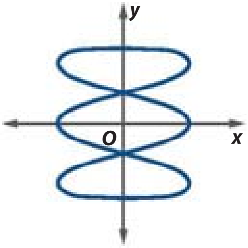


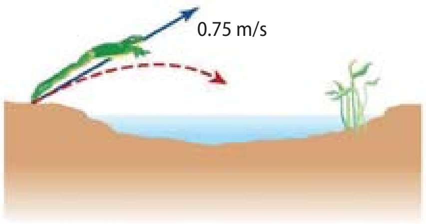
c.



d.



49. BIOLOGY A frog jumps off the bank of a creek with an initial velocity of 0.75 meter per second at an angle of 45° with the horizontal. The surface of the creek is 0.3 meter below the edge of the bank. Let g equal 9.8 meters per second squared.



a. Write the parametric equations to describe the position of the frog at time t. Assume that the surface of the water is located at the line $y = 0$.

b. If the creek is 0.5 meter wide, will the frog reach the other bank, which is level with the surface of the creek? If not, how far from the other bank will it hit the water?

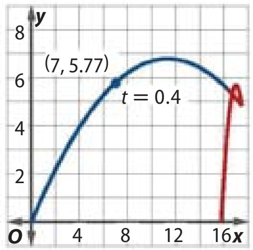
c. If the frog was able to jump on a lily pad resting on the surface of the creek 0.4 meter away and stayed in the air for 0.38 second, what was the initial speed of the frog?

50. RACE Luna and Ruby are competing in a 100-meter dash. When the starter gun fires, Luna runs 8.0 meters per second after a 0.1 second delay from the point (0, 2) and Ruby runs 8.1 meters per second after a 0.3 second delay from the point (0, 5).

a. Using the y-axis as the starting line and assuming that the women run parallel to the x-axis, write parametric equations to describe each runner's position after t seconds.

b. Who wins the race? If the women ran 200 meters instead of 100 meters, who would win? Explain your answer.

51. SOCCER The graph below models the path of a soccer ball kicked by one player and then headed back by another player. The path of the initial kick is shown in blue, and the path of the headed ball is shown in red.



a. If the ball is initially kicked at an angle of 50°, find the initial speed of the ball.

b. At what time does the ball reach the second player if the second player is standing about 17.5 feet away?

c. If the second player heads the ball at an angle of 75°, an initial speed of 8 feet per second, and at a height of 4.75 feet, approximately how long does the ball stay in the air from the time it is first kicked until it lands?

52. MULTIPLE REPRESENTATIONS In this problem, you will investigate a *cycloid*, the curve created by the path of a point on a circle with a radius of 1 unit as it is rolled along the x-axis.

a. GRAPHICAL Use a graphing calculator to graph the parametric equations $x = t - \sin t$ and $y = 1 - \cos t$, where t is measured in radians.

b. ANALYTICAL What is the distance between x-intercepts? Describe what the x-intercepts and the distance between them represent.

c. ANALYTICAL What is the maximum value of y? Describe what this value represents and how it would change for circles of differing radii.

H.O.T. Problems Use Higher-Order Thinking Skills

53. CHALLENGE Consider a line ℓ with parametric equations $x = 2 + 3t$ and $y = -t + 5$. Write a set of parametric equations for the line m perpendicular to ℓ containing the point (4, 10).

54. WRITING IN MATH Explain why there are infinitely many sets of parametric equations to describe one line in the xy-plane.

55. REASONING Determine whether parametric equations for projectile motion can apply to objects thrown at an angle of 90°. Explain your reasoning.

56. CHALLENGE A line in three-dimensional space contains the points $P(2, 3, -8)$ and $Q(-1, 5, -4)$. Find two sets of parametric equations for the line.

57. WRITING IN MATH Explain the advantage of using parametric equations versus rectangular equations when analyzing the horizontal/vertical components of a graph.

Spiral Review

Graph each equation at the indicated angle. (Lesson 7-4)

58. $\frac{(x')^2}{9} - \frac{(y')^2}{4} = 1$ at a 60° rotation from the xy-axis

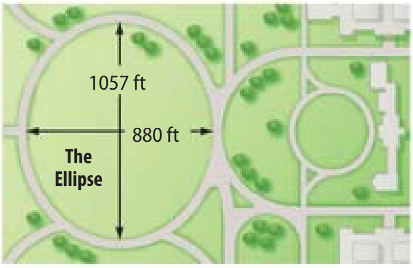
59. $(x')^2 - (y')^2 = 1$ at a 45° rotation from the xy-axis

Write an equation for the hyperbola with the given characteristics. (Lesson 7-3)

60. vertices $(5, 4)$, $(5, -8)$; conjugate axis length of 4

61. transverse axis length of 4; foci $(3, 5)$, $(3, -1)$

62. **WHITE HOUSE** There is an open area south of the White House known as The Ellipse. Write an equation to model The Ellipse. Assume that the origin is at its center. (Lesson 7-2)



Simplify each expression. (Lesson 5-1)

63. $\frac{\sin x}{\csc x - 1} + \frac{\sin x}{\csc x + 1}$

64. $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x}$

Use the properties of logarithms to rewrite each logarithm below in the form $a \ln 2 + b \ln 3$, where a and b are constants. Then approximate the value of each logarithm given that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$. (Lesson 3-3)

65. $\ln 54$

66. $\ln 24$

67. $\ln \frac{8}{3}$

68. $\ln \frac{9}{16}$

For each function, determine any asymptotes and intercepts. Then graph the function and state its domain. (Lesson 2-5)

69. $h(x) = \frac{x}{x + 6}$

70. $h(x) = \frac{x^2 + 6x + 8}{x^2 - 7x - 8}$

71. $f(x) = \frac{x^2 + 8x}{x + 5}$

72. $f(x) = \frac{x^2 + 4x + 3}{x^3 + x^2 - 6x}$

Solve each equation. (Lesson 2-1)

73. $\sqrt{3z - 5} - 3 = 1$

74. $\sqrt{5n - 1} = 0$

75. $\sqrt{2c + 3} - 7 = 0$

76. $\sqrt{4a + 8} + 8 = 5$

Skills Review for Standardized Tests

77. **SAT/ACT** With the exception of the shaded squares, every square in the figure contains the sum of the number in the square directly above it and the number in the square directly to its left. For example, the number 4 in the unshaded square is the sum of the 2 in the square above it and the 2 in the square directly to its left. What is the value of x?

0	1	2	3	4	5
1	2	4			
2					
3			x		
4					
5					

A 7 **B** 8 **C** 15 **D** 23 **E** 30

78. Jack and Graham are performing a physics experiment in which they will launch a model rocket. The rocket is supposed to release a parachute 300 feet in the air, 7 seconds after liftoff. They are firing the rocket at a 78° angle from the horizontal. To protect other students from the falling rockets, the teacher needs to place warning signs 50 yards from where the parachute is released. How far should the signs be from the point where the rockets are launched?

F 122 yards
G 127 yards
H 133 yards
J 138 yards

79. **FREE RESPONSE** An object moves along a curve according to $y = \frac{10\sqrt{3t} \pm \sqrt{496 - 2304t}}{62}$, $x = \sqrt{t}$.

a. Convert the parametric equations to rectangular form.
b. Identify the conic section represented by the curve.
c. Write an equation for the curve in the $x'y'$-plane, assuming it was rotated 30°.
d. Determine the eccentricity of the conic.
e. Identify the location of the foci in the $x'y'$-plane, if they exist.

EXTEND 7-5

Graphing Technology Lab
Modeling with Parametric Equations

Objective

- Use a graphing calculator to model functions parametrically.

As shown in Lesson 7-5, the independent variable *t* in parametric equations represents time. This parameter reflects the speed with which the graph is drawn. If one graph is completed for $0 \le t \le 5$, while an identical graph is completed for $0 \le t \le 10$, then the first graph is faster.

StudyTip

Setting Parameters Use the situation in the problem as a guide for setting the range of values for *x*, *y*, and *t*.

Activity Parametric Graph

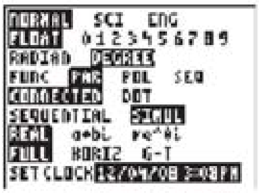
FOOTBALL Standing side by side, Neva and Owen throw a football at exactly the same time. Neva throws the ball with an initial velocity of 20 meters per second at 60°. Owen throws the ball 15 meters per second at 45°. Assuming that the footballs were thrown from the same initial height, simulate the throws on a graphing calculator.

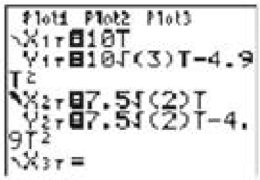
Step 1 The parametric equations for each throw are as follows.

Neva: $x = 20t \cos 60 = 10t$ $\qquad y = 20t \sin 60 - 4.9t^2 = 10\sqrt{3}t - 4.9t^2$

Owen: $x = 15t \cos 45 = 7.5\sqrt{2}t$ $\qquad y = 15t \sin 45 - 4.9t^2 = 7.5\sqrt{2}t - 4.9t^2$

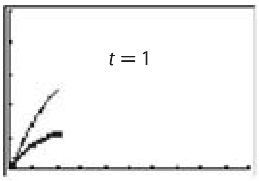
Step 2 Set the mode. In the MODE menu, select degree, par, and simul. This allows the equations to be graphed simultaneously. Enter the parametric equations. In parametric form, X,T,θ,n uses *t* instead of *x*. Set the second set of equations to shade dark to distinguish between the throws.





Step 3 Set the *t*-values to range from 0 to 8 as an estimate for the duration of the throws. Set tstep to 0.1 in order to watch the throws in the graph.

Step 4 Graph the equations.



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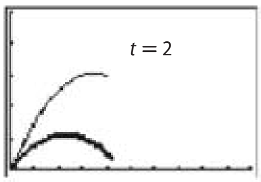

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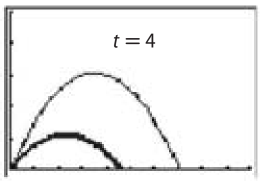

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Neva's throw goes higher and at a greater distance while Owen's lands first.

Exercises

1. **FOOTBALL** Owen's next throw is 21 meters per second at 50°. A second later, Neva throws her football 24 meters per second at 35°. Simulate the throws on a graphing calculator and interpret the results.

2. **BASEBALL** Neva throws a baseball 27 meters per second at 82°. A second later, Owen hits a ball 45 meters per second at 20°. Assuming they are still side by side and the initial height of the hit is one meter lower, simulate the situation on a graphing calculator and interpret the results.

Study Guide and Review

Chapter Summary

KeyConcepts

Parabolas (Lesson 7-1)

Equations	Orientation	Vertex	Focus
$(y-k)^2 = 4p(x-h)$	horizontal	(h, k)	$(h+p, k)$
$(x-h)^2 = 4p(y-k)$	vertical	(h, k)	$(h, k+p)$

- p is the distance from the vertex to the focus.

Ellipses and Circles (Lesson 7-2)

Equations	Major Axis	Vertex	Focus
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	horizontal	$(h \pm a, k)$	$(h \pm c, k)$
$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$	vertical	$(h, k \pm a)$	$(h, k \pm c)$

- The eccentricity of an ellipse is given by $e = \frac{c}{a}$, where $a^2 - b^2 = c^2$.
- The standard form of an equation for a circle with center (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$.

Hyperbolas (Lesson 7-3)

Equations	Transverse Axis	Vertex	Focus
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	horizontal	$(h \pm a, k)$	$(h \pm c, k)$
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	vertical	$(h, k \pm a)$	$(h, k \pm c)$

- The eccentricity of a hyperbola is given by $e = \frac{c}{a}$, where $a^2 + b^2 = c^2$.

Rotations of Conic Sections (Lesson 7-4)

- An equation in the xy-plane can be transformed to an equation in the $x'y'$-plane using $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.
- An equation in the $x'y'$ plane can be transformed to an equation in the xy plane using $x' = x\cos\theta + y\sin\theta$, and $y' = y\cos\theta - x\sin\theta$.

Parametric Equations (Lesson 7-5)

- Parametric equations are used to describe the horizontal and vertical components of an equation separately, generally in terms of the parameter t.
- For an object launched at an angle θ with the horizontal at an initial velocity of v_0, where g is the gravitational constant, t is time, and h_0 is the initial height, its horizontal distance is $x = tv_0 \cos\theta$, and its vertical distance is $y = tv_0 \sin\theta - \frac{1}{2}gt^2 + h_0$.

KeyVocabulary

axis of symmetry (p. 422)
center (p. 432)
conic section (p. 422)
conjugate axis (p. 442)
co-vertices (p. 432)
degenerate conic (p. 422)
directrix (p. 422)
eccentricity (p. 435)
ellipse (p. 432)
foci (p. 432)
focus (p. 422)
hyperbola (p. 442)
locus (p. 422)
major axis (p. 432)
minor axis (p. 432)
orientation (p. 464)
parabola (p. 422)
parameter (p. 464)
parametric curve (p. 464)
parametric equation (p. 464)
transverse axis (p. 442)
vertex (p. 422)
vertices (p. 432)

VocabularyCheck

Choose the correct term from the list above to complete each sentence.

1. A ________ is a figure formed when a plane intersects a double-napped right cone.
2. A circle is the ________ of points that fulfill the property that all points be in a given plane and a specified distance from a given point.
3. The ________ of a parabola is perpendicular to its axis of symmetry.
4. The co-vertices of a(n) ________ lie on its minor axis, while the vertices lie on its major axis.
5. From any point on an ellipse, the sum of the distances to the ________ of the ellipse remains constant.
6. The ________ of an ellipse is a ratio that determines how “stretched” or “circular” the ellipse is. It is found using the ratio $\frac{c}{a}$.
7. The ________ of a circle is a single point, and all points on the circle are equidistant from that point.
8. Like an ellipse, a ________ has vertices and foci, but it also has a pair of asymptotes and does not have a connected graph.
9. The equation for a graph can be written using the variables x and y, or using ________ equations, generally using t or the angle θ.
10. The graph of $f(t) = (\sin t, \cos t)$ is a ________ with a shape that is a circle traced clockwise.

Lesson-by-Lesson Review

7-1 Parabolas (pp. 422–431)

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

11. $(x + 3)^2 = 12(y + 2)$

12. $(y - 2)^2 = 8(x - 5)$

13. $(x - 2)^2 = -4(y + 1)$

14. $(x - 5) = \frac{1}{12}(y - 3)^2$

Write an equation for and graph a parabola with the given focus F and vertex V.

15. $F(1, 1)$, $V(1, 5)$

16. $F(-3, 6)$, $V(7, 6)$

17. $F(-2, -3)$, $V(-2, 1)$

18. $F(3, -4)$, $V(3, -2)$

Write an equation for and graph each parabola with focus F and the given characteristics.

19. $F(-4, -4)$; concave left; contains $(-7, 0)$

20. $F(-1, 4)$; concave down; contains $(7, -2)$

21. $F(3, -6)$; concave up; contains $(9, 2)$

Example 1

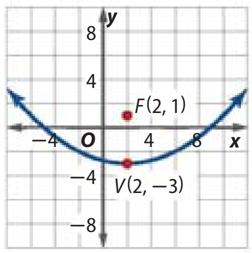
Write an equation for and graph the parabola with focus (2, 1) and vertex (2, −3).

Since the focus and vertex share the same x-coordinate, the graph opens vertically. The focus is $(h, k + p)$, so the value of p is $1 - (-3)$ or 4. Because p is positive, the graph opens up.

Write an equation for the parabola in standard form using the values of h, p, and k.

$4p(y - k) = (x - h)^2$	Standard form
$4(4)(y + 3) = (x - 2)^2$	$p = 4$, $k = -3$, and $h = 2$
$16(y + 3) = (x - 2)^2$	Simplify.

The standard form of the equation is $(x - 2)^2 = 16(y + 3)$. Graph the vertex and focus. Use a table of values to graph the parabola.



7-2 Ellipses and Circles (pp. 432–441)

Graph the ellipse given by each equation.

22. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

23. $\frac{(x - 3)^2}{16} + \frac{(y + 6)^2}{25} = 1$

Write an equation for the ellipse with each set of characteristics.

24. vertices $(7, -3)$, $(3, -3)$; foci $(6, -3)$, $(4, -3)$

25. foci $(1, 2)$, $(9, 2)$; length of minor axis equals 6

26. major axis $(-4, 4)$ to $(6, 4)$; minor axis $(1, 1)$ to $(1, 7)$

Write an equation in standard form. Identify the related conic.

27. $x^2 - 2x + y^2 - 4y - 25 = 0$

28. $4x^2 + 24x + 25y^2 - 300y + 836 = 0$

29. $x^2 - 4x + 4y + 24 = 0$

Example 2

Write an equation for the ellipse with a major axis from (−9, 4) to (11, 4) and a minor axis from (1, 12) to (1, −4).

Use the major and minor axes to determine a and b.

Half length of major axis	Half length of minor axis
$a = \frac{11 - (-9)}{2}$ or 10	$b = \frac{12 - (-4)}{2}$ or 8

The center of the ellipse is at the midpoint of the major axis.

$(h, k) = \left(\frac{11 + (-9)}{2}, \frac{4 + 4}{2}\right)$	Midpoint Formula
$= (1, 4)$	Simplify.

The y-coordinates are the same for both endpoints of the major axis, so the major axis is horizontal and the value of a belongs with the x^2 term. Therefore, the equation of the ellipse is $\frac{(x - 1)^2}{100} + \frac{(y - 4)^2}{64} = 1$.

7-3 Hyperbolas (pp. 442–452)

Graph the hyperbola given by each equation.

30. $\frac{(y+3)^2}{30} - \frac{(x-6)^2}{8} = 1$

31. $\frac{(x+7)^2}{18} - \frac{(y-6)^2}{36} = 1$

32. $\frac{(y-1)^2}{4} - (x+1)^2 = 1$

33. $x^2 - y^2 - 2x + 4y - 7 = 0$

Write an equation for the hyperbola with the given characteristics.

34. vertices (7, 0), (−7, 0); conjugate axis length of 8

35. foci (0, 5), (0, −5); vertices (0, 3), (0, −3)

36. foci (1, 15), (1, −5); transverse axis length of 16

37. vertices (2, 0), (−2, 0); asymptotes $y = \pm\frac{3}{2}x$

Use the discriminant to identify each conic section.

38. $x^2 - 4y^2 - 6x - 16y - 11 = 0$

39. $4y^2 - x - 40y + 107 = 0$

40. $9x^2 + 4y^2 + 162x + 8y + 732 = 0$

Example 3

Graph $\frac{(y+3)^2}{16} - \frac{(x+1)^2}{4} = 1$.

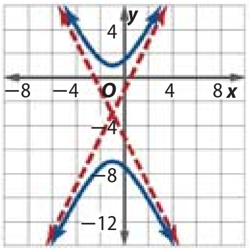
In this equation, $h = -1$, $k = -3$, $a = \sqrt{16}$ or 4, $b = \sqrt{4}$ or 2, and $c = \sqrt{16+4}$ or $2\sqrt{5}$.

Determine the characteristics of the hyperbola.

orientation:	vertical	
center:	$(-1, -3)$	(h, k)
vertices:	$(-1, 1), (-1, -7)$	$(h, k \pm a)$
foci:	$(-1, -3 + 2\sqrt{5})$, $(-1, -3 - 2\sqrt{5})$	$(h, k \pm c)$
asymptotes:	$y + 3 = 2(x+1)$, $y + 3 = -2(x+1)$	$y - k = \pm\frac{a}{b}(x - h)$

Make a table of values.

x	y
−6	7.77, −13.77
−2	1.47, −7.47
2	4.21, −10.21
6	11.56, −17.56



7-4 Rotations of Conic Sections (pp. 454–461)

Use a graphing calculator to graph the conic given by each equation.

41. $x^2 - 4xy + y^2 - 2y - 2x = 0$

42. $x^2 - 3xy + y^2 - 3y - 6x + 5 = 0$

43. $2x^2 + 2y^2 - 8xy + 4 = 0$

44. $3x^2 + 9xy + y^2 = 0$

45. $4x^2 - 2xy + 8y^2 - 7 = 0$

Write each equation in the $x'y'$-plane for the given value of θ. Then identify the conic.

46. $x^2 + y^2 = 4$; $\theta = \frac{\pi}{4}$

47. $x^2 - 2x + y = 5$; $\theta = \frac{\pi}{3}$

48. $x^2 - 4y^2 = 4$; $\theta = \frac{\pi}{2}$

49. $9x^2 + 4y^2 = 36$; $\theta = 90°$

Example 4

Use a graphing calculator to graph $x^2 + 2xy + y^2 + 4x - 2y = 0$.

$x^2 + 2xy + y^2 + 4x - 2y = 0$ — Original equation

$1y^2 + (2x - 2)y + (x^2 + 4x) = 0$ — Quadratic form

Use the Quadratic Formula.

$$y = \frac{-(2x-2) \pm \sqrt{(2x-2)^2 - 4(1)(x^2+4x)}}{2(1)}$$

$$= \frac{-2x + 2 \pm \sqrt{4x^2 - 8x + 4 - 4x^2 - 16x}}{2}$$

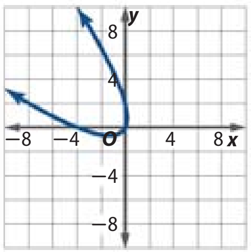
$$= \frac{-2x + 2 \pm 2\sqrt{1 - 6x}}{2}$$

$$= -x + 1 \pm \sqrt{1 - 6x}$$

Graph as

$y_1 = -x + 1 + \sqrt{1 - 6x}$ and

$y_2 = -x + 1 - \sqrt{1 - 6x}$.



7-5 Parametric Equations (pp. 464–471)

Sketch the curve given by each pair of parametric equations over the given interval.

50. $x = \sqrt{t},\ y = 1 - t;\ 0 \leq t \leq 9$

51. $x = t + 2,\ y = t^2 - 4;\ -4 \leq t \leq 4$

Write each pair of parametric equations in rectangular form. Then graph the equation.

52. $x = t + 5$ and $y = 2t - 6$

53. $x = 2t$ and $y = t^2 - 2$

54. $x = t^2 + 3$ and $y = t^2 - 4$

55. $x = t^2 - 1$ and $y = 2t + 1$

Example 5

Write $x = 5 \cos t$ and $y = 9 \sin t$ in rectangular form. Then graph the equation.

$x = 5 \cos t \qquad y = 9 \sin t$

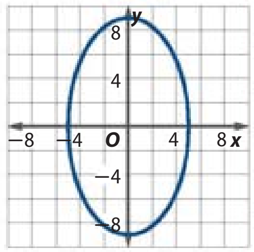
$\cos t = \frac{x}{5} \qquad \sin t = \frac{y}{9}$ Solve for sin t and cos t

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{9}\right)^2 = 1$

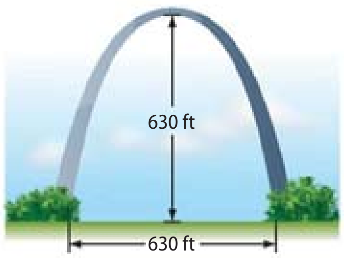
$\frac{x^2}{25} + \frac{y^2}{81} = 1$

The parametric equations represent the graph of an ellipse.



Applications and Problem Solving

56. MONUMENTS The St. Louis Arch is in the shape of a *catenary*, which resembles a parabola. (Lesson 7-1)



a. Write an equation for a parabola that would approximate the shape of the arch.

b. Find the location of the focus of this parabola.

57. WATER DYNAMICS A rock dropped in a pond will produce ripples of water made up of concentric expanding circles. Suppose the radii of the circles expand at 3 inches per second. (Lesson 7-2)

a. Write an equation for the circle 10 seconds after the rock is dropped in the pond. Assume that the point where the rock is dropped is the origin.

b. One concentric circle has equation $x^2 + y^2 = 225$. How many seconds after the rock is dropped does it take for the circle to have this equation?

58. ENERGY Cooling towers at a power plant are in the shape of a hyperboloid. The cross section of a hyperboloid is a hyperbola. (Lesson 7-3)

a. Write an equation for the cross section of a tower that is 50 feet tall and 30 feet wide.

b. If the ratio of the height to the width of the tower increases, how is the equation affected?

59. SOLAR DISH Students building a parabolic device to capture solar energy for cooking marshmallows placed at the focus must plan for the device to be easily oriented. Rotating the device directly toward the Sun's rays maximizes the heat potential. (Lesson 7-4)

a. After the parabola is rotated 30° toward the Sun, the equation of the parabola used to create the device in the $x'y'$-plane is $y' = 0.25(x')^2$. Find the equation of the parabola in the xy-plane.

b. Graph the rotated parabola.

60. GEOMETRY Consider $x_1(t) = 4 \cos t$, $y_1(t) = 4 \sin t$, $x_2(t) = 4 \cos 2t$, and $y_2(t) = 4 \sin 2t$. (Lesson 7-5)

a. Compare the graphs of the two sets of equations: x_1 and y_1; and x_2 and y_2.

b. Write parametric equations for a circle of radius 6 that complete its graph in half the time of $x_1(t)$ and $y_1(t)$.

c. Write the equations from part **b** in rectangular form.

CHAPTER 7 Practice Test

Write an equation for an ellipse with each set of characteristics.

1. vertices $(7, -4)$, $(-3, -4)$; foci $(6, -4)$, $(-2, -4)$

2. foci $(-2, 1)$, $(-2, -9)$; length of major axis is 12

3. **MULTIPLE CHOICE** What value must c be so that the graph of $4x^2 + cy^2 + 2x - 2y - 18 = 0$ is a circle?

   A -8

   B -4

   C 4

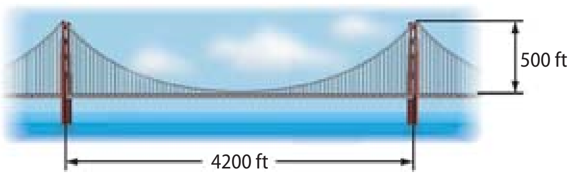
   D 8

Write each pair of parametric equations in rectangular form. Then graph the equation.

4. $x = t - 5$ and $y = 3t - 4$

5. $x = t^2 - 1$ and $y = 2t + 1$

6. **BRIDGES** At 1.7 miles long, San Francisco's Golden Gate Bridge was the longest suspension bridge in the world when it was constructed.



   a. Suppose the design of the bridge can be modeled by a parabola and the lowest point of the cable is 15 feet above the road. Write an equation for the design of the bridge.

   b. Where is the focus located in relation to the vertex?

Write an equation for the hyperbola with the given characteristics.

7. vertices $(3, 0)$, $(-3, 0)$; asymptotes $y = \pm\frac{2}{3}x$

8. foci $(8, 0)$, $(8, 8)$; vertices $(8, 2)$, $(8, 6)$

Write an equation for each conic in the xy-plane for the given equation in $x'y'$ form and the given value of θ.

9. $7(x' - 3) = (y')^2$, $\theta = 60°$

10. $\frac{(x')^2}{2} + \frac{(y')^2}{10} = 1$, $\theta = \frac{\pi}{6}$

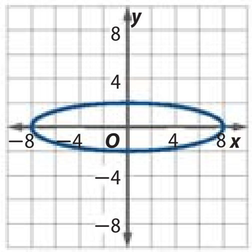
Graph the hyperbola given by each equation.

11. $\frac{x^2}{64} - \frac{(y-4)^2}{25} = 1$

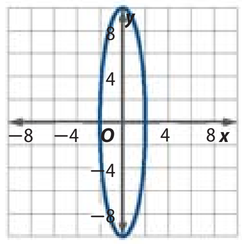
12. $\frac{(y+3)^2}{4} - \frac{(x+6)^2}{36} = 1$

13. **MULTIPLE CHOICE** Which ellipse has the greatest eccentricity?

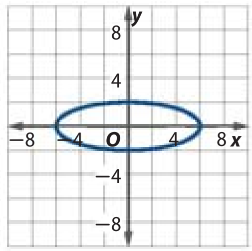
A



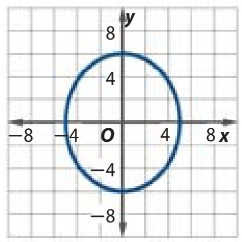
C



B



D



Write an equation for and graph a parabola with the given focus F and vertex V.

14. $F(2, 8)$, $V(2, 10)$

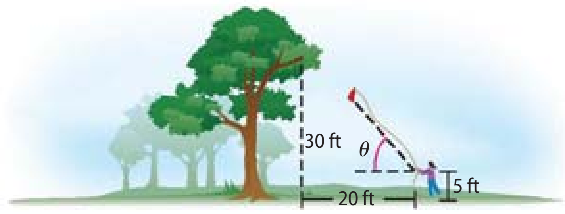
15. $F(2, 5)$, $V(-1, 5)$

Graph the ellipse given by each equation.

16. $\frac{(x-5)^2}{49} + \frac{(y+3)^2}{9} = 1$

17. $(x+3)^2 + \frac{(y+6)^2}{81} = 1$

18. **CAMPING** In many U.S. parks, campers must secure food and provisions from bears and other animals. One method is to secure food using a bear bag, which is done by tossing a bag tied to a rope over a tall tree branch and securing the rope to the tree. Suppose a tree branch is 30 feet above the ground, and a person 20 feet from the branch throws the bag from 5 feet above the ground.



   a. Will a bag thrown at a speed of 40 feet per second at an angle of 60° go over the branch?

   b. Will a bag thrown at a speed of 45 feet per second at an angle of 75° go over the branch?

Use a graphing calculator to graph the conic given by each equation.

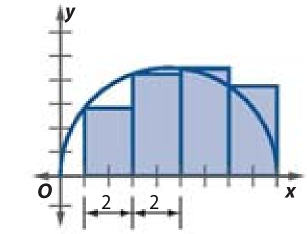
19. $x^2 - 6xy + y^2 - 4y - 8x = 0$

20. $x^2 + 4y^2 - 2xy + 3y - 6x + 5 = 0$

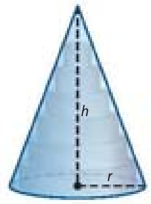
CHAPTER 7

Connect to AP Calculus
Solids of Revolution

In Chapter 2, you learned that integral calculus is a branch of mathematics that focuses on the processes of finding lengths, areas, and volumes. You used rectangles to approximate the areas of irregular shapes, such as those created by a curve and the *x*-axis. A similar technique can be used to approximate volumes of irregular shapes.



Consider a cone with a height of *h* and a base with a radius of *r*. If we did not already know the formula for the volume of a cone, we could approximate the volume by drawing several cylinders of equal height inside the cone. We could then calculate the volume of each cylinder, and find the sum.



Activity 1 Sphere

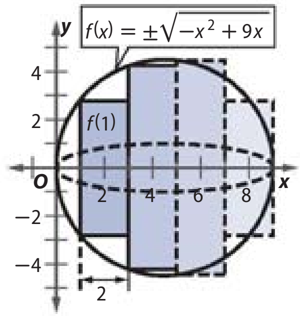
Approximate the volume of the sphere with a radius of 4.5 units and a great circle defined by $f(x) = \pm\sqrt{-x^2 + 9x}$.

Step 1 Sketch a diagram of the sphere.

Step 2 Inscribe a cylinder in the sphere with a base perpendicular to the *x*-axis and a height of 2 units. Allow for the left edge of the cylinder to begin at $x = 1$ and to extend to the great circle. The radius of the cylinder is $f(1)$.

Step 3 Draw 3 more cylinders all with a height of 2 units. Allow for the left edge of each cylinder to extend to the great circle.

Step 4 Calculate the volume of each cylinder.



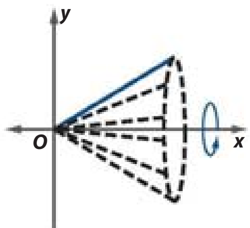
StudyTip

Volume The formula for the volume of a sphere is $V = \pi r^2 h$.

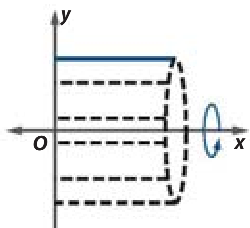
Analyze the Results

1. What is the approximation for the volume of the sphere?
2. Calculate the actual volume of the sphere using the radius. How does the approximation compare with the actual volume? What could be done to improve upon the accuracy of the approximation?

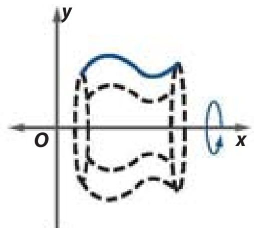
When the region between a graph and the *x*-axis is rotated about the *x*-axis, a *solid of revolution* is formed. The shape of the graph dictates the shape of the three-dimensional figure formed.



cone

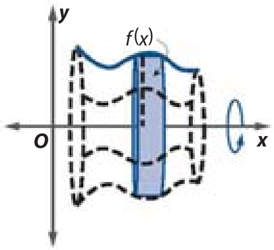


cylinder

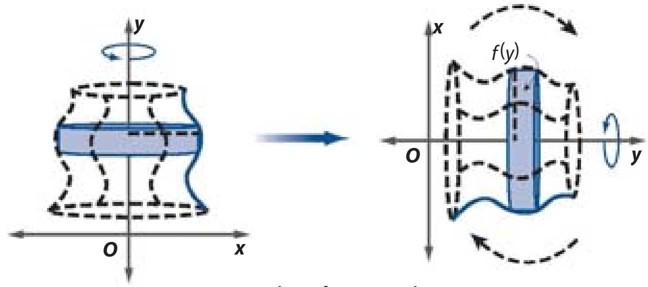


solid

A solid of revolution can be formed by rotating a region in a plane about any fixed line, called the *axis of revolution*. The axis of revolution will dictate the direction and the radii of the cylinders used to approximate the area. If revolving about the x-axis, the cylinders will be parallel to the y-axis and the radii will be given by $f(x)$. If revolving about the y-axis, the cylinders will be parallel to the x-axis and the radii will be given by $f(y)$.



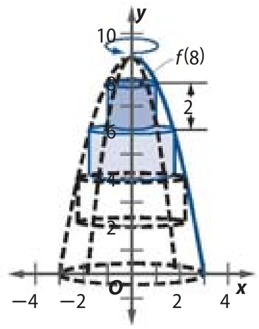
rotating about x-axis



rotating about y-axis

Activity 2 Paraboloid

Approximate the volume of the paraboloid created by revolving the region between $f(x) = -x^2 + 9$, the x-axis, and the y-axis about the y-axis.



Step 1 Sketch a diagram of the paraboloid.

Step 2 Inscribe a cylinder in the paraboloid with a base parallel to the x-axis and a height of 2 units. Allow for the top edge of the cylinder to begin at $y = 8$ and to extend to the edge of the paraboloid.

Step 3 When revolving about the y-axis, the radius is given as $f(y)$. To find $f(y)$, write $f(x)$ as $y = -x^2 + 9$ and solve for y.

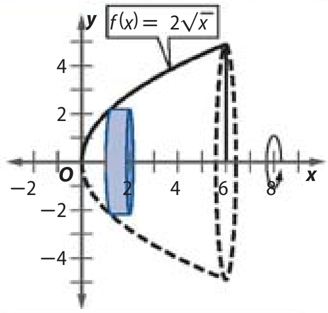
Step 4 Draw 3 more cylinders all with a height of 2 units. Allow for the top of each cylinder to extend to the edge of the paraboloid.

Step 5 Calculate the volume of each cylinder.

Analyze the Results

3. What is the approximation for the volume of the paraboloid?
4. Find approximations for the volume of the paraboloid using 8 cylinders with heights of 1 unit and again using 17 cylinders with heights of 0.5 units.
5. As the heights of the cylinders decrease and approach 0, what is happening to the approximations? Explain your reasoning.
6. What shape do the cylinders start to resemble as h approaches 0? Explain your reasoning.

Model and Apply



7. Approximate the volume of the paraboloid created by revolving the region between $f(x) = 2\sqrt{x}$, the x-axis, and the line $x = 6$ about the x-axis. Use 5 cylinders with heights of 1 unit. Let the first cylinder begin at $x = 1$ and the left edge of each cylinder extend to the edge of the paraboloid.

CHAPTER 8

Vectors

Fuse/Getty Images

Then

In **Chapter 4,** you used trigonometry to solve triangles.

Now

In **Chapter 8,** you will:

- Represent and operate with vectors algebraically in the two- and three-dimensional coordinate systems.
- Find the projection of one vector onto another.
- Find cross products of vectors in space and find volumes of parallelepipeds.
- Find the dot products of and angles between vectors.

Why?

ROWING Vectors are often used to model changes in direction due to water and air currents. For example, a vector can be used to determine the resultant speed and direction of a kayak that is traveling 8 miles per hour against a 3-mile-per-hour river current.

PREREAD Scan the lesson titles and Key Concept boxes in Chapter 8. Use this information to predict what you will learn in this chapter.

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