CHAPTER 6

Systems of Equations and Matrices

Then

In **Chapter 0**, you solved systems of equations and performed matrix operations.

Now

In **Chapter 6**, you will:

- Multiply matrices, and find determinants and inverses of matrices.
- Solve systems of linear equations.
- Write partial fraction decompositions of rational expressions.
- Use linear programming to solve applications.

Why?

BUSINESS Linear programming has become a standard tool for many businesses, like farming. Farmers must take into account many constraints in order to maximize profits from the sale of crops or livestock, including the cost of labor, land, and feed.

PREREAD Discuss what you already know about solving equations with a classmate. Then scan the lesson titles and write two or three predictions about what you will learn in this chapter.

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Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

Use any method to solve each system of equations. (Lesson 0-5)

1. $2x - y = 7$
$3x + 2y = 14$

2. $3x + y = 14$
$2x - 2y = -4$

3. $x + 3y = 10$
$-2x + 3y = 16$

4. $4x + 2y = -34$
$-3x - y = 24$

5. $2x + 5y = -16$
$3x + 4y = -17$

6. $-5x + 2y = -33$
$6x - 3y = 42$

7. **VETERINARY** A veterinarian charges different amounts to trim the nails of dogs and cats. On Monday, she made \$96 trimming 4 dogs and 3 cats. On Tuesday, she made \$126 trimming 6 dogs and 3 cats. What is the charge to trim the nails of each animal? (Lesson 0-5)

Find each of the following for
$A = \begin{bmatrix} 3 & -5 & 1 \\ -7 & 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -9 & 1 \\ 10 & 8 & -1 \end{bmatrix}$, and
$C = \begin{bmatrix} 0 & 11 & -3 \\ 9 & -3 & 5 \end{bmatrix}$. (Lesson 0-6)

8. $A + 3C$

9. $2(B - A)$

10. $2A - 3B$

11. $3C + 2A$

12. $A + B - C$

13. $2(B + C) - A$

Divide. (Lesson 2-3)

14. $(x^4 - 2x^3 + 4x^2 - 5x - 5) \div (x - 1)$

15. $(2x^4 + 4x^3 - x^2 + 2x - 4) \div (x + 2)$

16. $(3x^4 - 6x^3 - 12x - 36) \div (x - 4)$

17. $(2x^5 - x^3 + 2x^2 + 9x + 6) \div (x - 2)$

18. $(2x^6 + 2x^5 + 3x^3 + x^2 - 8x - 6) \div (x + 1)$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

NewVocabulary

English		Español
multivariable linear system	p. 364	sistema lineal multivariable
Gaussian elimination	p. 364	Eliminación de Gaussian
augmented matrix	p. 366	matriz aumentada
coefficient matrix	p. 366	matriz de coefficent
row-echelon form	p. 364	forma de grado de fila
reduced row-echelon form	p. 369	reducir fila escalón forma
Gauss-Jordan elimination	p. 369	Eliminación de Gauss-Jordania
identity matrix	p. 378	matriz de identidad
inverse matrix	p. 379	matriz inversa
inverse	p. 379	inverso
invertible	p. 379	invertible
singular matrix	p. 379	matriz singular
determinant	p. 381	determinante
square system	p. 388	sistema cuadrado
Cramer's Rule	p. 390	La Regla de Cramer
partial fraction	p. 398	fracción parcial
optimization	p. 405	optimización
linear programming	p. 405	programación lineal
objective function	p. 405	función objetivo
feasible solutions	p. 405	soluciones viables
constraint	p. 405	coacción
multiple optimal solutions	p. 408	múltiples soluciones óptimas
unbounded	p. 408	no acotado

ReviewVocabulary

system of equations p. P19 sistema de ecuaciones a set of two or more equations

matrix p. P24 matriz a rectangular array of elements

square matrix p. P24 matriz cuadrada a matrix that has the same number of rows as columns

scalar p. P25 escalar a constant that a matrix is multiplied by

LESSON 6-1

Multivariable Linear Systems and Row Operations

Then

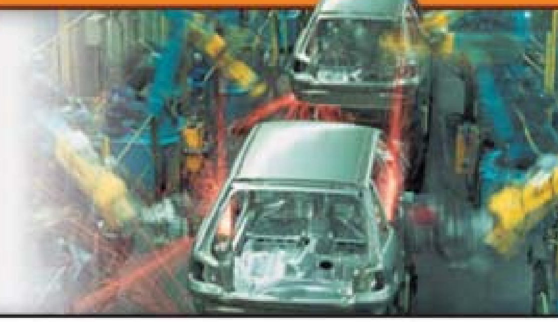
- You solved systems of equations algebraically and represented data using matrices. (Lessons 0-5 and 0-6)

Now

1. Solve systems of linear equations using matrices and Gaussian elimination.
2. Solve systems of linear equations using matrices and Gauss-Jordan elimination.

Why?

- Metal alloys are often developed in the automotive industry to improve the performance of cars. You can solve a system of equations to determine what percent of each metal is needed for a specific alloy.



NewVocabulary
multivariable linear system
row-echelon form
Gaussian elimination
augmented matrix
coefficient matrix
reduced row-echelon form
Gauss-Jordan elimination

1 Gaussian Elimination

A **multivariable linear system**, or *multivariate* linear system, is a system of linear equations in two or more variables. In previous courses, you may have used the *elimination method* to solve such systems. One elimination method begins by rewriting a system using an inverted triangular shape in which the leading coefficients are 1.

The substitution and elimination methods you have previously learned can be used to convert a multivariable linear system into an equivalent system in *triangular* or **row-echelon form**.

System in Row-Echelon Form

$$\begin{aligned} x - y - 2z &= 5 \\ y + 4z &= -5 \\ z &= -2 \end{aligned}$$

Notice that the left side of the system forms a triangle in which the leading coefficients are 1. The last equation contains only one variable, and each equation above it contains the variables from the equation immediately below it.

Once a system is in this form, the solutions can be found by substitution. The final equation determines the final variable. In the example above, the final equation determines that $z = -2$.

Substitute the value for z in the second equation to find y.

$y + 4z = -5$	Second equation
$y + 4(-2) = -5$	$z = -2$
$y = 3$	Solve for y.

Substitute the values for y and z in the first equation to find x.

$x - y - 2z = 5$	First equation
$x - 3 - 2(-2) = 5$	$y = 3$ and $z = -2$
$x = 4$	Solve for x.

So, the solution of the system is $x = 4$, $y = 3$, and $z = -2$.

The algorithm used to transform a system of linear equations into an equivalent system in row-echelon form is called **Gaussian elimination**, named after the German mathematician Carl Friedrich Gauss. The operations used to produce equivalent systems are given below.

KeyConcept Operations that Produce Equivalent Systems

Each of the following operations produces an equivalent system of linear equations.

- Interchange any two equations.
- Multiply one of the equations by a nonzero real number.
- Add a multiple of one equation to another equation.

The specific algorithm for Gaussian elimination is outlined in the example below.

> **StudyTip**
>
> **Geometric Interpretation** The solution set of a two-by-two system can be represented by the intersection of a pair of lines in a plane while the solution set of a three-by-three system can be represented by the intersection of three planes in space.

Example 1 Gaussian Elimination with a System

Write the system of equations in triangular form using Gaussian elimination. Then solve the system.

$5x - 5y - 5z = 35$ Equation 1
$-x + 2y - 3z = -12$ Equation 2
$3x - 2y + 7z = 30$ Equation 3

Step 1 The leading coefficient in Equation 1 is not 1, so multiply this equation by the reciprocal of its leading coefficient.

$$\begin{aligned} x - y - z &= 7 && \leftarrow \tfrac{1}{5}(5x - 5y - 5z = 35) \\ -x + 2y - 3z &= -12 \\ 3x - 2y + 7z &= 30 \end{aligned}$$

Step 2 Eliminate the x-term in Equation 2. To do this, replace Equation 2 with (Equation 1 + Equation 2).

$$\begin{aligned} x - y - z &= 7 \\ y - 4z &= -5 \\ 3x - 2y + 7z &= 30 \end{aligned}$$

$$\begin{aligned} x - y - z &= 7 \\ (+)\ -x + 2y - 3z &= -12 \\ \hline y - 4z &= -5 \end{aligned}$$

Step 3 Eliminate the x-term in Equation 3 by replacing Equation 3 with [−3(Equation 1) + Equation 3].

$$\begin{aligned} x - y - z &= 7 \\ y - 4z &= -5 \\ y + 10z &= 9 \end{aligned}$$

$$\begin{aligned} -3x + 3y + 3z &= -21 \\ (+)\ 3x - 2y + 7z &= 30 \\ \hline y + 10z &= 9 \end{aligned}$$

Step 4 The leading coefficient in Equation 2 is 1, so next eliminate the y-term from Equation 3 by replacing Equation 3 with [−1(Equation 2) + Equation 3].

$$\begin{aligned} x - y - z &= 7 \\ y - 4z &= -5 \\ 14z &= 14 \end{aligned}$$

$$\begin{aligned} -y + 4z &= 5 \\ (+)\ y + 10z &= 9 \\ \hline 14z &= 14 \end{aligned}$$

Step 5 The leading coefficient in Equation 3 is not 1, so multiply this equation by the reciprocal of its leading coefficient.

$$\begin{aligned} x - y - z &= 7 \\ y - 4z &= -5 \\ z &= 1 && \leftarrow \tfrac{1}{14}(14z = 14) \end{aligned}$$

You can use substitution to find that $y = -1$ and $x = 7$. So, the solution of the system is $x = 7$, $y = -1$, and $z = 1$, or the ordered triple $(7, -1, 1)$.

> **StudyTip**
>
> **Check Your Solution** When solving a system of equations, you should check your solution using substitution in the original equations. The check for Example 1 is shown below.
>
> Equation 1:
> $5(7) - 5(-1) - 5(1) = 35$ ✔
>
> Equation 2:
> $-7 + 2(-1) - 3(1) = -12$ ✔
>
> Equation 3:
> $3(7) - 2(-1) + 7(1) = 30$ ✔

GuidedPractice

Write each system of equations in triangular form using Gaussian elimination. Then solve the system.

1A. $x + 2y - 3z = -28$
$3x - y + 2z = 3$
$-x + y - z = -5$

1B. $3x + 5y + 8z = -20$
$-x + 2y - 4z = 18$
$-6x + 4z = 0$

Solving a system of linear equations using Gaussian elimination only affects the coefficients of the variables to the left and the constants to the right of the equals sign, so it is often easier to keep track of just these numbers using a matrix.

> **ReadingMath**
> **Augmented Matrix** Notice that a dashed line separates the coefficient matrix from the column of constants in an augmented matrix.

The **augmented matrix** of a system is derived from the coefficients and constant terms of the linear equations, each written in standard form with the constant terms to the right of the equals sign. If the column of constant terms is not included, the matrix reduces to that of the **coefficient matrix** of the system. You will use this type of matrix in Lesson 6-3.

System of Equations	Augmented Matrix	Coefficient Matrix
$5x - 5y - 5z = 35$ $-x + 2y - 3z = -12$ $3x - 2y + 7z = 30$	$\left[\begin{array}{rrr\|r} 5 & -5 & -5 & 35 \\ -1 & 2 & -3 & -12 \\ 3 & -2 & 7 & 30 \end{array}\right]$	$\begin{bmatrix} 5 & -5 & -5 \\ -1 & 2 & -3 \\ 3 & -2 & 7 \end{bmatrix}$

Example 2 Write an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\mathbf{w + 4x + z = 2}$$
$$\mathbf{x + 2y - 3z = 0}$$
$$\mathbf{w - 3y - 8z = -1}$$
$$\mathbf{3w + 2x + 3y = 9}$$

While each linear equation is in standard form, not all of the four variables of the system are represented in each equation, so the like terms do not align. Rewrite the system, using the coefficient 0 for missing terms. Then write the augmented matrix.

System of Equations	Augmented Matrix
$w + 4x + 0y + z = 2$ $0w + x + 2y - 3z = 0$ $w + 0x - 3y - 8z = -1$ $3w + 2x + 3y + 0z = 9$	$\left[\begin{array}{rrrr\|r} 1 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & -3 & 0 \\ 1 & 0 & -3 & -8 & -1 \\ 3 & 2 & 3 & 0 & 9 \end{array}\right]$

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Write the augmented matrix for each system of linear equations.

2A. $4w - 5x + 7z = -11$
$-w + 8x + 3y = 6$
$15x - 2y + 10z = 9$

2B. $-3w + 7x + y = 21$
$4w - 12y + 8z = 5$
$16w - 14y + z = -2$
$w + x + 2y = 7$

The three operations used to produce equivalent systems have corresponding matrix operations that can be used to produce an equivalent augmented matrix. Each row in an augmented matrix corresponds to an equation of the original system, so these operations are called *elementary row operations*.

KeyConcept Elementary Row Operations

Each of the following row operations produces an equivalent augmented matrix.

- Interchange any two rows.
- Multiply one row by a nonzero real number.
- Add a multiple of one row to another row.

Row operations are termed *elementary* because they are simple to perform. However, it is easy to make a mistake, so you should record each step using the notation illustrated below.

1 Row 1, Row 2, Row 3

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{rrr|r} 5 & -5 & -5 & 35 \\ -1 & 2 & -3 & -12 \\ 3 & -2 & 7 & 30 \end{array}\right]$$

2 Interchange Rows 2 and 3.

$$\begin{array}{c} \\ R_3 \rightarrow \\ R_2 \rightarrow \end{array}\left[\begin{array}{rrr|r} 5 & -5 & -5 & 35 \\ 3 & -2 & 7 & 30 \\ -1 & 2 & -3 & -12 \end{array}\right]$$

3 Multiply Row 1 by $\frac{1}{5}$.

$$\begin{array}{c} \frac{1}{5}R_1 \rightarrow \\ \\ \\ \end{array}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 7 \\ 3 & -2 & 7 & 30 \\ -1 & 2 & -3 & -12 \end{array}\right]$$

4 Add −3 times Row 1 to Row 2.

$$\begin{array}{c} \\ -3R_1 + R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 7 \\ 0 & 1 & 10 & 9 \\ -1 & 2 & -3 & -12 \end{array}\right]$$

StudyTip

Row-Equivalent If one matrix can be obtained by a sequence of row operations on another, the two matrices are said to be *row-equivalent.*

Compare the Gaussian elimination from Example 1 to its matrix version using row operations.

System of Equations		Augmented Matrix	
$\frac{1}{5}$(Eqn. 1) →	$x - y - z = 7$ $-x + 2y - 3z = -12$ $3x - 2y + 7z = 30$	$\frac{1}{5}R_1$ →	$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 7 \\ -1 & 2 & -3 & -12 \\ 3 & -2 & 7 & 30 \end{array}\right]$
Eqn. 1 + Eqn. 2 →	$x - y - z = 7$ $y - 4z = -5$ $3x - 2y + 7z = 30$	$R_1 + R_2$ →	$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 7 \\ 0 & 1 & -4 & -5 \\ 3 & -2 & 7 & 30 \end{array}\right]$
−3(Eqn. 1) + Eqn. 3 →	$x - y - z = 7$ $y - 4z = -5$ $y + 10z = 9$	$-3R_1 + R_3$ →	$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 7 \\ 0 & 1 & -4 & -5 \\ 0 & 1 & 10 & 9 \end{array}\right]$
−(Eqn. 2) + Eqn. 3 →	$x - y - z = 7$ $y - 4z = -5$ $14z = 14$	$-R_2 + R_3$ →	$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 7 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 14 & 14 \end{array}\right]$
$\frac{1}{14}$(Eqn. 3) →	$x - y - z = 7$ $y - 4z = -5$ $z = 1$	$\frac{1}{14}R_3$ →	$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 7 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$

The augmented matrix that corresponds to the row-echelon form of the original system of equations is also said to be in row-echelon form.

StudyTip

Row-Echelon Form The row-echelon form of a matrix is not unique because there are many combinations of row operations that can be performed. However, the final solution of the system of equations will always be the same.

KeyConcept Row-Echelon Form

A matrix is in row-echelon form if the following conditions are met.

- Rows consisting entirely of zeros (if any) appear at the bottom of the matrix.
- The first nonzero entry in a row is 1, called a *leading 1*.
- For two successive rows with nonzero entries, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

$$\left[\begin{array}{ccc:c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 3 Identify an Augmented Matrix in Row-Echelon Form

Determine whether each matrix is in row-echelon form.

a. $\left[\begin{array}{ccc:c} 1 & 2 & 0 & -1 \\ 0 & 1 & 4 & 2 \end{array}\right]$

There is a zero below the leading one in the first row. The matrix is in row-echelon form.

b. $\left[\begin{array}{cccc:c} 1 & 6 & 2 & -11 & 10 \\ 0 & 1 & -5 & 8 & -7 \\ 0 & 0 & 0 & 1 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

There is a zero below each of the leading ones in each row. The matrix is in row-echelon form.

c. $\left[\begin{array}{ccc:c} 1 & 5 & -6 & 10 \\ 0 & 1 & 9 & -3 \\ 0 & 1 & 0 & 14 \end{array}\right]$

There is not a zero below the leading one in Row 2. The matrix is not in row-echelon form.

GuidedPractice

3A. $\left[\begin{array}{ccc:c} 1 & -6 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 9 \end{array}\right]$

3B. $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 19 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -20 \end{array}\right]$

3C. $\left[\begin{array}{cccc:c} 0 & 1 & 0 & 4 & 10 \\ 1 & 0 & -3 & 10 & -7 \\ 0 & 1 & 6 & 0 & 8 \\ 0 & 0 & 1 & -2 & -4 \end{array}\right]$

To solve a system of equations using an augmented matrix and Gaussian elimination, use row operations to transform the matrix so that it is in row-echelon form. Then write the corresponding system of equations and use substitution to finish solving the system. Remember, if you encounter a false equation, this means that the system has no solution.

Real-WorldLink

In a recent year, Italy was the fourth most-visited country in the world, with over 40 million tourists.

Source: World Tourism Organization

Real-World Example 4 Gaussian Elimination with a Matrix

TRAVEL Manuel went to Italy during spring break. The average daily hotel, food, and transportation costs for each city he visited are shown. Write and solve a system of equations to determine how many days Manuel spent in each city. Interpret your solution.

Expense	Venice	Rome	Naples	Total
hotels	\$60	\$120	\$60	\$720
food	\$40	\$90	\$30	\$490
transportation	\$15	\$10	\$20	\$130

Write the information as a system of equations. Let x, y, and z represent the number of days Manuel spent in Venice, Rome, and Naples, respectively.

$$60x + 120y + 60z = 720$$
$$40x + 90y + 30z = 490$$
$$15x + 10y + 20z = 130$$

Next, write the augmented matrix and apply elementary row operations to obtain a row-echelon form of the matrix.

Augmented matrix $\left[\begin{array}{ccc|c} 60 & 120 & 60 & 720 \\ 40 & 90 & 30 & 490 \\ 15 & 10 & 20 & 130 \end{array}\right]$

$\frac{1}{60}R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 40 & 90 & 30 & 490 \\ 15 & 10 & 20 & 130 \end{array}\right]$

$-40R_1 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 10 & -10 & 10 \\ 15 & 10 & 20 & 130 \end{array}\right]$

$\frac{1}{10}R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 1 & -1 & 1 \\ 15 & 10 & 20 & 130 \end{array}\right]$

$-15R_1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & -20 & 5 & -50 \end{array}\right]$

$20R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -15 & -30 \end{array}\right]$

$-\frac{1}{15}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

You can use substitution to find that $y = 3$ and $x = 4$. Therefore, the solution of the system is $x = 4$, $y = 3$, and $z = 2$, or the ordered triple (4, 3, 2).

Manuel spent 4 days in Venice, 3 days in Rome, and 2 days in Naples.

StudyTip

Types of Solutions Recall that a system of equations can have one solution, no solution, or infinitely many solutions.

GuidedPractice

4. **TRAVEL** The following year, Manuel traveled to France for spring break. The average daily hotel, food, and transportation costs for each city in France that he visited are shown. Write and solve a system of equations to determine how many days Manuel spent in each city. Interpret your solution.

Expense	Paris	Lyon	Marseille	Total
hotels	\$80	\$70	\$80	\$500
food	\$50	\$40	\$50	\$330
transportation	\$10	\$10	\$10	\$70

2 Gauss-Jordan Elimination

If you continue to apply elementary row operations to the row-echelon form of an augmented matrix, you can obtain a matrix in which the first nonzero element of each row is 1 and the rest of the elements in the same column as this element are 0. This is called the **reduced row-echelon form** of the matrix and is shown at the right. The reduced row-echelon form of a matrix is always unique, regardless of the order of operations that were performed.

Reduced Row-Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving a system by transforming an augmented matrix so that it is in reduced row-echelon form is called **Gauss-Jordan elimination**, named after Carl Friedrich Gauss and Wilhelm Jordan.

StudyTip

Patterns While different elementary row operations can be used to solve the same system of equations, a general pattern can be used as a guide to help avoid wasteful operations. For the system at the right, begin by producing a 0 in the first term of the second row and work your way around the matrix in the order shown, producing 0s and 1s. Once this is completed, the terms in the first row can be converted to 0s and 1s as well.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

Example 5 Use Gauss-Jordan Elimination

Solve the system of equations.

$x - y + z = 0$

$-x + 2y - 3z = -5$

$2x - 3y + 5z = 8$

Write the augmented matrix. Apply elementary row operations to obtain a reduced row-echelon form.

Augmented matrix $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 2 & -3 & -5 \\ 2 & -3 & 5 & 8 \end{array}\right]$

$R_1 + R_2 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 2 & -3 & 5 & 8 \end{array}\right]$

$-2R_1 + R_3 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & -1 & 3 & 8 \end{array}\right]$

$R_2 + R_3 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$ Row-echelon form

$R_2 + R_1 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -5 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

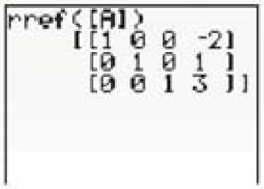
$R_3 + R_1 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$2R_3 + R_2 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ Reduced row-echelon form

The solution of the system is $x = -2$, $y = 1$, and $z = 3$ or the ordered triple $(-2, 1, 3)$. Check this solution in the original system of equations.

TechnologyTip

You can check the reduced row-echelon form of a matrix by using the rref(feature on a graphing calculator.



GuidedPractice

Solve each system of equations.

5A. $x + 2y - 3z = 7$
$-3x - 7y + 9z = -12$
$2x + y - 5z = 8$

5B. $4x + 9y + 16z = 2$
$-x - 2y - 4z = -1$
$2x + 4y + 9z = -5$

When solving a system of equations, if a matrix cannot be written in reduced row-echelon form, then the system either has no solution or infinitely many solutions.

Example 6 No Solution and Infinitely Many Solutions

Solve each system of equations.

a. $-5x - 2y + z = 2$
$4x - y - 6z = 2$
$-3x - y + z = 1$

Write the augmented matrix. Then apply elementary row operations to obtain a reduced row-echelon matrix.

Augmented matrix $\left[\begin{array}{rrr|r} -5 & -2 & 1 & 2 \\ 4 & -1 & -6 & 2 \\ -3 & -1 & 1 & 1 \end{array}\right]$

$-2R_3 + R_1 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 4 & -1 & -6 & 2 \\ -3 & -1 & 1 & 1 \end{array}\right]$

$-4R_1 + R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 2 \\ -3 & -1 & 1 & 1 \end{array}\right]$

$3R_1 + R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$-2R_3 + R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$R_2 + R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

According to the last row, $0x + 0y + 0z = 1$. This is impossible, so the system has no solution.

b. $3x + 5y - 8z = -3$
$2x + 5y - 2z = -7$
$-x - y + 4z = -1$

Write the augmented matrix. Then apply elementary row operations to obtain a reduced row-echelon matrix.

Augmented matrix $\left[\begin{array}{rrr|r} 3 & 5 & -8 & -3 \\ 2 & 5 & -2 & -7 \\ -1 & -1 & 4 & -1 \end{array}\right]$

$R_1 - R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -6 & 4 \\ 2 & 5 & -2 & -7 \\ -1 & -1 & 4 & -1 \end{array}\right]$

$2R_3 + R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -6 & 4 \\ 0 & 3 & 6 & -9 \\ -1 & -1 & 4 & -1 \end{array}\right]$

$R_1 + R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -6 & 4 \\ 0 & 3 & 6 & -9 \\ 0 & -1 & -2 & 3 \end{array}\right]$

$2R_3 + R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -6 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & -1 & -2 & 3 \end{array}\right]$

$R_2 + R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -6 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Write the corresponding system of linear equations for the reduced row-echelon form of the augmented matrix.

$$\begin{aligned} x \quad - 6z &= 4 \\ y + 2z &= -3 \end{aligned}$$

Because the value of z is not determined, this system has infinitely many solutions. Solving for x and y in terms of z, you have $x = 6z + 4$ and $y = -2z - 3$.

So, a solution of the system can be expressed as $(6z + 4, -2z - 3, z)$, where z is any real number.

StudyTip

Infinitely Many Solutions The solution of the system in Example 6b is not a unique answer because the solution could be expressed in terms of any of the variables in the system.

GuidedPractice

Solve each system of equations.

6A. $3x - y - 5z = 9$
$4x + 2y - 3z = 6$
$-7x - 11y - 3z = 3$

6B. $x + 3y + 4z = 8$
$4x - 2y - z = 6$
$8x - 18y - 19z = -2$

When a system has fewer equations than variables, the system either has no solution or infinitely many solutions. When solving a system of equations with three or more variables, it is important to check your answer using all of the original equations. This is necessary because it is possible for an incorrect solution to work for some of the equations but not the others.

Math HistoryLink

Wilhelm Jordan

(1842–1899)

A German geodesist, Wilhelm Jordan is credited with simplifying the Gaussian method of solving a system of linear equations so it could be applied to minimizing the squared error in surveying.

Example 7 Infinitely Many Solutions

Solve the system of equations.

$$3x - 8y + 19z - 12w = 6$$
$$2x - 4y + 10z = -8$$
$$x - 3y + 5z - 2w = -1$$

Write the augmented matrix. Then apply elementary row operations to obtain leading 1s in each row and zeros below these 1s in each column.

$$\left[\begin{array}{cccc|c} 3 & -8 & 19 & -12 & 6 \\ 2 & -4 & 10 & 0 & -8 \\ 1 & -3 & 5 & -2 & -1 \end{array}\right]$$

$$R_3 \leftrightarrow R_1 \quad \left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & -1 \\ 2 & -4 & 10 & 0 & -8 \\ 3 & -8 & 19 & -12 & 6 \end{array}\right]$$

$$-2R_1 + R_2 \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & -1 \\ 0 & 2 & 0 & 4 & -6 \\ 3 & -8 & 19 & -12 & 6 \end{array}\right]$$

$$-3R_1 + R_3 \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & -1 \\ 0 & 2 & 0 & 4 & -6 \\ 0 & 1 & 4 & -6 & 9 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & -1 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 1 & 4 & -6 & 9 \end{array}\right]$$

$$-R_2 + R_3 \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & -1 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 4 & -8 & 12 \end{array}\right]$$

$$3R_2 + R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & 4 & -10 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 4 & -8 & 12 \end{array}\right]$$

$$\tfrac{1}{4}R_3 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & 4 & -10 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

$$-5R_3 + R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 14 & -25 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

Write the corresponding system of linear equations for the reduced row-echelon form of the augmented matrix.

$$x + 14w = -25$$
$$y + 2w = -3$$
$$z - 2w = 3$$

This system of equations has infinitely many solutions because for every value of w there are three equations that can be used to find the corresponding values of x, y, and z. Solving for x, y, and z in terms of w, you have $x = -14w - 25$, $y = -2w - 3$, and $z = 2w + 3$.

So, a solution of the system can be expressed as $(-14w - 25, -2w - 3, 2w + 3, w)$, where w is any real number.

CHECK Using different values for w, calculate a few solutions and check them in the original system of equations. For example, if $w = 1$, a solution of the system is $(-39, -5, 5, 1)$. This solution checks in each equation of the original system.

$$3(-39) - 8(-5) + 19(5) - 12(1) = 6 \checkmark$$
$$2(-39) - 4(-5) + 10(5) = -8 \checkmark$$
$$(-39) - 3(-5) + 5(5) - 2(1) = -1 \checkmark$$

GuidedPractice

Solve each system of equations.

7A. $-5w + 10x + 4y + 54z = 15$
$w - 2x - y - 9z = -1$
$-2w + 3x + y + 19z = 9$

7B. $3w + x - 2y - 3z = 14$
$-w + x - 10y + z = -11$
$-2w - x + 4y + 2z = -9$

Exercises

= Step-by-Step Solutions begin on page R29.

Write each system of equations in triangular form using Gaussian elimination. Then solve the system. (Example 1)

1. $5x = -3y - 31$
$2y = -4x - 22$

2. $4y + 17 = -7x$
$8x + 5y = -19$

3. $12x = 21 - 3y$
$2y = 6x + 7$

4. $4y = 12x - 3$
$9x = 20y - 2$

5. $-3x + y + 6z = 15$
$2x + 2y - 5z = 9$
$4x - 5y + 2z = -3$

6. $8x - 24y + 16z = -7$
$40x - 9y + 2z = 10$
$32x + 8y - z = -2$

7. $3x + 9y - 6z = 17$
$-2x - y + 24z = 12$
$2x - 5y + 12z = -30$

8. $5x - 50y + z = 24$
$2x + 10y + 3z = 23$
$-5x - 20y + 10z = 13$

Write the augmented matrix for each system of linear equations. (Example 2)

9. $12x - 5y = -9$
$-3x + 8y = 10$

10. $-4x - 6y = 25$
$7x + 2y = 16$

11. $3x - 5y + 7z = 9$
$-10x + y + 8z = 6$
$4x - 15z = -8$

12. $4x - z = 27$
$-8x + 7y - 6z = -35$
$12x - 3y + 5z = 20$

13. $w - 8x + 5y = 11$
$7w + 2x - 3y + 9z = -5$
$6w + 12y - 15z = 4$
$3x + 4y - 8z = -13$

14. $14x - 2y + 3z = -22$
$5w - 4x + 11z = -8$
$2w - 6y + 3z = 15$
$3w + 7x - y = 1$

15. BAKE SALE Members of a youth group held a bake sale to raise money for summer trip. They sold 30 cakes, 40 pies, and 200 giant cookies and raised \$684.50. A pie cost \$2 less than a cake and cake cost 5 times as much as a giant cookie. (Example 2)

a. Let c = number of cakes, p = number of pies, and g = number of giant cookies. Write a system of three linear equations to represent the problem.

b. Write the augmented matrix for the system of linear equations that you wrote in part **a.**

c. Solve the system of equations. Interpret your solution.

Determine whether each matrix is in row-echelon form. (Example 3)

16. $\left[\begin{array}{cc|c} 1 & 8 & 7 \\ 0 & 1 & 3 \end{array}\right]$

17. $\left[\begin{array}{cc|c} 1 & -2 & 9 \\ 0 & 0 & 1 \end{array}\right]$

18. $\left[\begin{array}{cc|c} 1 & -8 & 12 \\ 1 & 3 & -7 \\ 0 & 1 & 4 \end{array}\right]$

19. $\left[\begin{array}{cc|c} 1 & -4 & 10 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 0 & 1 & -7 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$

21. $\left[\begin{array}{ccc|c} 0 & 1 & -8 & 5 \\ 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solve each system of equations using Gaussian or Gauss-Jordan elimination. (Examples 4 and 5)

22. $2x = -10y + 11$
$-8y = -9x + 23$

23. $4y + 17 = -7x$
$8x + 5y = -19$

24. $x + 7y = 10$
$3x + 9y = -6$

25. $7y = 9 - 5x$
$8x = 2 - 5y$

26. $3x - 4y + 8z = 27$
$9x - y - z = 3$
$x + 8y - 2z = 9$

27. $x + 9y + 8z = 0$
$5x + 8y + z = 35$
$x - 4y - z = 17$

28. $4x + 8y - z = 10$
$3x - 8y + 9z = 14$
$7x + 6y + 5z = 0$

29. $2x - 10y + z = 28$
$-5x + 11y + 7z = 18$
$6x - y - 12z = 14$

30. COFFEE A local coffee shop specializes in espresso drinks. The table shows the cups of each drink sold throughout the day. Write and solve a system of equations to determine the price of each espresso drink. Interpret your solution. (Example 4)

Hours	Cappuccino	Latte	Macchiato	Earnings ($)
8–11	103	86	79	1040.25
11–2	48	32	26	406.50
2–5	45	25	18	334.00

31. FLORIST An advertisement for a floral shop shows the price of several flower arrangements and a list of the flowers included in each arrangement as shown below. Write and solve a system of equations to determine the price of each type of flower. Interpret your solution. (Example 6)

Birthday bouquet **\$35.00**
4 roses, 12 lilies, and 5 irises

Sunny Garden **\$50.25**
6 roses, 9 lilies, and 12 irises

Summer Expressions **\$83.75**
10 roses, 15 lilies, and 20 irises

Solve each system of equations. (Examples 6 and 7)

32. $-2x + y - 3z = 0$
$3x - 4y + 10z = -7$
$5x + 2y + 8z = 23$

33. $4x - 5y - 9z = -25$
$-6x + y + 7z = -21$
$7x - 3y - 10z = 8$

34. $-x + 3y + 10z = 8$
$4x - 9y - 34z = -17$
$3x + 5y - 2z = 46$

35. $5x - 4y - 7z = -31$
$2x + y - 8z = 11$
$-4x + 3y + 6z = 23$

36. $-3x + 4y - z = -10$
$6x - y - 5z = -29$
$4x - 5y + z = 11$

37. $8x - 9y - 4z = -33$
$-2x + 3y - 2z = 9$
$-7x + 6y + 11z = 27$

38. $2x - 5y + 4z + 4w = 2$
$-3x + 6y - 2z - 7w = 11$
$5x - 4y + 8z - 5w = 29$

39. $x - 4y + 4z + 3w = 2$
$-2x - 3y + 7z - 3w = -9$
$3x - 5y + z + 10w = 15$

GRAPHING CALCULATOR **Find the row-echelon form and reduced row-echelon form of each system.**

40. $3x + 2.5y = 18$
$6.8x - 4y = 29.2$

41. $\frac{2}{5}x - \frac{1}{2}y = 8$
$\frac{3}{4}x + \frac{5}{8}y = \frac{5}{2}$

42. $7x + \frac{2}{3}y + \frac{1}{6}z = -\frac{13}{3}$
$-\frac{3}{5}x + y - \frac{1}{3}z = \frac{11}{10}$
$2x - \frac{2}{5}y - \frac{1}{2}z = -6$

43. $15.9x - y + 4.3z = 14.8$
$-8.2x + 14y = 14.6$
$-11x + 0.5y - 1.6z = -20.4$

44. FINANCIAL LITERACY A sports equipment company took out three different loans from a bank to buy treadmills. The bank statement after the first year is shown below. The amount borrowed at the 6.5% rate was \$50,000 less than the amounts borrowed at the other two rates combined.

Bay Bank Co.

STATEMENT SUMMARY

Amount Borrowed	\$350,000
Loan 1	6.5% Interest Rate
Loan 2	7% Interest Rate
Loan 3	9% Interest Rate
Interest Paid	\$24,950

a. Write a system of three linear equations to represent this situation.

b. Use a graphing calculator to solve the system of equations. Interpret the solution.

Determine the row operation performed to obtain each matrix.

45. $\left[\begin{array}{ccc|c} 1 & 5 & -6 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -6 & 3 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -1 & -1 \end{array}\right]$

46. $\left[\begin{array}{ccc|c} 3 & 1 & -5 & 4 \\ 9 & -1 & 4 & -2 \\ 8 & 4 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -5 & 4 \\ 9 & -1 & 4 & -2 \\ 2 & 2 & 7 & -7 \end{array}\right]$

47. $\left[\begin{array}{cccc|c} 1 & 15 & 2 & 4 & 14 \\ 0 & 8 & 5 & -5 & 15 \\ 2 & 1 & 0 & 16 & 20 \\ -3 & -11 & -1 & 6 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 15 & 2 & 4 & 14 \\ 0 & 8 & 5 & -5 & 15 \\ 2 & 1 & 0 & 16 & 20 \\ 0 & 34 & 5 & 18 & 38 \end{array}\right]$

48. $\left[\begin{array}{ccccc|c} 8 & -2 & 0 & 2 & 12 & -2 \\ 8 & 5 & -7 & 1 & 6 & 9 \\ -1 & 0 & 9 & 3 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 8 & -2 & 0 & 2 & 12 & -2 \\ 0 & 7 & -7 & -1 & -6 & 11 \\ -1 & 0 & 9 & 3 & 3 & 2 \end{array}\right]$

49. MEDICINE A diluted saline solution is needed for routine procedures in a hospital. The supply room has a large quantity of 20% saline solution and 40% saline solution, but needs 10 liters of 25% saline solution.

a. Write a system of equations to represent this situation.

b. Solve the system of equations. Interpret the solution.

H.O.T. Problems Use Higher-Order Thinking Skills

50. OPEN ENDED Create a system of 3 variable equations that has infinitely many solutions. Explain your reasoning.

51. CHALLENGE Consider the following system of equations. What value of k would make the system consistent and independent?

$$2x + 2y = 5$$
$$5y - kz = -22$$
$$2x + 5z = 26$$
$$-2x + ky + z = -8$$

52. ERROR ANALYSIS Ken and Sari are writing the augmented matrix of the system below in row-echelon form.

$$2x - y + z = 0$$
$$x + y - 2z = -7$$
$$x - 3y + 4z = 9$$

Ken

$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$

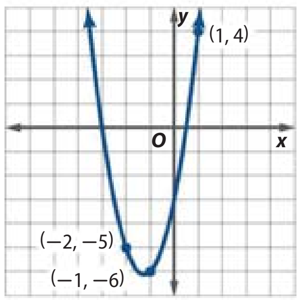
Sari

$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Is either of them correct? Explain your reasoning.

53. REASONING *True* or *false*: If an augmented square matrix in row-echelon form has a row of zeros as its last row, then the corresponding system of equations has no solution. Explain your reasoning.

54. CHALLENGE A parabola passes through the three points shown on the graph below.



a. Write a system of equations that can be used to find the equation of the parabola in the form $f(x) = ax^2 + bx + c$.

b. Use matrices to solve the system of equations that you wrote in part **a.**

c. Use the solution you found in part **b** to write an equation of the parabola. Verify your results using a graphing calculator.

55. WRITING IN MATH Compare and contrast Gaussian elimination and Gauss-Jordan elimination.

Spiral Review

Verify each identity. (Lesson 5-5)

56. $2\cos^2\frac{x}{2} = 1 + \cos x$

57. $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$

58. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$

Find the exact value of each trigonometric expression. (Lesson 5-4)

59. $\cos 105°$

60. $\sin 165°$

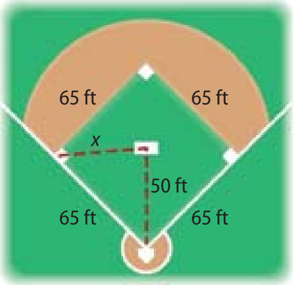
61. $\cos\frac{7\pi}{12}$

62. $\sin\frac{\pi}{12}$

63. $\cot\frac{113\pi}{12}$

64. $\sec 1275°$

65. SOFTBALL In slow-pitch softball, the diamond is a square that is 65 feet on each side. The distance between the pitcher's mound and home plate is 50 feet. How far does the pitcher have to throw the softball from the pitcher's mound to third base to stop a player who is trying to steal third base? (Lesson 4-7)



66. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls? (Lesson 4-1)

67. RABBITS Rabbits reproduce at a tremendous rate and their population increases exponentially in the absence of natural enemies. Suppose there were originally 65,000 rabbits in a region, and two years later there were 2,500,000. (Lesson 3-1)

a. Write an exponential function that could be used to model the rabbit population y in that region. Write the function in terms of x, the number of years since the original year.

b. Assume that the rabbit population continued to grow at that rate. Estimate the rabbit population in that region seven years after the initial year.

Solve each equation. (Lesson 2-5)

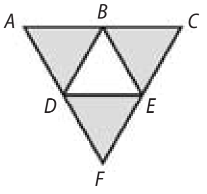
68. $\frac{3}{x} + \frac{2}{x-1} = \frac{17}{12}$

69. $\frac{4}{x+3} - \frac{2}{x+1} = \frac{2}{15}$

70. $\frac{4}{3} - \frac{1}{x-2} = \frac{13}{2x}$

Skills Review for Standardized Tests

71. SAT/ACT $\triangle ACF$ is equilateral with sides of length 4. If B, D, and E are the midpoints of their respective sides, what is the sum of the areas of the shaded regions?



A $3\sqrt{2}$ **C** $4\sqrt{2}$ **E** $6\sqrt{3}$

B $3\sqrt{3}$ **D** $4\sqrt{3}$

72. REVIEW The caterer for a lunch bought several pounds of chicken and tuna salad. The chicken salad cost \$9 per pound, and the tuna salad cost \$6 per pound. He bought a total of 14 pounds of salad and paid a total of \$111. How much chicken salad did the caterer buy?

F 6 pounds **H** 8 pounds

G 7 pounds **J** 9 pounds

73. The Yogurt Shoppe sells small cones for \$0.89, medium cones for \$1.19, and large cones for \$1.39. One day, Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many medium cones did he sell?

A 11 **C** 24

B 17 **D** 36

74. REVIEW To practice at home, Tate purchased a basketball and a volleyball that cost a total of \$67, not including tax. If the cost of the basketball b was \$4 more than twice the cost of the volleyball v, which system of linear equations could be used to determine the cost of each ball?

F $b + v = 67$, $b = 2v - 4$

G $b + v = 67$, $b = 2v + 4$

H $b + v = 4$, $b = 2v - 67$

J $b + v = 4$, $b = 2v + 67$

LESSON 6-2

Matrix Multiplication, Inverses, and Determinants

Then

- You performed operations on matrices. (Lesson 0-5)

Now

1. Multiply matrices.
2. Find determinants and inverses of 2 × 2 and 3 × 3 matrices.

Why?

- Matrices are used in many industries as a convenient method to store data. In the restaurant business, matrix multiplication can be used to determine the amount of raw materials that are necessary to produce the desired final product, or items on the menu.

NewVocabulary

identity matrix
inverse matrix
inverse
invertible
singular matrix
determinant

1 Multiply Matrices

The three basic matrix operations are matrix addition, scalar multiplication, and matrix multiplication. You have seen that adding matrices is similar to adding real numbers, and multiplying a matrix by a scalar is similar to multiplying real numbers.

Matrix Addition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

Scalar Multiplication

$$k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$$

Matrix multiplication has no operational counterpart in the real number system. To multiply matrix A by matrix B, the number of columns in A *must* be equal to the number of rows in B. This can be determined by examining the dimensions of A and B. If it exists, product matrix AB has the same number of rows as A and the same number of columns as B.

matrix A	·	matrix B	=	AB
3×2		2×4		3×4

Equal

Dimensions of AB

KeyConcept Matrix Multiplication

Words If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is an $m \times n$ matrix obtained by adding the products of the entries of a row in A to the corresponding entries of a column in B.

Symbols If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is an $m \times n$ matrix in which

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}.$$

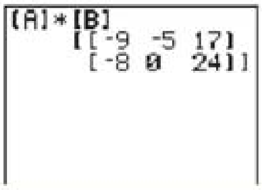
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \dots & b_{rj} & \dots & b_{rn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots & & \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{in} \\ \vdots & \vdots & \dots & \vdots & & \\ c_{m1} & c_{m2} & & c_{mj} & \dots & c_{mn} \end{bmatrix}$$

Each entry in the product of two matrices can be thought of as the product of a $1 \times r$ row matrix and an $r \times 1$ column matrix. Consider the product of the 1×3 row matrix and 3×1 column matrix shown.

$$\begin{bmatrix} -2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -2(4) + 1(-6) + 3(5) \end{bmatrix} \text{ or } \begin{bmatrix} 1 \end{bmatrix}$$

Technology Tip

Multiplying Matrices You can use a graphing calculator to multiply matrices. Define *A* and *B* in the matrix list, and then multiply the matrices using their letter references. Notice that the calculator displays rows of the product in Example 1a using 1×3 matrices.



Example 1 Multiply Matrices

Use matrices $A = \begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix}$ to find each product, if possible.

a. AB

$$AB = \begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix}$$ Dimensions of *A*: 2×2, Dimensions of *B*: 2×3

A is a 2×2 matrix and *B* is a 2×3 matrix. Because the number of columns for *A* is equal to the number of rows for *B*, the product *AB* exists.

To find the first entry in *AB*, write the sum of the products of the entries in row 1 of *A* and in column 1 of *B*.

$$\begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3(-2) + (-1)(3) & & \\ & & \end{bmatrix}$$

Follow this same procedure to find the entry for row 1, column 2 of *AB*.

$$\begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3(-2) + (-1)(3) & 3(0) + (-1)(5) & \\ & & \end{bmatrix}$$

Continue multiplying each row by each column to find the sum for each entry.

$$\begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3(-2) + (-1)(3) & 3(0) + (-1)(5) & 3(6) + (-1)(1) \\ 4(-2) + 0(3) & 4(0) + 0(5) & 4(6) + 0(1) \end{bmatrix}$$

Finally, simplify each sum.

$$\begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -5 & 17 \\ -8 & 0 & 24 \end{bmatrix}$$

b. BA

$$BA = \begin{bmatrix} -2 & 0 & 6 \\ 3 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix}$$ Dimensions of *B*: 2×3, Dimensions of *A*: 2×2

Because the number of columns for *B* is not the same as the number of rows for *A*, the product *BA* does *not* exist. *BA* is undefined.

Guided Practice

Find *AB* and *BA*, if possible.

1A. $A = \begin{bmatrix} 3 & 1 & -5 \\ -2 & 0 & 4 \end{bmatrix}$

$B = \begin{bmatrix} -6 & 1 & 7 \\ 4 & -5 & 3 \end{bmatrix}$

1B. $A = \begin{bmatrix} -2 & 0 & 3 \\ 5 & -7 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 0 \\ -1 & 0 \\ 9 & 3 \end{bmatrix}$

Notice in Example 1 that the two products *AB* and *BA* are different. In most cases, even when both products are defined, $AB \neq BA$. This means that the Commutative Property does not hold for matrix multiplication. However, some of the properties of real numbers do hold for matrix multiplication.

KeyConcept Properties of Matrix Multiplication

For any matrices A, B, and C for which the matrix product is defined and any scalar k, the following properties are true.

Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$k(AB) = (kA)B = A(kB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

You will prove these properties in Exercises 72–75.

Matrix multiplication can be used to solve real-world problems.

Real-WorldLink

In the 2008 election, Barack Obama received 66,882,230 votes or 53% of the popular vote.

Source: CNN

Real-World Example 2 Multiply Matrices

VOTING **The percent of voters of different ages who were registered as Democrats, Republicans, or Independents in a recent city election are shown. Use this information to determine whether there were more male voters registered as Democrats than there were female voters registered as Republicans.**

Distribution by Party and Age (%)

Party	18–25	26–40	41–50	50+
Democrat	0.55	0.50	0.35	0.40
Republican	0.30	0.40	0.45	0.55
Independent	0.15	0.10	0.20	0.05

Distribution by Age and Gender

Age	Female	Male
18–25	18,500	16,000
26–40	20,000	24,000
41–50	24,500	22,500
50+	16,500	14,000

Let matrix X represent the distribution by party and age, and let matrix Y represent the distribution by age and gender. Then find the product XY.

$$XY = \begin{bmatrix} 0.55 & 0.50 & 0.35 & 0.40 \\ 0.30 & 0.40 & 0.45 & 0.55 \\ 0.15 & 0.10 & 0.20 & 0.05 \end{bmatrix} \cdot \begin{bmatrix} 18{,}500 & 16{,}000 \\ 20{,}000 & 24{,}000 \\ 24{,}500 & 22{,}500 \\ 16{,}500 & 14{,}000 \end{bmatrix} = \begin{bmatrix} 35{,}350 & 34{,}275 \\ 33{,}650 & 32{,}225 \\ 10{,}500 & 10{,}000 \end{bmatrix}$$

The product XY represents the distribution of male and female voters that were registered in each party. You can use the product matrix to find the number of male voters that were registered as Democrat and the number of female voters registered as Republican.

	Female	Male
Democrat	35,350	**34,275**
Republican	**33,650**	32,225
Independent	10,500	10,000

More male voters were registered as Democrat than female voters registered as Republican because 34,275 > 33,650.

GuidedPractice

2. **SALES** The number of laptops that a company sold in the first three months of the year is shown, as well as the price per model during those months. Use this information to determine which model generated the most income for the first three months.

Month	Model 1	Model 2	Model 3
Jan.	150	250	550
Feb.	200	625	100
Mar.	600	100	350

Model	Jan.	Feb.	Mar.
1	$650	$575	$485
2	$800	$700	$775
3	$900	$1050	$925

Paul J. Richards/AFP/Getty Images

You know that the *multiplicative identity* for real numbers is 1, because for any real number a, $a \cdot 1 = a$. The multiplicative identity $n \times n$ square matrices is called the **identity matrix**.

ReadingMath

Identity Matrix The notation I_n is used to represent the identity for $n \times n$ matrices. The notation I is used instead of I_2, I_3, I_4, etc., when the order of the identity is known.

KeyConcept Identity Matrix

Words The identity matrix of order n, denoted I_n, is an $n \times n$ matrix consisting of all 1s on its main diagonal, from upper left to lower right, and 0s for all other elements.

Symbols

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

So, if A is an $n \times n$ matrix, then $AI_n = I_nA = A$. You may find an identity matrix as the left side of many augmented matrices in reduced row-echelon form. In general, if A is the coefficient matrix of a system of equations, X is the column matrix of variables, and B is the column matrix of constants, then you can write the system of equations as an equation of matrices.

System of Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

→

Matrix Equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A \cdot X = B$

Example 3 Solve a System of Linear Equations

Write the system of equations as a matrix equation, $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix to solve the system.

$$-x_1 + x_2 - 2x_3 = 2$$
$$-2x_1 + 3x_2 - 4x_3 = 5$$
$$3x_1 - 4x_2 + 7x_3 = -1$$

Write the system matrix in form, $AX = B$.

$$\begin{bmatrix} -1 & 1 & -2 \\ -2 & 3 & -4 \\ 3 & -4 & 7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$ $A \cdot X = B$

ReadingMath

Augmented Matrices The notation $[A \vdots B]$, read *A augmented with B*, represents the augmented matrix that results when matrix B is attached to matrix A.

Write the augmented matrix $[A \vdots B]$. Use Gauss-Jordan elimination to solve the system.

$$[A \vdots B] = \left[\begin{array}{ccc:c} -1 & 1 & -2 & 2 \\ -2 & 3 & -4 & 5 \\ 3 & -4 & 7 & -1 \end{array}\right]$$ Augmented matrix

$$[I \vdots X] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{array}\right]$$ Use elementary row operations to transform A into I.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \\ 6 \end{bmatrix}$$ The solution of the equation is given by X.

Therefore, the solution of the system of equations is $(-13, 1, 6)$.

GuidedPractice

Write each system of equations as a matrix equation, $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve the system.

3A. $x_1 - 2x_2 - 3x_3 = 9$
$-4x_1 + x_2 + 8x_3 = -16$
$2x_1 + 3x_2 + 2x_3 = 6$

3B. $x_1 + x_2 + x_3 = 2$
$2x_1 - x_2 + 2x_3 = 4$
$-x_1 + 4x_2 + x_3 = 3$

2 Inverses and Determinants

You know that if a is a nonzero real number, then $\frac{1}{a}$ or a^{-1} is the multiplicative inverse of a because $a\left(\frac{1}{a}\right) = a \cdot a^{-1} = 1$. The multiplicative inverse of a square matrix is called its **inverse matrix**.

ReadingMath

Inverse Matrix The notation A^{-1} is read *A inverse*.

KeyConcept Inverse of a Square Matrix

Let A be an $n \times n$ matrix. If there exists a matrix B such that $AB = BA = I_n$, then B is called the **inverse** of A and is written as A^{-1}. So, $AA^{-1} = A^{-1}A = I_n$.

Example 4 Verify an Inverse Matrix

Determine whether $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ are inverse matrices.

If A and B are inverse matrices, then $AB = BA = I$.

$$AB = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3+4 & 6+(-6) \\ -2+2 & 4+(-3) \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3+4 & 2+(-2) \\ -6+6 & 4+(-3) \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $AB = BA = I$, it follows that $B = A^{-1}$ and $A = B^{-1}$.

GuidedPractice

Determine whether A and B are inverse matrices.

4A. $A = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

4B. $A = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$

StudyTip

Singular Matrix If a matrix is singular, then the matrix equation $AB = I$ will have no solution.

If a matrix A has an inverse, then A is said to be **invertible** or *nonsingular*. A **singular matrix** does not have an inverse. Not all square matrices are invertible. To find the inverse of a square matrix A, you need to find a matrix A^{-1}, assuming A^{-1} exists, such that the product of A and A^{-1} is the identity matrix. In other words, you need to solve the matrix equation $AA^{-1} = I_n$ for B. Once B is determined, you will then need to confirm that $AA^{-1} = A^{-1}A = I_n$.

One method for finding the inverse of a square matrix is to use a system of equations. Let $A = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$, and suppose A^{-1} exists. Write the matrix equation $AA^{-1} = I_2$, where $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^{-1} = I_2$$

$$\begin{bmatrix} 8a-5c & 8b-5d \\ -3a+2c & -3b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Matrix multiplication}$$

$$\begin{aligned} 8a-5c &= 1 & 8b-5d &= 0 \\ -3a+2c &= 0 & -3b+2d &= 1 \end{aligned} \qquad \text{Equate corresponding elements.}$$

From this set of four equations, you can see that there are two systems of equations that each have two unknowns. Write the corresponding augmented matrices.

$$\left[\begin{array}{cc|c} 8 & -5 & 1 \\ -3 & 2 & 0 \end{array}\right] \left[\begin{array}{cc|c} 8 & -5 & 0 \\ -3 & 2 & 1 \end{array}\right]$$

Notice that the augmented matrix of each system has the same coefficient matrix, $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$.

Because the coefficient matrix of the systems is the same, we can perform row reductions on the two augmented matrices simultaneously by creating a *doubly augmented matrix*, $[A \mid I]$. To find A^{-1}, use the doubly augmented matrix $\left[\begin{array}{cc|cc} 8 & -5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right]$.

Example 5 Inverse of a Matrix

Find A^{-1}, if it exists. If A^{-1} does not exist, write *singular*.

a. $A = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

Step 1 Create the doubly augmented matrix $[A \,\vdots\, I]$.

$$[A \,\vdots\, I] = \left[\begin{array}{cc:cc} 8 & -5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right]$$ Doubly augmented matrix

Step 2 Apply elementary row operations to write the matrix in reduced row-echelon form.

$$\left[\begin{array}{cc:cc} 8 & -5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right]$$

$$3R_1 + 8R_2 \rightarrow \left[\begin{array}{cc:cc} 8 & -5 & 1 & 0 \\ 0 & 1 & 3 & 8 \end{array}\right]$$

$$R_1 + 5R_2 \rightarrow \left[\begin{array}{cc:cc} 8 & 0 & 16 & 40 \\ 0 & 1 & 3 & 8 \end{array}\right]$$

$$\tfrac{1}{8}R_1 \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 8 \end{array}\right] = [I \,\vdots\, A^{-1}]$$

The first two columns are the identity matrix. Therefore, A is invertible and $A^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$.

CHECK Confirm that $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } I \checkmark$$

$$A^{-1}A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } I \checkmark$$

b. $A = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$

Step 1 $[A \,\vdots\, I] = \left[\begin{array}{cc:cc} 2 & 4 & 1 & 0 \\ -3 & -6 & 0 & 1 \end{array}\right]$ Doubly augmented matrix

Step 2 $\tfrac{1}{2}R_1 \rightarrow \left[\begin{array}{cc:cc} 1 & 2 & \frac{1}{2} & 0 \\ -3 & -6 & 0 & 1 \end{array}\right]$ Apply elementary row operations to write the matrix in reduced row-echelon form.

$3R_1 + R_2 \rightarrow \left[\begin{array}{cc:cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 1 \end{array}\right]$ Notice that it is impossible to obtain the identity matrix I on the left-side of the doubly augmented matrix

Therefore, A is singular.

Guided Practice

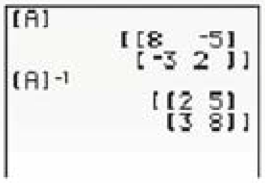
5A. $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

5B. $\begin{bmatrix} -1 & -2 & -2 \\ 3 & 7 & 9 \\ 1 & 4 & 7 \end{bmatrix}$

5C. $\begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}$

Technology Tip

Inverse You can use x^{-1} on your graphing calculator to find the inverse of a square matrix.



The process used to find the inverse of a square matrix is summarized below.

Technology Tip

Singular Matrices If a matrix is singular, your graphing calculator will display the following error message.

ERR: SINGULAR MAT

Concept Summary Finding the Inverse of a Square Matrix

Let A be an $n \times n$ matrix.

1. Write the augmented matrix $[A \,\vdots\, I_n]$.
2. Perform elementary row operations on the augmented matrix to reduce A to its reduced row-echelon form.
3. Decide whether A is invertible.
   - If A can be reduced to the identity matrix I_n, then A^{-1} is the matrix on the right of the transformed augmented matrix, $[I_n \,\vdots\, A^{-1}]$.
   - If A cannot be reduced to the identity matrix I_n, then A is singular.

While the method of finding an inverse matrix used in Example 5 works well for any square matrix, you may find the following formula helpful when finding the inverse of a 2 × 2 matrix.

KeyConcept Inverse and Determinant of a 2 × 2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A is invertible if and only if $ad - cb \neq 0$.

If A is invertible, then $A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The number $ad - cb$ is called the **determinant** of the 2 × 2 matrix and is denoted

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

You will prove this Theorem in Exercise 66.

Therefore, the determinant of a 2 × 2 matrix provides a test for determining if the matrix is invertible.

Notice that the determinant of a 2 × 2 matrix is the difference of the product of the two diagonals of the matrix.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

StudyTip

Inverse of a 2 × 2 Matrix The formula for the inverse of a 2 × 2 matrix is sometimes written as $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 6 Determinant and Inverse of a 2 × 2 Matrix

Find the determinant of each matrix. Then find the inverse of the matrix, if it exists.

a. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix}$ — $a = 2, b = -3, c = 4$, and $d = 4$

$= 2(4) - 4(-3)$ or 20 — $ad - cb$

Because $\det(A) \neq 0$, A is invertible. Apply the formula for the inverse of a 2 × 2 matrix.

$A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ — Inverse of 2 × 2 matrix

$= \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$ — $a = 2, b = -3, c = 4, d = 4$, and $ad - cb = 20$

$= \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$ — Scalar multiplication

CHECK $AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✔

b. $B = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$

$\det(B) = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 6(6) - 9(4)$ or 0

Because $\det(B) = 0$, B is not invertible. Therefore, B^{-1} does not exist.

GuidedPractice

6A. $\begin{bmatrix} -4 & 6 \\ 8 & -12 \end{bmatrix}$

6B. $\begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix}$

Technology Tip

Determinants You can use the det(function on a graphing calculator to find the determinant of a square matrix. If you try to find the determinant of a matrix with dimensions other than $n \times n$, your calculator will display the following error message.

ERR:INVALID DIM

The determinant for a 3 ×3 matrix is defined using 2 × 2 determinants as shown.

Key Concept Determinant of a 3 × 3 Matrix

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then $\det(A) = |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

As with 2 × 2 matrices, a 3 × 3 matrix A has an inverse if and only if $\det(A) \neq 0$. A formula for calculating the inverse of 3 × 3 and higher order matrices exists. However, due to the complexity of this formula, we will use a graphing calculator to calculate the inverse of 3 × 3 and higher-order square matrices.

Example 7 Determinant and Inverse of a 3 × 3 Matrix

Find the determinant of $C = \begin{bmatrix} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \end{bmatrix}$. Then find C^{-1}, if it exists.

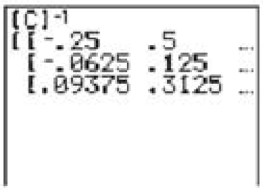
$$\det(C) = \begin{vmatrix} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \end{vmatrix}$$

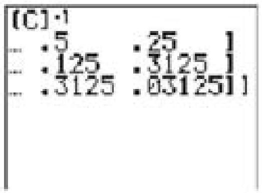
$$= -3\begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= -3[(-1)(0) - 4(2)] - 2[1(0) - (-1)(2)] + 4[1(4) - (-1)(-1)]$$

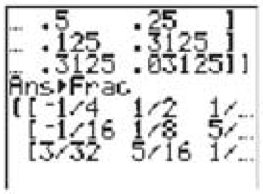
$$= -3(-8) - 2(2) + 4(3) \text{ or } 32$$

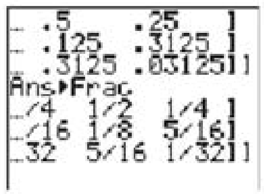
Because $\det(A)$ does not equal zero, C^{-1} exists. Use a graphing calculator to find C^{-1}.





You can use the ▸Frac feature under the MATH menu to write the inverse using fractions, as shown below.





Therefore, $C^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{16} & \frac{1}{8} & \frac{5}{16} \\ \frac{3}{32} & \frac{5}{16} & \frac{1}{32} \end{bmatrix}$.

Guided Practice

Find the determinant of each matrix. Then find its inverse, if it exists.

7A. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

7B. $\begin{bmatrix} -1 & -2 & 1 \\ 4 & 0 & 3 \\ -3 & 1 & -2 \end{bmatrix}$

Exercises

= Step-by-Step Solutions begin on page R29.

Find AB and BA, if possible. (Example 1)

1. $A = \begin{bmatrix} 8 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -7 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 9 \\ -7 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -4 \\ 0 & 3 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & -2 \\ 1 & -3 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 & -10 & 9 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 & -1 \\ -4 & 9 & 8 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 0 \\ -4 & -3 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 & -5 \\ 2 & -7 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 3 & 4 \\ -7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 & -8 \\ -6 & 0 & 9 \end{bmatrix}$

8. $A = \begin{bmatrix} 6 & -9 & 10 \\ 4 & 3 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -8 \\ 3 & -9 \\ -2 & 5 \\ 4 & 1 \end{bmatrix}$

9. **BASKETBALL** Different point values are awarded for different shots in basketball. Use the information to determine the total amount of points scored by each player. (Example 2)

Player	FT	2-pointer	3-pointer
Rey	44	32	25
Chris	37	24	31
Jerry	35	39	29

Shots	Points
free throw	1
2-pointer	2
3-pointer	3

10. **CARS** The number of vehicles that a company manufactures each day from two different factories is shown, as well as the price of the vehicle during each sales quarter of the year. Use this information to determine which factory produced the highest sales in the 4th quarter. (Example 2)

Factory	Model			
	Coupe	Sedan	SUV	Mini Van
1	500	600	150	250
2	250	350	250	400

Model	Quarter			
	1st ($)	2nd ($)	3rd ($)	4th ($)
Coupe	18,700	17,100	16,200	15,600
Sedan	25,400	24,600	23,900	23,400
SUV	36,300	35,500	34,900	34,500
Mini Van	38,600	37,900	37,400	36,900

Write each system of equations as a matrix equation, $AX = B$. Then use Gauss-Jordan elimination on the augmented matrix to solve the system. (Example 3)

11. $2x_1 - 5x_2 + 3x_3 = 9$
$4x_1 + x_2 - 6x_3 = 35$
$-3x_1 + 9x_2 - 7x_3 = -6$

12. $3x_1 - 10x_2 - x_3 = 6$
$-5x_1 + 12x_2 + 2x_3 = -5$
$-4x_1 - 8x_2 + 3x_3 = 16$

13. $2x_1 - 10x_2 + 7x_3 = 7$
$6x_1 - x_2 + 5x_3 = -2$
$-4x_1 + 8x_2 - 3x_3 = -22$

14. $x_1 + 5x_2 + 5x_3 = -18$
$-7x_1 - 3x_2 + 2x_3 = -3$
$6x_1 + 7x_2 - x_3 = 42$

15. $2x_1 + 6x_2 - 5x_3 = -20$
$8x_1 - 12x_2 + 7x_3 = 28$
$-4x_1 + 10x_2 - x_3 = 7$

16. $3x_1 - 5x_2 + 12x_3 = 9$
$2x_1 + 4x_2 - 11x_3 = 1$
$-5x_1 + 7x_2 - 15x_3 = -28$

17. $-x_1 - 3x_2 + 9x_3 = 25$
$-5x_1 + 11x_2 + 8x_3 = 33$
$2x_1 + x_2 - 13x_3 = -45$

18. $x_1 - 8x_2 - 3x_3 = -4$
$-3x_1 + 10x_2 + 5x_3 = -42$
$2x_1 + 7x_2 + 3x_3 = 20$

Determine whether A and B are inverse matrices. (Example 4)

19. $A = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

20. $A = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

21. $A = \begin{bmatrix} -5 & 3 \\ 6 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

22. $A = \begin{bmatrix} -8 & 4 \\ 6 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

23. $A = \begin{bmatrix} 9 & 2 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 5 & -9 \end{bmatrix}$

24. $A = \begin{bmatrix} 7 & 5 \\ -6 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -5 \\ 6 & 7 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -3 \\ -3 & -2 \end{bmatrix}$

26. $A = \begin{bmatrix} 9 & -7 \\ 8 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -6 \\ 4 & 10 \end{bmatrix}$

Find A^{-1}, if it exists. If A^{-1} does not exist, write *singular*. (Example 5)

27. $A = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$

28. $A = \begin{bmatrix} -4 & 8 \\ 1 & -2 \end{bmatrix}$

29. $A = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$

30. $A = \begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}$

31. $A = \begin{bmatrix} -1 & -1 & -3 \\ 3 & 6 & 4 \\ 2 & 1 & 8 \end{bmatrix}$

32. $A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & 3 & 5 \\ 6 & -1 & -4 \end{bmatrix}$

33. $A = \begin{bmatrix} 5 & 2 & -1 \\ 4 & 7 & -3 \\ 1 & -5 & 2 \end{bmatrix}$

34. $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}$

Find the determinant of each matrix. Then find the inverse of the matrix, if it exists. (Examples 6 and 7)

35. $\begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}$

36. $\begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix}$

37. $\begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix}$

38. $\begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}$

39. $\begin{bmatrix} 3 & 1 & -2 \\ 8 & -5 & 2 \\ -4 & 3 & -1 \end{bmatrix}$

40. $\begin{bmatrix} 1 & -1 & -2 \\ 5 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$

41. $\begin{bmatrix} 9 & 3 & 7 \\ -6 & -2 & -5 \\ 3 & 1 & 4 \end{bmatrix}$

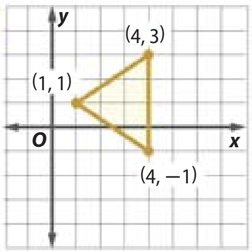
42. $\begin{bmatrix} 2 & 3 & -1 \\ -4 & -5 & 2 \\ 6 & 1 & 3 \end{bmatrix}$

43. $\begin{bmatrix} -1 & 3 & 2 \\ 3 & -5 & -3 \\ 4 & 2 & 6 \end{bmatrix}$

44. $\begin{bmatrix} 6 & -1 & 2 \\ 1 & -2 & -4 \\ -3 & 1 & -5 \end{bmatrix}$

Find the area A of each triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), by using $A = \frac{1}{2}|\det(X)|$, where X is $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$.

45.



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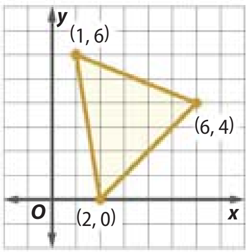

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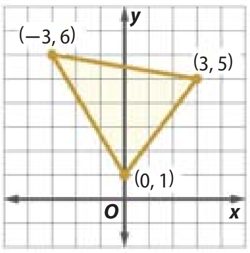
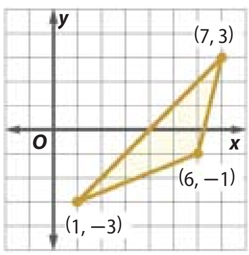


48.



Given A and AB, find B.

49. $A = \begin{bmatrix} 8 & -4 \\ 3 & 6 \end{bmatrix}$, $AB = \begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix}$

50. $A = \begin{bmatrix} 5 & 0 & 1 \\ 2 & -3 & 2 \\ 1 & -1 & 4 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 4 \\ -16 & -6 \\ -2 & -5 \end{bmatrix}$

Find x and y.

51. $A = \begin{bmatrix} 2x & -y \\ -3y & 5x \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, and $AB = \begin{bmatrix} -2 \\ 31 \end{bmatrix}$

Find the determinant of each matrix.

52. $\begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix}$

53. $\begin{bmatrix} c & c & c \\ 0 & c & c \\ 0 & 0 & c \end{bmatrix}$

54. FUNDRAISER Hawthorne High School had a fundraiser selling popcorn. The school bought the four flavors of popcorn by the case. The prices paid for the different types of popcorn and the selling prices are shown.

Class	Cases of Popcorn			
	Butter	Kettle	Cheese	Caramel
freshman	152	80	125	136
sophomore	112	92	112	150
junior	176	90	118	122
senior	140	102	106	143

Flavor	Price Paid per Case ($)	Selling Price per Case ($)	Profit per Case ($)
butter	18.90	42.00	
kettle	21.00	45.00	
cheese	23.10	48.00	
caramel	25.20	51.00	

a. Complete the last column of the second table.

b. Which class had the highest total sales?

c. How much more profit did the seniors earn than the sophomores?

55. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$.

a. Find A^2, A^3, and A^4. Then use the pattern to write a matrix for A^n.

b. Find B^2, B^3, B^4, and B^5. Then use the pattern to write a general formula for B^n.

c. Find C^2, C^3, C^4, C^5, … until you notice a pattern. Then use the pattern to write a general formula for C^n.

d. Use the formula that you wrote in part **c** to find C^7.

56. HORSES The owner of each horse stable listed below buys bales of hay and bags of feed each month. In May, hay cost \$2.50 per bale and feed cost \$7.95 per bag. In June, the cost per bale of hay was \$3.00 and the cost per bag of feed was \$6.75.

Stables	Bales of Hay	Bags of Feed
Galloping Hills	45	5
Amazing Acres	75	9
Fairwind Farms	135	16
Saddle-Up Stables	90	11

a. Write a matrix X to represent the bales of hay i and bags of feed j that are bought monthly by each stable.

b. Write a matrix Y to represent the costs per bale of hay and bags of feed for May and June.

c. Find the product YX, and label its rows and columns.

d. How much more were the total costs in June for Fairwind Farms than the total costs in May for Galloping Hills?

Evaluate each expression.

$A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 2 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} -1 & 9 & -6 \\ 7 & 5 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & -2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 7 & 2 \\ -4 & -1 \end{bmatrix}$

57. $BD + B$ **58.** $DC - A$

59. $B(A + C)$ **60.** $AB + CB$

Solve each equation for X, if possible.

$A = \begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 5 & 3 \\ -6 & -2 \end{bmatrix}$

$B = \begin{bmatrix} 8 & 4 \\ 1 & -3 \\ 6 & 7 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & -2 \end{bmatrix}$

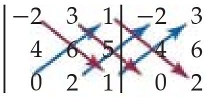
61. $A + C = 2X$ **62.** $4X + A = C$

63. $B - 3X = D$ **64.** $DA = 7X$

65. 3 × 3 DETERMINANTS In this problem, you will investigate an alternative method for calculating the determinant of a 3 × 3 matrix.

a. Calculate $\det(A) = \begin{vmatrix} -2 & 3 & 1 \\ 4 & 6 & 5 \\ 0 & 2 & 1 \end{vmatrix}$ using the method shown in this lesson.

b. Adjoin the first two columns to the right of det(A) as shown. Then find the difference between the sum of the products along the indicated downward diagonals and the sum of the products along the indicated upward diagonals.



c. Compare your answers in parts **a** and **b.**

d. Show that, in general, the determinant of a 3 × 3 matrix can be found using the procedure described above.

e. Does this method work for a 4 × 4 matrix? If so, explain your reasoning. If not, provide a counterexample.

66. PROOF Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$.

Use the matrix equation $AA^{-1} = I_2$ to derive the formula for the inverse of a 2 × 2 matrix.

67. PROOF Write a paragraph proof to show that if a square matrix has an inverse, that inverse is unique. (*Hint:* Assume that a square matrix A has inverses B and C. Then show that $B = C$.)

68. MULTIPLE REPRESENTATIONS In this problem, you will explore square matrices. A square matrix is called *upper triangular* if all elements below the main diagonal are 0, and *lower triangular* if all elements above the main diagonal are 0. If all elements not on the diagonal of a matrix are 0, then the matrix is called *diagonal*. In this problem, you will investigate the determinants of 3 × 3 upper triangular, lower triangular, and diagonal matrices.

a. ANALYTICAL Write one upper triangular, one lower triangular, and one diagonal 2 × 2 matrix. Then find the determinant of each matrix.

b. ANALYTICAL Write one upper triangular, one lower triangular, and one diagonal 3 × 3 matrix. Then find the determinant of each matrix.

c. VERBAL Make a conjecture as to the value of the determinant of any 3 × 3 upper triangular, lower triangular, or diagonal matrix.

d. ANALYTICAL Find the inverse of each of the diagonal matrices you wrote in part **a** and **b.**

e. VERBAL Make a conjecture about the inverse of any 3 × 3 diagonal matrix.

H.O.T. Problems Use Higher-Order Thinking Skills

69. CHALLENGE Given A and AB, find B.

$A = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 0 & 2 \\ 6 & 4 & 1 \end{bmatrix}, AB = \begin{bmatrix} 14 & 6 & 33 \\ 4 & 4 & 13 \\ 1 & 18 & 12 \end{bmatrix}$

70. REASONING Explain why a nonsquare matrix cannot have an inverse.

71. OPEN ENDED Write two matrices A and B such that $AB = BA$, but neither A nor B is the identity matrix.

PROOF Show that each property is true for all 2 × 2 matrices.

72. Right Distributive Property

73. Left Distributive Property

74. Associative Property of Matrix Multiplication

75. Associative Property of Scalar Multiplication

76. ERROR ANALYSIS Alexis and Paul are discussing determinants. Alexis theorizes that the determinant of a 2 × 2 matrix A remains unchanged if two rows of the matrix are interchanged. Paul theorizes that the determinant of the new matrix will have the same absolute value but will be different in sign. Is either of them correct? Explain your reasoning.

77. REASONING If AB has dimensions 5 × 8, with A having dimensions 5 × 6, what are the dimensions of B?

78. WRITING IN MATH Explain why order is important when finding the product of two matrices A and B. Give some general examples to support your answer.

Spiral Review

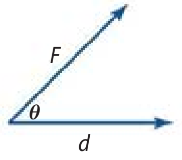
Write the augmented matrix for each system of linear equations. (Lesson 6-1)

79. $10x - 3y = -12$
$-6x + 4y = 20$

80. $15x + 7y - 2z = 41$
$9x - 8y + z = 32$
$5x + y - 11z = 36$

81. $w - 6x + 14y = 19$
$3w + 2x - 4y + 8z = -2$
$9w + 18y - 12z = 3$
$5x + 10y - 16z = -26$

82. PHYSICS The work done to move an object is given by $W = Fd \cos \theta$, where θ is the angle between the displacement d and the force exerted F. If Lisa does 2400 joules of work while exerting a force of 120 newtons over 40 meters, at what angle was she exerting the force? (Lesson 5-5)



Write each degree measure in radians as a multiple of π and each radian measure in degrees. (Lesson 4-2)

83. $-10°$

84. $485°$

85. $\frac{3\pi}{4}$

Solve each equation. (Lesson 3-4)

86. $\log_{10} \sqrt[3]{10} = x$

87. $2 \log_5 (x - 2) = \log_5 36$

88. $\log_5 (x + 4) + \log_5 8 = \log_5 64$

89. $\log_4 (x - 3) + \log_4 (x + 3) = 2$

90. $\frac{1}{2}(\log_7 x + \log_7 8) = \log_7 16$

91. $\log_{12} x = \frac{1}{2} \log_{12} 9 + \frac{1}{3} \log_{12} 27$

92. AERONAUTICS The data below represent the lift of a jet model's wing in a wind tunnel at certain angles of attack. The angle of attack α of the wing is the angle between the wing and the flow of the wind. (Lesson 2-1)

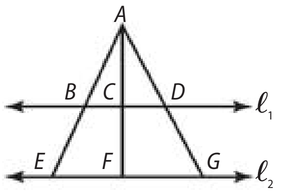
Angle of Attack α	0.1	0.5	1.0	1.5	2.0	3.0	5.0	10.0
Lift (lbs)	11.7	236.0	476.2	711.6	950.3	1782.6	2384.4	4049.3

a. Determine a power function to model the data.

b. Use the function to predict the lift of the wing at 4.0 degrees.

Skills Review for Standardized Tests

93. SAT/ACT In the figure, $\ell_1 \parallel \ell_2$. If $EF = x$, and $EG = y$, which of the following represents the ratio of CD to BC?



A $1 - \frac{y}{x}$

B $1 + \frac{y}{x}$

C $\frac{y}{x} - 1$

D $1 - \frac{x}{y}$

E $1 + \frac{x}{y}$

94. What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

F 1×3

G 3×1

H 3×3

J 4×3

95. REVIEW Shenae spent $42 on 1 can of primer and 2 cans of paint for her room. If the price of one can of paint p is 150% of the price of one can of primer r, which system of equations can be used to find the price of paint and primer?

A $p = r + \frac{1}{2}r, r + 2p = 42$

B $p = r + 2r, r + \frac{1}{2}p = 42$

C $r = p + \frac{1}{2}p, r + 2r = 42$

D $r = p + 2p, r + \frac{1}{2} = 42$

96. REVIEW To join the football team, a student must have a GPA of at least 2.0 and must have attended at least five after-school practices. Which system of inequalities best represents this situation if x represents a student's GPA, and y represents the number of after-school practices the student attended?

F $x \geq 2, y \geq 5$

G $x \leq 2, y \leq 5$

H $x < 2, y < 5$

J $x > 2, y > 5$

EXTEND 6-2

Graphing Technology Lab
Determinants and Areas of Polygons

Objective

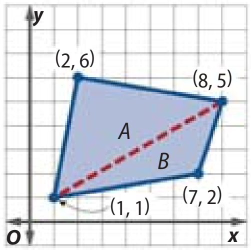
- Use a graphing calculator to find areas of polygons using determinants.

In Lesson 6-2, you learned that the area of a triangle X with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) can be found by calculating $\frac{1}{2}|\det(X)|$. This process can be used to find the area of any polygon.

Activity Area of a Quadrilateral

a. Find the area of the quadrilateral with vertices (1, 1), (2, 6), (8, 5), and (7, 2).

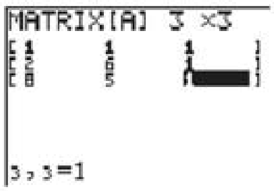
Step 1 Sketch the quadrilateral, and divide it into two triangles.

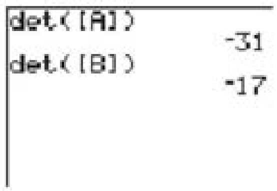


Step 2 Create a matrix for each triangle.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 6 & 1 \\ 8 & 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 5 & 1 \\ 7 & 2 & 1 \end{bmatrix}$$

Step 3 Enter each matrix into your graphing calculator, and find det(A) and det(B).

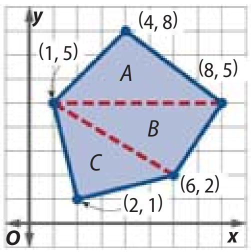




Step 4 Multiply the absolute value of each determinant by $\frac{1}{2}$, and find the sum. The area is $\frac{1}{2}|-31| + \frac{1}{2}|-17|$ or 24 square units.

b. Find the area of the polygon with vertices (1, 5), (4, 8), (8, 5), (6, 2), and (2, 1).

Step 1 Sketch the pentagon, and divide it into three triangles.



Step 2 Create a matrix for each triangle.

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 4 & 8 & 1 \\ 8 & 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 1 \\ 8 & 5 & 1 \\ 6 & 2 & 1 \end{bmatrix}$$

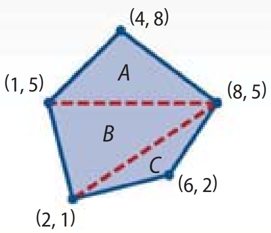
$$C = \begin{bmatrix} 1 & 5 & 1 \\ 6 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Step 3 Enter each matrix into your graphing calculator, and find the determinants. The determinants are −21, −21, and −17.

Step 4 Multiply the absolute value of each determinant by $\frac{1}{2}$ and find the sum. The area is $\frac{1}{2}|-21| + \frac{1}{2}|-21| + \frac{1}{2}|-17|$ or 29.5 square units.

StudyTip

Dividing Polygons There may be various ways to divide a given polygon into triangles. For instance, the quadrilateral in Example 2 could have also been divided as shown below.



Exercises

Find the area of the polygon with the given vertices.

1. (3, 2), (1, 9), (10, 12), (8, 3)
2. (−2, −4), (−11, −1), (−9, −8), (−1, −12)
3. (1, 3), (2, 9), (10, 11), (13, 7), (6, 2)
4. (−7, −6), (−10, 2), (−9, 8), (−5, 10), (8, 6), (13, 2)

LESSON 6-3 Solving Linear Systems using Inverses and Cramer's Rule

Then

- You found determinants and inverses of 2 × 2 and 3 × 3 matrices. (Lesson 6-2)

Now

1. Solve systems of linear equations using inverse matrices.
2. Solve systems of linear equations using Cramer's Rule.

Why?

- Marcella downloads her favorite shows to her portable media player. A nature show requires twice as much memory as a sitcom, and a movie requires twice as much memory as a nature show. When given the amount of memory that has been used, you can use an inverse matrix to solve a system of equations to find the number of each type of show that Marcella downloaded.

NewVocabulary
square system
Cramer's Rule

1 Use Inverse Matrices

If a system of linear equations has the same number of equations as variables, then its coefficient matrix is square and the system is said to be a **square system**. If this square coefficient matrix is invertible, then the system has a unique solution.

KeyConcept Invertible Square Linear Systems

Let A be the coefficient matrix of a system of n linear equations in n variables given by $AX = B$, where X is the matrix of variables and B is the matrix of constants. If A is invertible, then the system of equations has a unique solution given by $X = A^{-1}B$.

Example 1 Solve a 2 × 2 System Using an Inverse Matrix

Use an inverse matrix to solve the system of equations, if possible.

$$2x - 3y = -1$$
$$-3x + 5y = 3$$

Write the system in matrix form $AX = B$.

$$\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad AX = B$$

Use the formula for the inverse of a 2 × 2 matrix to find the inverse A^{-1}.

$$A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{Formula for the inverse of a } 2 \times 2 \text{ matrix } \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{2(5) - (-3)(-3)}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad a = 2, b = -3, c = -3, \text{ and } d = 5$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad \text{Simplify.}$$

Multiply A^{-1} by B to solve the system.

$$X = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad X = A^{-1}B$$

Therefore, the solution of the system is (4, 3).

GuidedPractice

Use an inverse matrix to solve the system of equations, if possible.

1A. $6x + y = -8$
$-4x - 5y = -12$

1B. $-3x + 9y = 36$
$7x - 8y = -19$

Jeff Chiu/AP Images

To solve a 3 × 3 system of equations using an inverse matrix, use a calculator.

Real-WorldLink

A bond is essentially an IOU issued by a company or government to fund its day-to-day operations or a specific project. If you invest in bonds, you are loaning your money for a certain period of time to the issuer. In return, you will receive your money back plus interest.

Source: CNN

Real-World Example 2 Solve a 3 × 3 System Using an Inverse Matrix

FINANCIAL LITERACY Belinda is investing $20,000 by purchasing three bonds with expected annual returns of 10%, 8%, and 6%. Investments with a higher expected return are often riskier than other investments. She wants an average annual return of $1340. If she wants to invest three times as much money in the bond with a 6% return than the other two combined, how much money should she invest in each bond?

Her investment can be represented by

$x + y + z = 20{,}000$

$3x + 3y - z = 0$

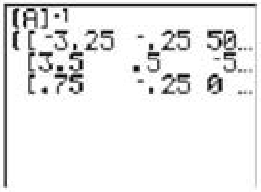
$0.10x + 0.08y + 0.06z = 1340,$

where x, y, and z represent the amounts invested in the bonds with 10%, 8%, and 6% annual returns, respectively.

Write the system in matrix form $AX = B$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 0.10 & 0.08 & 0.06 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 20{,}000 \\ 0 \\ 1340 \end{bmatrix}}$$

Use a graphing calculator to find A^{-1}.



$$A^{-1} = \begin{bmatrix} -3.25 & -0.25 & 50 \\ 3.5 & 0.5 & -50 \\ 0.75 & -0.25 & 0 \end{bmatrix}$$

Multiply A^{-1} by B to solve the system.

$$X = A^{-1}B$$

$$= \begin{bmatrix} -3.25 & -0.25 & 50 \\ 3.5 & 0.5 & -50 \\ 0.75 & -0.25 & 0 \end{bmatrix} \cdot \begin{bmatrix} 20{,}000 \\ 0 \\ 1340 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 3000 \\ 15{,}000 \end{bmatrix}$$

The solution of the system is (2000, 3000, 15,000). Therefore, Belinda invested $2000 in the bond with a 10% annual return, $3000 in the bond with an 8% annual return, and $15,000 in the bond with a 6% annual return.

CHECK You can check the solution by substituting back into the original system.

$$2000 + 3000 + 15{,}000 = 20{,}000$$
$$20{,}000 = 20{,}000 \checkmark$$

$$3(2000) + 3(3000) - 15{,}000 = 0$$
$$0 = 0 \checkmark$$

$$0.10(2000) + 0.08(3000) + 0.06(15{,}000) = 1340$$
$$1340 = 1340 \checkmark$$

GuidedPractice

2. INDUSTRY During three consecutive years, an auto assembly plant produced a total of 720,000 cars. If 50,000 more cars were made in the second year than the first year, and 80,000 more cars were made in the third year than the second year, how many cars were made in each year?

2 Use Cramer's Rule

Another method for solving square systems, known as **Cramer's Rule**, uses determinants instead of row reduction or inverse matrices.

Consider the following 2×2 system.

$ax + by = e$

$cx + dy = f$

Use the elimination method to solve for x.

Multiply by d. →	$adx + bdy = ed$
Multiply by $-b$. →	$(+) \quad -bcx - bdy = -fb$
	$(ad - bc)x = ed - fb$

So, $x = \dfrac{ed - fb}{ad - bc}$.

Similarly, it can be shown that $y = \dfrac{af - ce}{ad - bc}$. You should recognize the denominator of each fraction as the determinant of the system's coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Both the numerator and denominator of each solution can be expressed using determinants.

$$x = \frac{ed - fb}{ad - bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|A_x|}{|A|} \qquad y = \frac{af - ce}{ad - bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|A_y|}{|A|}$$

Notice that numerators $|A_x|$ and $|A_y|$ are the determinants of the matrices formed by replacing the coefficients of x or y, respectively, in the coefficient matrix with the column of constant terms $\begin{matrix} e \\ f \end{matrix}$ from the original system $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$.

Cramer's Rule can be generalized to systems of n equations in n variables.

KeyConcept Cramer's Rule

Let A be the coefficient matrix of a system of n linear equations in n variables given by $AX = B$. If $\det(A) \neq 0$, then the unique solution of the system is given by

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ x_3 = \frac{|A_3|}{|A|},\ \ldots,\ x_n = \frac{|A_n|}{|A|},$$

where A_i is obtained by replacing the ith column of A with the column of constant terms B. If $\det(A) = 0$, then $AX = B$ has either no solution or infinitely many solutions.

Example 3 Use Cramer's Rule to Solve a 2 × 2 System

Use Cramer's Rule to find the solution of the system of linear equations, if a unique solution exists.

$$3x_1 + 2x_2 = 6$$
$$-4x_1 - x_2 = -13$$

The coefficient matrix is $A = \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix}$. Calculate the determinant of A.

$$A = \begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix} = 3(-1) - (-4)(2) \text{ or } 5$$

Because the determinant of A does not equal zero, you can apply Cramer's Rule.

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 6 & 2 \\ -13 & -1 \end{vmatrix}}{5} = \frac{6(-1) - (-13)(2)}{5} = \frac{20}{5} \text{ or } 4$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & 6 \\ -4 & -13 \end{vmatrix}}{5} = \frac{3(-13) - (-4)(6)}{5} = \frac{-15}{5} \text{ or } -3$$

So, the solution is $x_1 = 4$ and $x_2 = -3$ or $(4, -3)$. Check your answer in the original system.

WatchOut!

Division by Zero Remember that Cramer's Rule does not apply when the determinant of the coefficient matrix is 0, because this would introduce division by zero, which is undefined.

Guided Practice

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.

3A. $2x - y = 4$
$5x - 3y = -6$

3B. $-9x + 3y = 8$
$2x - y = -3$

3C. $12x - 9y = -5$
$4x - 3y = 11$

Example 4 Use Cramer's Rule to Solve a 3 × 3 System

Use Cramer's Rule to find the solution of the system of linear equations, if a unique solution exists.

$-x - 2y = -4z + 12$
$3x - 6y + z = 15$
$2x + 5y + 1 = 0$

The coefficient matrix is $A = \begin{bmatrix} -1 & -2 & 4 \\ 3 & -6 & 1 \\ 2 & 5 & 0 \end{bmatrix}$. Calculate the determinant of A.

$$|A| = \begin{vmatrix} -1 & -2 & 4 \\ 3 & -6 & 1 \\ 2 & 5 & 0 \end{vmatrix} = -1\begin{vmatrix} -6 & 1 \\ 5 & 0 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} + 4\begin{vmatrix} 3 & -6 \\ 2 & 5 \end{vmatrix}$$ Formula for the determinant of a 3 × 3 matrix

$$= -1[-6(0) - 5(1)] - (-2)[3(0) - 1(2)] + 4[3(5) - 2(-6)]$$ Simplify.

$$= -1(-5) + 2(-2) + 4(27) \text{ or } 109$$ Simplify.

Because the determinant of A does not equal zero, you can apply Cramer's Rule.

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 12 & -2 & 4 \\ 15 & -6 & 1 \\ -1 & 5 & 0 \end{vmatrix}}{109} = \frac{12[(-6)(0) - 5(1)] - (-2)[15(0) - (-1)(1)] + 4[15(5) - (-1)(-6)]}{109} = \frac{218}{109} \text{ or } 2$$

$$y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} -1 & 12 & 4 \\ 3 & 15 & 1 \\ 2 & -1 & 0 \end{vmatrix}}{109} = \frac{(-1)[15(0) - 1(-1)] - 12[3(0) - 2(1)] + 4[3(-1) - 2(15)]}{109} = \frac{-109}{109} \text{ or } -1$$

$$z = \frac{|A_z|}{|A|} = \frac{\begin{vmatrix} -1 & -2 & 12 \\ 3 & -6 & 15 \\ 2 & 5 & -1 \end{vmatrix}}{109} = \frac{(-1)[(-6)(-1) - 5(15)] - (-2)[3(-1) - 2(15)] + 12[3(5) - 2(-6)]}{109} = \frac{327}{109} \text{ or } 3$$

Therefore, the solution is $x = 2$, $y = -1$, and $z = 3$ or $(2, -1, 3)$.

CHECK Check the solution by substituting back into the original system.

$-(2) - 2(-1) = -4(3) + 12$
$0 = 0$ ✔

$3(2) - 6(-1) + 3 = 15$
$15 = 15$ ✔

$2(2) + 5(-1) + 1 = 0$
$0 = 0$ ✔

Reading Math

Replacing Columns The notation $|A_x|$ is read as *the determinant of the coefficient matrix A with the column of x-coefficients replaced with the column of constants.*

Guided Practice

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.

4A. $8x + 12y - 24z = -40$
$3x - 8y + 12z = 23$
$2x + 3y - 6z = -10$

4B. $-2x + 4y - z = -3$
$3x + y + 2z = 6$
$x - 3y = 1$

Exercises

= Step-by-Step Solutions begin on page R29.

Use an inverse matrix to solve each system of equations, if possible. (Examples 1 and 2)

1. $5x - 2y = 11$
$-4x + 7y = 2$

2. $2x + 3y = 2$
$x - 4y = -21$

3. $-3x + 5y = 33$
$2x - 4y = -26$

4. $-4x + y = 19$
$3x - 2y = -18$

5. $2x + y - z = -13$
$3x + 2y - 4z = -36$
$x + 6y - 3z = 12$

6. $3x - 2y + 8z = 38$
$6x + 3y - 9z = -12$
$4x + 4y + 20z = 0$

7. $x + 2y - z = 2$
$2x - y + 3z = 4$
$3x + y + 2z = 6$

8. $4x + 6y + z = -1$
$-x - y + 8z = 8$
$6x - 4y + 11z = 21$

9. DOWNLOADING Marcela downloaded some programs on her portable media player. In general, a 30-minute sitcom uses 0.3 gigabyte of memory, a 1-hour talk show uses 0.6 gigabyte, and a 2-hour movie uses 1.2 gigabytes. She downloaded 9 programs totaling 5.4 gigabytes. If she downloaded two more sitcoms than movies, what number of each type of show did Marcela download? (Example 2)

10. BASKETBALL Trevor knows that he has scored 37 times for a total of 70 points thus far this basketball season. He wants to know how many free throws, 2-point and 3-point field goals he has made. The sum of his 2- and 3-point field goals equals twice the number of free throws minus two. How many free throws, 2-point field goals, and 3-point field goals has Trevor made? (Example 2)

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists. (Examples 3 and 4)

11. $-3x + y = 4$
$2x + y = -6$

12. $2x + 3y = 4$
$5x + 6y = 5$

13. $5x + 4y = 7$
$-x - 4y = -3$

14. $4x + \frac{1}{3}y = 8$
$3x + y = 6$

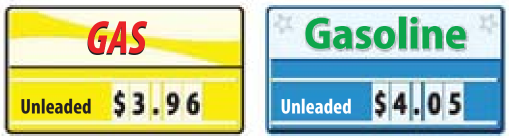
15. $2x - y + z = 1$
$x + 2y - 4z = 3$
$4x + 3y - 7z = -8$

16. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

17. $x + 2y = 12$
$3y - 4z = 25$
$x + 6y + z = 20$

18. $9x + 7y = -30$
$8y + 5z = 11$
$-3x + 10z = 73$

19. ROAD TRIP Dena stopped for gasoline twice during a road trip. The price of gasoline at each station is shown below. She bought a total of 33.5 gallons and spent \$134.28. Use Cramer's Rule to determine the number of gallons of gasoline Dena bought for \$3.96 a gallon. (Example 3)



20. GROUP PLANNING A class reunion committee is planning for 400 guests for its 10-year reunion. The guests can choose one of the three options for dessert that are shown below. The chef preparing the desserts must spend 5 minutes on each pie, 8 minutes on each trifle, and 12 minutes on each cheesecake. The total cost of the desserts was \$1170, and the chef spends exactly 45 hours preparing them. Use Cramer's Rule to determine how many servings of each dessert were prepared. (Example 4)

21. PHONES Megan, Emma, and Mora all went over their allotted phone plans. For an extra 30 minutes of gaming, 12 minutes of calls, and 40 text messages, Megan paid \$52.90. Emma paid \$48.07 for 18 minutes of gaming, 15 minutes of calls, and 55 text messages. Mora only paid \$13.64 for 6 minutes of gaming and 7 minutes of calls. If they all have the same plan, find the cost of each service. (Example 4)

Find the solution to each matrix equation.

22. $\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ 0 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 6 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$

25. $\begin{bmatrix} -2 & 1 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 17 & -9 \end{bmatrix}$

26. FITNESS Eva is training for a half-marathon and consumes energy gels, bars, and drinks every week. This week, she consumed 12 energy items for a total of 1450 Calories and 310 grams of carbohydrates. The nutritional content of each item is shown.

Energy Item	gel	bar	drink
Calories	100	250	50
Carbohydrates (g)	25	43	14

How many energy gels, bars, and drinks did Eva consume this week?

GRAPHING CALCULATOR **Solve each system of equations using inverse matrices.**

27. $2a - b + 4c = 6$
$a + 5b - 2c = -6$
$3a - 2b + 6c = 8$

28. $3x - 5y + 2z = 22$
$2x + 3y - z = -9$
$4x + 3y + 3z = 1$

29. $r + 5s - 2t = 16$
$-2r - s + 3t = 3$
$3r + 2s - 4t = -2$

30. $-4m + n + 6p = 17$
$3m - n - p = 5$
$-5m - 2n + 3p = 2$

Find the values of *n* such that the system represented by the given augmented matrix cannot be solved using an inverse matrix.

31. $\left[\begin{array}{cc|c} n & -8 & 6 \\ 1 & 2 & 3 \end{array}\right]$

32. $\left[\begin{array}{cc|c} 3 & n & 4 \\ n & 2 & -5 \end{array}\right]$

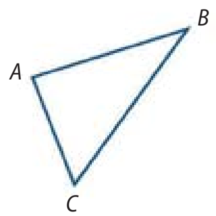
33. $\left[\begin{array}{cc|c} -5 & -9 & 3 \\ n & n & 11 \end{array}\right]$

34. $\left[\begin{array}{cc|c} n & -n & 0 \\ 7 & n & -8 \end{array}\right]$

35. CHEMICALS Three alloys of copper and silver contain 35% pure silver, 55% pure silver, and 60% pure silver, respectively. How much of each type should be mixed to produce 2.5 kilograms of an alloy containing 54.4% silver if there is to be 0.5 kilogram more of the 60% alloy than the 55% alloy?

36. DELI A Greek deli sells the gyros shown below. During one lunch, the deli sold a total of 74 gyros and earned \$320.50. The total amount of meat used for the small, large, and jumbo gyros was 274 ounces. The number of large gyros sold was one more than twice the number of jumbo gyros sold. How many of each type of gyro did the deli sell during lunch?

37. GEOMETRY The perimeter of $\triangle ABC$ is 89 millimeters. The length of $\overline{AC}$ is 47 millimeters less than the sum of the lengths of the other two sides. The length of $\overline{BC}$ is 20 millimeters more than half the length of $\overline{AB}$. Use a system of equations to find the length of each side.



Find the inverse of each matrix, if possible.

38. $\begin{bmatrix} e^{2x} & e^{-x} \\ e^{x} & e^{-3x} \end{bmatrix}$

39. $\begin{bmatrix} \frac{1}{x} & \frac{3}{x} \\ x & 2 \end{bmatrix}$

40. $\begin{bmatrix} \pi^{x} & 1 \\ 0 & \pi^{-2x} \end{bmatrix}$

41. $\begin{bmatrix} i & -3 \\ i^2 & 2i \end{bmatrix}$

Let *A* and *B* be $n \times n$ matrices and let *C*, *D*, and *X* be $n \times 1$ matrices. Solve each equation for *X*. Assume that all inverses exist.

42. $AX = BX - C$

43. $D = AX + BX$

44. $AX + BX = 2C - X$

45. $X + C = AX - D$

46. $3X - D = C - BX$

47. $BX = AD + AX$

48. CALCULUS In calculus, systems of equations can be obtained using *partial derivatives*. These equations contain λ, which is called a *Lagrange multiplier*. Find values of x and y that satisfy $x + \lambda + 1 = 0$; $2y + \lambda = 0$; $x + y + 7 = 0$.

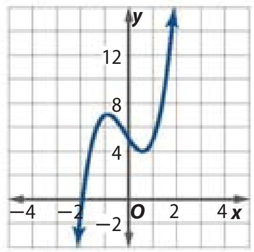
H.O.T. Problems Use Higher-Order Thinking Skills

49. ERROR ANALYSIS Trent and Kate are trying to solve the system below using Cramer's Rule. Is either of them correct? Explain your reasoning.

$2x + 7y = 10$
$6x + 21y = 30$

Trent	Kate
The system has no solution because the determinant of the coefficient matrix is 0.	The system has one solution but cannot be found by using Cramer's Rule.

50. CHALLENGE The graph shown below goes through points at $(-2, -1)$, $(-1, 7)$, $(1, 5)$, and $(2, 19)$. The equation of the graph is of the form $f(x) = ax^3 + bx^2 + cx + d$.



Find the equation of the graph by solving a system of equations using an inverse matrix.

51. REASONING If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and A is nonsingular, does $(A^2)^{-1} = (A^{-1})^2$? Explain your reasoning.

52. OPEN ENDED Give an example of a system of equations in two variables that does not have a unique solution, and demonstrate how the system expressed as a matrix equation would have no solution.

53. WRITING IN MATH Describe what types of systems can be solved using each method. Explain your reasoning.

a. Gauss-Jordan elimination

b. inverse matrices

c. Cramer's Rule

Spiral Review

Find *AB* and *BA*, if possible. (Lesson 6-2)

54. $A = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

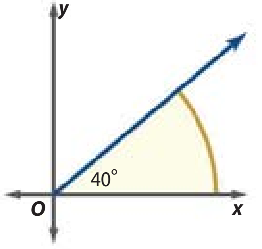
55. $A = \begin{bmatrix} 4 & -2 & 6 \\ 7 & 8 & 3 \\ 11 & -5 & -1 \end{bmatrix}, B = \begin{bmatrix} 17 & 2 & -4 \\ 10 & -9 & 6 \\ 1 & 0 & -8 \end{bmatrix}$

Determine whether each matrix is in row-echelon form. (Lesson 6-1)

56. $\left[\begin{array}{ccc|c} 0 & -3 & -6 & 4 \\ 9 & -1 & -2 & -1 \\ 3 & 1 & -2 & -3 \\ 0 & 3 & -1 & 1 \end{array}\right]$

57. $\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

58. **TRACK AND FIELD** A shot put must land in a 40° sector. The vertex of the sector is at the origin, and one side lies along the *x*-axis. If an athlete puts the shot at a point with coordinates (18, 17), will the shot land in the required region? Explain your reasoning. (Lesson 4-6)



59. **STARS** Some stars appear bright only because they are very close to us. Absolute magnitude M is a measure of how bright a star would appear if it were 10 parsecs, or about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where d is the star's distance from Earth measured in parsecs and m is its apparent magnitude. (Lesson 3-3)

Star	Apparent Magnitude	Distance (parsecs)
Sirius	−1.44	2.64
Vega	0.03	7.76

a. Sirius and Vega are two of the brightest stars. Which star appears brighter?

b. Find the absolute magnitudes of Sirius and Vega.

c. Which star is actually brighter? That is, which has a lower absolute magnitude?

Skills Review for Standardized Tests

60. **SAT/ACT** Point *C* is the center of the circle in the figure below. The shaded region has an area of 3π square centimeters. What is the perimeter of the shaded region in centimeters?

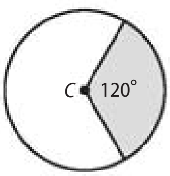
A $2\pi + 6$

B $2\pi + 9$

C $2\pi + 12$

D $3\pi + 6$

E $3\pi + 12$



61. In March, Claudia bought 2 standard and 2 premium ring tones from her cell phone provider for $8.96. In May, she paid $9.46 for 1 standard and 3 premium ring tones. What are the prices for standard and premium ring tones?

F $1.99, $2.49

G $2.29, $2.79

H $1.99, $2.79

J $2.49, $2.99

62. **REVIEW** Each year, the students at Capital High School vote for a homecoming dance theme. The theme "A Night Under the Stars" received 225 votes. "The Time of My Life" received 480 votes. If 40% of girls voted for the star theme, 75% of boys voted for the life theme and all of the students voted, how many girls and boys are there at Capital High School?

A 854 boys and 176 girls

B 705 boys and 325 girls

C 395 boys and 310 girls

D 380 boys and 325 girls

63. **REVIEW** What is the solution of $\frac{1}{8}x - \frac{2}{3}y + \frac{5}{6}z = -8$, $\frac{3}{4}x + \frac{1}{6}y - \frac{1}{3}z = -12$, and $\frac{3}{16}x - \frac{5}{8}y - \frac{7}{12}z = -25$?

F (−4, 6, 3)

G (−8, 12, 6)

H (−16, 24, 12)

J no solution

EXTEND 6-3

Graphing Technology Lab
Matrices and Cryptography

Objective

- Use a graphing calculator and matrices to encode and decode messages.

Cryptography is the study of coded messages. Matrices can be used to code messages so that they can only be read after being deciphered by a *key.*

The first step is to decide on a key that can be used to encode the matrix. The key must be an $n \times n$ invertible matrix. The next step is to convert the message to numbers and write it as a matrix. Each letter of the alphabet is represented by a number. The number 0 is used to represent a blank space.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Finally, the message is encoded by multiplying it by the key.

Activity 1 Encode a Message

Use $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ to encode the message SATURDAY AT NOON.

Step 1 Convert the message to numbers and write it as a matrix.

S	A	T	U	R	D	A	Y	_	A	T	_	N	O	O	N
19	1	20	21	18	4	1	25	0	1	20	0	14	15	15	14

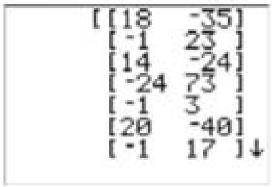
The key is a 2 × 2 matrix. To make the matrix multiplication possible, write the message as an 8 × 2 matrix.

$$\begin{bmatrix} 19 & 1 \\ 20 & 21 \\ 18 & 4 \\ 1 & 25 \\ 0 & 1 \\ 20 & 0 \\ 14 & 15 \\ 15 & 14 \end{bmatrix}$$

StudyTip

Conversion Add zeros to the end of a message if additional entries are needed to fill a matrix.

Step 2 Multiply the converted message by the key using a graphing calculator.



Step 3 Remove the matrix notation to reveal the encoded message.

18 −35 −1 23 14 −24 −24 73 −1 3 20 −40 −1 17 1 12

Exercises

Use $\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$ to encode each message.

1. CALL ME

2. SEE YOU LATER

3. ORDER PIZZA

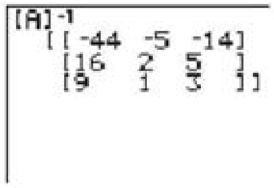
4. **CHALLENGE** Use $\begin{bmatrix} 2 & -4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & 4 \end{bmatrix}$ to encode the message MEET ME AT FIVE.

To decode a message, the inverse of the key must be found. The coded message is then written in matrix form to make the multiplication possible. For instance, if the key is an $n \times n$ matrix, the message is written as a $k \times n$ matrix, where k is the number of rows necessary to include each number in the matrix. If there are not enough characters to fill a row, insert "0"s as spaces. Finally, the coded matrix is multiplied by the inverse of the key.

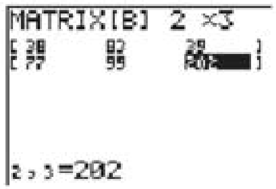
Activity 2 Decode a Message

Use the inverse of $\begin{bmatrix} -1 & -1 & -3 \\ 3 & 6 & 4 \\ 2 & 1 & 8 \end{bmatrix}$ to decode the message 38 83 39 77 99 202.

Step 1 Use a graphing calculator to find the inverse of the key.

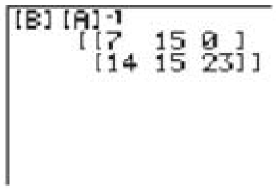


Step 2 Write the coded message as a matrix. The coded matrix will have 3 columns because the key is a 3 × 3 matrix. It is a 2 × 3 matrix because there are enough numbers to fill two rows. Enter it into your graphing calculator.



Step 3 Use a graphing calculator to multiply the coded matrix by the inverse of the key.

Step 4 Remove the matrix notation and convert the numbers to letters.



7	15	0	14	15	23
G	O	_	N	O	W

Exercises

Use the inverse of $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$ to decode each message.

5. 128 −73 232 −135 300 −175 99 −56 83 −48 180 −104 300 −175

6. −27 17 38 −21 84 −49 21 −11 131 −76 201 −116 161 −93

7. 151 −88 150 −86 93 −54 −35 22 −5 3 191 −111 −30 18 182 −105

8. 102 −58 45 −26 −48 29 −69 42 39 −21 228 −133 141 −81 −19 12 228 −133

9. **CHALLENGE** Use the inverse of $\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 8 & -4 & -6 \\ 7 & 6 & -5 & 3 \\ 1 & 7 & 9 & 2 \end{bmatrix}$ to decode

126 265 −49 −34 198 347 193 96 174 239 49 72 177 286 −61 −27 48 200 70 −76 122 162 −21 35 81 190 −37 −63 130 331 214 17 67 267 94 −25 93 161 120 25.

Mid-Chapter Quiz

Lessons 6-1 through 6-3

Write each system of equations in triangular form using Gaussian elimination. Then solve the system. (Lesson 6-1)

1. $2x - y = 13$
$2x + y = 23$

2. $x + y + z = 6$
$2x - y - z = -3$
$3x - 5y + 7z = 14$

Solve each system of equations. (Lesson 6-1)

3. $3x + 3y = -8$
$6x - 5y = 28$

4. $-x + 8y - 2z = -37$
$2x + 5y - 11z = -7$
$4x - 7y + 6z = 4$

5. $-2x + 2y + z = 5$
$3x - 2y + 2z = 7$
$5x - y + 4z = 8$

6. $x - 5y + 8z = 7$
$-8x + 3y + 12z = -9$
$5x - 4y - 3z = 9$

7. **PET CARE** Amelia purchased 25 total pounds of dog food, bird seed, and cat food for $100. She purchased 10 pounds more dog food than bird seed. The cost per pound for each type of food is shown. (Lesson 6-1)

$4.00/lb $7.00/lb $3.00/lb

a. Write a set of linear equations for this situation.

b. Determine the number of pounds of each type of food Amelia purchased.

8. **MULTIPLE CHOICE** Which matrix is nonsingular? (Lesson 6-2)

A $\begin{bmatrix} 2 & 3 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 6 & -5 & 0 \\ 4 & 4 & 3 & 4 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 0 & 0 & 5 \\ 2 & 0 & 0 & 5 \\ 5 & 1 & -7 & 8 \end{bmatrix}$

D $\begin{bmatrix} 5 & 3 & 1 & 0 \\ 10 & 6 & 2 & 0 \\ 4 & 3 & -1 & 5 \\ 7 & 7 & 3 & 9 \end{bmatrix}$

Find *AB* and *BA*, if possible. (Lesson 6-2)

9. $A = \begin{bmatrix} 1 & -3 & 4 \\ -2 & 5 & 1 \\ 0 & -4 & -6 \end{bmatrix}$
$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$
$B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 0 \\ 5 & -8 & 2 \end{bmatrix}$

Find the determinant of each matrix. Then find the inverse of the matrix, if it exists. (Lesson 6-2)

11. $\begin{bmatrix} 3 & 8 \\ -1 & -2 \end{bmatrix}$

12. $\begin{bmatrix} -9 & -5 \\ -7 & -4 \end{bmatrix}$

13. $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

14. $\begin{bmatrix} 5 & -10 \\ 4 & -6 \end{bmatrix}$

15. **NURSING** Troy is an Emergency Room nurse. He earns $24 per hour during regular shifts and $30 per hour when working overtime. The table shows the hours Troy worked during the past three weeks. (Lesson 6-2)

Week	Regular Hours	Overtime Hours
1	35	7
2	38	0
3	40	9

a. Use matrices to determine how much Troy earned during each week.

b. During week 4, Troy worked four times more regular hours than overtime hours. Determine the number of hours he worked if he earned $1008.

Use an inverse matrix to solve each system of equations, if possible. (Lesson 6-3)

16. $2x - y = 6$
$3x + 2y = 37$

17. $2x + y + z = 19$
$3x - 2y + 3z = 2$
$4x - 6y + 5z = -26$

18. **MULTIPLE CHOICE** Which of the augmented matrices represents the solutions of the system of equations? (Lesson 6-3)

$$x + y = 13$$
$$2x - 3y = -9$$

F $\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$

H $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 7 \end{array}\right]$

G $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -3 \end{array}\right]$

J $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 5 \end{array}\right]$

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists. (Lesson 6-3)

19. $2x - y = 6$
$4x - 2y = 12$

20. $3x - y - z = 13$
$3x - 2y + 3z = 16$
$-x + 4y - 8z = -9$

LESSON 6-4 Partial Fractions

Then

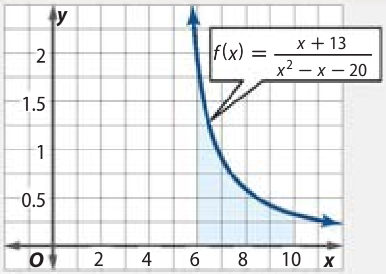
- You graphed rational functions. (Lesson 2-4)

Now

1. Write partial fraction decompositions of rational expressions with linear factors in the denominator.
2. Write partial fraction decompositions of rational expressions with prime quadratic factors.

Why?

- In calculus, you will learn to find the area under the graph of a function over a specified interval. To find the area under the curve of a rational function such as $f(x) = \frac{x+13}{x^2 - x - 20}$, you will first need to *decompose* the rational expression or rewrite it as the sum of two simpler expressions.



NewVocabulary
partial fraction
partial fraction decomposition

1 Linear Factors

In Lesson 2-3, you learned that many polynomial functions with real coefficients can be expressed as the *product* of linear and quadratic factors. Similarly, many rational functions can be expressed as the *sum* of two or more simpler rational functions with numerators that are real constants and with denominators that are a power of a linear factor or an irreducible quadratic factor. For example, the rational function $f(x)$ below can be written as the sum of two fractions with denominators that are linear factors of the original denominator.

$$f(x) = \frac{x+13}{x^2 - x - 20} = \frac{2}{x-5} + \frac{-1}{x+4}$$

Each fraction in the sum is a **partial fraction**. The sum of these partial fractions makes up the **partial fraction decomposition** of the original rational function.

Example 1 Denominator with Nonrepeated Linear Factors

Find the partial fraction decomposition of $\frac{x+13}{x^2 - x - 20}$.

Rewrite the expression as partial fractions with constant numerators, A and B, and denominators that are the linear factors of the original denominator.

$$\frac{x+13}{x^2 - x - 20} = \frac{A}{x-5} + \frac{B}{x+4}$$ Form of partial fraction decomposition

$$x + 13 = A(x+4) + B(x-5)$$ Multiply each side by the LCD, $x^2 - x - 20$.

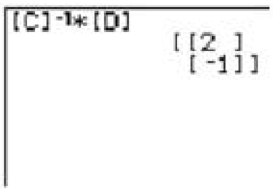
$$x + 13 = Ax + 4A + Bx - 5B$$ Distributive Property

$$1x + 13 = (A + B)x + (4A - 5B)$$ Group like terms.

Equate the coefficients on the left and right side of the equation to obtain a system of two equations. To solve the system, you can write it in matrix form $CX = D$ and solve for X.

$$\begin{aligned} A + B &= 1 \\ 4A - 5B &= 13 \end{aligned} \quad \rightarrow \quad \overset{C}{\begin{bmatrix} 1 & 1 \\ 4 & -5 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} A \\ B \end{bmatrix}} = \overset{D}{\begin{bmatrix} 1 \\ 13 \end{bmatrix}}$$

You can use a graphing calculator to find $X = C^{-1}D$. So, $A = 2$ and $B = -1$. Use substitution to find the partial fraction decomposition.



$$\frac{x+13}{x^2 - x - 20} = \frac{A}{x-5} + \frac{B}{x+4}$$ Form of partial fraction decomposition

$$\frac{x+13}{x^2 - x - 20} = \frac{2}{x-5} + \frac{-1}{x+4}$$ $A = 2$ and $B = -1$

GuidedPractice

Find the partial fraction decomposition of each rational expression.

1A. $\frac{2x+5}{x^2 - x - 2}$

1B. $\frac{x+11}{2x^2 - 5x - 3}$

If a rational expression $\frac{f(x)}{d(x)}$ is improper, with the degree of $f(x)$ greater than or equal to the degree of $d(x)$, you must first use the division algorithm $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ to rewrite the expression as the sum of a polynomial and a proper rational expression. Then decompose the remaining rational expression.

Example 2 Improper Rational Expression

Find the partial fraction decomposition of $\frac{2x^2 + 5x - 4}{x^2 - x}$.

Because the degree of the numerator is greater than or equal to the degree of the denominator, the rational expression is improper. To rewrite the expression, divide the numerator by the denominator using polynomial division.

$$\begin{array}{r} 2 \\ x^2 - x \overline{)2x^2 + 5x - 4} \\ (-)\ \underline{2x^2 - 2x} \\ 7x - 4 \end{array}$$

Multiply the divisor by 2 because $\frac{2x^2}{x^2} = 2$.

Subtract and bring down next term.

So, the original expression is equal to $2 + \frac{7x - 4}{x^2 - x}$.

Because the remaining rational expression is now proper, you can factor its denominator as $x(x - 1)$ and rewrite the expression using partial fractions.

$\frac{7x - 4}{x^2 - x} = \frac{A}{x} + \frac{B}{x - 1}$ — Form of decomposition

$7x - 4 = A(x - 1) + B(x)$ — Multiply by the LCD, $x^2 - x$.

$7x - 4 = Ax - A + Bx$ — Distributive Property

$7x - 4 = (A + B)x - A$ — Group like terms.

Write and solve the system of equations obtained by equating coefficients.

$$\begin{aligned} A + B &= 7 \\ -A &= -4 \end{aligned} \quad \rightarrow \quad \begin{aligned} A &= 4 \\ B &= 3 \end{aligned}$$

Therefore, $\frac{2x^2 + 5x - 4}{x^2 - x} = 2 + \frac{7x - 4}{x^2 - x}$ or $2 + \frac{4}{x} + \frac{3}{x - 1}$.

CHECK You can check your answer by simplifying the expression on the right side of the equation.

$\frac{2x^2 + 5x - 4}{x^2 - x} = 2 + \frac{4}{x} + \frac{3}{x - 1}$ — Partial fraction decomposition

$= \frac{2x(x - 1)}{x(x - 1)} + \frac{4(x - 1)}{x(x - 1)} + \frac{3x}{x(x - 1)}$ — Rewrite using LCD, $x(x - 1)$.

$= \frac{2(x^2 - x) + 4(x - 1) + 3x}{x(x - 1)}$ — Add.

$= \frac{2x^2 - 2x + 4x - 4 + 3x}{x^2 - x}$ — Multiply.

$= \frac{2x^2 + 5x - 4}{x^2 - x}$ ✓ — Simplify.

StudyTip

Alternate Method By design, the equation $7x - 4 = A(x - 1) + B(x)$ obtained after clearing the fractions in Example 2 is true for all x. Therefore, you can substitute any *convenient values* of x to find the values of A and B. Convenient values are those that are zeros of the original denominator.
If $x = 0$, $A = 4$. If $x = 1$, $B = 3$.

GuidedPractice

Find the partial fraction decomposition of each rational expression.

2A. $\frac{3x^2 + 12x + 4}{x^2 + 2x}$

2B. $\frac{x^4 - 3x^3 + x^2 - 9x + 4}{x^2 - 4x}$

If the denominator of a rational expression has a linear factor that is repeated n times, the partial fraction decomposition must include a partial fraction with its own constant numerator for each power 1 to n of the linear factor. For example, to find the partial fraction decomposition of $\frac{5x-1}{x^3(x-1)^2}$, you would write

$$\frac{5x-1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}.$$

Example 3 Denominator with Repeated Linear Factors

Find the partial fraction decomposition of $\frac{-x^2-3x-8}{x^3+4x^2+4x}$.

This rational expression is proper, so begin by factoring the denominator as $x(x^2+4x+4)$ or $x(x+2)^2$. Because the factor $(x+2)$ has multiplicity 2, include partial fractions with denominators of x, $(x+2)$, and $(x+2)^2$.

$$\frac{-x^2-3x-8}{x^3+4x^2+4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$ Form of partial fraction decomposition

$$-x^2-3x-8 = A(x+2)^2 + Bx(x+2) + Cx$$ Multiply each side by the LCD, $x(x+2)^2$.

$$-x^2-3x-8 = Ax^2+4Ax+4A+Bx^2+2Bx+Cx$$ Distributive Property

$$-1x^2-3x-8 = (A+B)x^2+(4A+2B+C)x+4A$$ Group like terms.

Once the system of equations obtained by equating coefficients is found, there are two methods that can be used to find the values of A, B, and C.

Method 1 You can write and solve the system of equations using the same method as Example 2.

$A + B = -1$ → $A = -2$

$4A + 2B + C = -3$ → $B = 1$

$4A = -8$ → $C = 3$

Method 2 Another way to solve this system is to let x equal a convenient value to eliminate a variable in the equation found by multiplying each side by the LCD.

$$-x^2-3x-8 = A(x+2)^2 + Bx(x+2) + Cx$$ Original equation

$$-(0)^2-3(0)-8 = A(0+2)^2 + B(0)(0+2) + C(0)$$ Let $x = 0$ to eliminate B and C.

$$-8 = 4A$$

$$-2 = A$$

$$-x^2-3x-8 = A(x+2)^2 + Bx(x+2) + Cx$$ Original equation

$$-(-2)^2-3(-2)-8 = A(-2+2)^2 + B(-2)(-2+2) + C(-2)$$ Let $x = -2$ to eliminate A and B.

$$-6 = -2C$$

$$3 = C$$

Substitute these values for A and C and any value for x into the equation to solve for B.

$$-x^2-3x-8 = A(x+2)^2 + Bx(x+2) + Cx$$ Original equation

$$-1^2-3(1)-8 = -2(1+2)^2 + B(1)(1+2) + 3(1)$$ Let $x = 1$, $A = -2$, and $C = 3$.

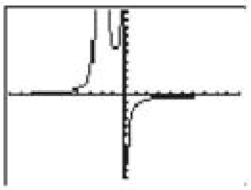
$$-12 = -15 + 3B$$ Simplify.

$$B = 1$$ Solve for B.

Therefore, $\frac{-x^2-3x-8}{x^3+4x^2+4x} = \frac{-2}{x} + \frac{1}{x+2} + \frac{3}{(x+2)^2}$.

StudyTip

Check Graphically You can check the solution to Example 3 by graphing $y_1 = \frac{-x^2-3x-8}{x^3+4x^2+4x}$ and $y_2 = \frac{-2}{x} + \frac{1}{x+2} + \frac{3}{(x+2)^2}$ in the same viewing window. The graphs should coincide. ✔



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GuidedPractice

Find the partial fraction decomposition of each rational expression.

3A. $\frac{x+2}{x^3-2x^2+x}$

3B. $\frac{x+18}{x^3-6x^2+9x}$

2 Prime Quadratic Factors

If the denominator of a rational expression contains a prime quadratic factor, the partial fraction decomposition must include a partial fraction with a *linear numerator* of the form $Bx + C$ for each power of this factor.

Example 4 Denominator with Prime Quadratic Factors

Find the partial fraction decomposition of $\frac{x^4 - 2x^3 + 8x^2 - 5x + 16}{x(x^2+4)^2}$.

This expression is proper. The denominator has one linear factor and one prime quadratic factor of multiplicity 2.

$$\frac{x^4 - 2x^3 + 8x^2 - 5x + 16}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2+4)^2}$$

$$x^4 - 2x^3 + 8x^2 - 5x + 16 = A(x^2+4)^2 + (Bx + C)x(x^2+4) + (Dx + E)x$$

$$x^4 - 2x^3 + 8x^2 - 5x + 16 = Ax^4 + 8Ax^2 + 16A + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex$$

$$1x^4 - 2x^3 + 8x^2 - 5x + 16 = (A + B)x^4 + Cx^3 + (8A + 4B + D)x^2 + (4C + E)x + 16A$$

Write and solve the system of equations obtained by equating coefficients.

$A + B = 1$	$A = 1$
$C = -2$	$B = 0$
$8A + 4B + D = 8$	$C = -2$
$4C + E = -5$	$D = 0$
$16A = 16$	$E = 3$

Therefore, $\frac{x^4 - 2x^3 + 8x^2 - 5x + 16}{x(x^2+4)^2} = \frac{1}{x} - \frac{2}{x^2+4} + \frac{3}{(x^2+4)^2}$.

WatchOut!

Prime Quadratic Factors The alternate method presented for Examples 2 and 3 is not as efficient as the method presented in Example 4 when the denominator of a rational expression involves a prime quadratic factor. This is because there are not enough or no convenient values for x.

GuidedPractice

Find the partial fraction decomposition of each rational expression.

4A. $\frac{x^3 + 2x}{(x^2+1)^2}$

4B. $\frac{4x^3 - 7x}{(x^2 + x + 1)^2}$

ConceptSummary Partial Fraction Decomposition of $f(x)/d(x)$

1. If the degree of $f(x) \geq$ the degree of $d(x)$, use polynomial long division and the division algorithm to write $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. Then apply partial fraction decomposition to $\frac{r(x)}{d(x)}$.

2. If $\frac{f(x)}{d(x)}$ is proper, factor $d(x)$ into a product of linear and/or prime quadratic factors.

3. For each factor of the form $(ax + b)^n$ in the denominator, the partial fraction decomposition must include the sum of n fractions
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{A_n}{(ax+b)^n},$$
where $A_1, A_2, A_3, \dots, A_n$ are real numbers.

4. For each prime quadratic factor that occurs n times in the denominator, the partial fraction decomposition must include the sum of n fractions
$$\frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \frac{B_3x + C_3}{(ax^2+bx+c)^3} + \dots + \frac{B_nx + C_n}{(ax^2+bx+c)^n},$$
where $B_1, B_2, B_3, \dots, B_n$ and $C_1, C_2, C_3, \dots, C_n$ are real numbers.

5. The partial fraction decomposition of the original function is the sum of $q(x)$ from part 1 and the fractions in parts 3 and 4.

Exercises

= Step-by-Step Solutions begin on page R29.

Find the partial fraction decomposition of each rational expression. (Example 1)

1. $\dfrac{x+1}{x^2+5x+6}$

2. $\dfrac{x-18}{x^2-13x+42}$

3. $\dfrac{x+13}{x^2+7x+12}$

4. $\dfrac{x+12}{x^2+14x+48}$

5. $\dfrac{x+6}{-2x^2-19x-45}$

6. $\dfrac{x+7}{2x^2+15x+28}$

Find the partial fraction decomposition of each improper rational expression. (Example 2)

7. $\dfrac{3x^2+x-4}{x^2-2x}$

8. $\dfrac{-5x^2-30x-21}{x^2+7x}$

9. $\dfrac{-2x^3+4x^2+22x-32}{x^3+2x^2-8x}$

10. $\dfrac{x^4-2x^3-2x^2+8x-6}{x^2-2x}$

11. $\dfrac{x^3+12x^2+33x+2}{x^2+8x+15}$

12. $\dfrac{x^4-9x^3+24x^2-4x-12}{x^3-6x^2+8x}$

Find the partial fraction decomposition of each rational expression with repeated factors in the denominator. (Example 3)

13. $\dfrac{x^2-3}{x^3+2x^2+x}$

14. $\dfrac{5x^2-18x+24}{x^3-4x^2+4x}$

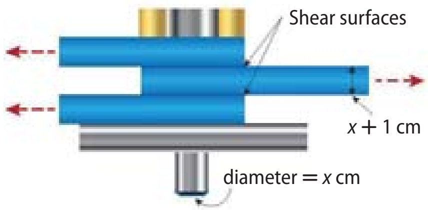
15. $\dfrac{-x^2-22x-50}{x^3+10x^2+25x}$

16. $\dfrac{-5x^2-6x+16}{x^3+8x^2+16x}$

17. $\dfrac{17x+256}{x^3-16x^2+64x}$

18. $\dfrac{-10x-108}{x^3+12x^2+36x}$

19. **ENGINEERING** The sum of the average tensile and shear stresses in the bar shown below can be approximated by $s(x)=\dfrac{20x+10\pi x+20}{\pi x^3+\pi x^2}$, where x is the diameter of the pin. (Example 3)



a. Find the partial fraction decomposition.

b. Graph $s(x)$ and the answer to part **a** in the same viewing window.

Find the partial fraction decomposition of each rational expression with prime quadratic factors in the denominator. (Example 4)

20. $\dfrac{x^3+5x-5}{(x^2+4)^2}$

21. $\dfrac{3x^4+4x^2+8x+18}{x(x^2+3)^2}$

22. $\dfrac{4x^4+x^2-25x+32}{x^5-4x^3+4x}$

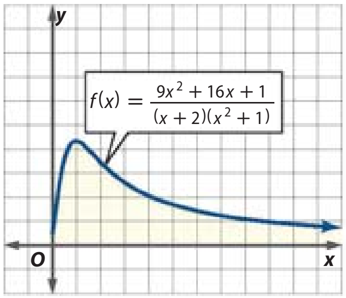
23. $\dfrac{8x^3-48x+7}{(x^2-6)^2}$

24. $\dfrac{-5x^3-10x^2-6x+4}{(x^2+2x+3)^2}$

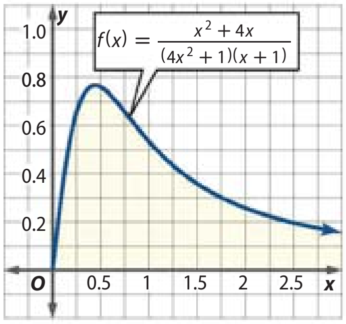
25. $\dfrac{4x^3-12x^2-5x+20}{(x^2-3x+3)^2}$

CALCULUS In calculus, you can find the area of the region between the graph of a rational function and the x-axis on a restricted domain. The first step in this process is to write the partial fraction decomposition of the rational expression. Find the partial fraction decomposition of each rational expression.

26.



27.



Find the partial fraction decomposition of each rational expression. Then use a graphing calculator to check your answer.

28. $\dfrac{x+4}{3x^2-x-2}$

29. $\dfrac{5x^2-2x+8}{x^3-4x}$

30. $\dfrac{4x^2-3x+3}{4x(x-1)^2}$

31. $\dfrac{x^2+x+5}{(x^2+3)^2}$

32. $\dfrac{2x^3}{(x-1)^2(x+1)^2}$

33. $\dfrac{2x^3+12x^2-3x+3}{x^2+6x+5}$

34. Find two rational expressions that have the sum $\dfrac{x+4}{3x^2-x-2}$.

35. Find three rational expressions that have the sum $\dfrac{6-x}{x^3+2x^2+x}$.

Find A, B, C, and D in terms of r and t.

36. $\dfrac{rx-t}{x^2-x-2}=\dfrac{A}{x-2}+\dfrac{B}{x+1}$

37. $\dfrac{4x^2+rx+2t}{x^2+3x}=4+\dfrac{A}{x}+\dfrac{B}{x+3}$

38. $\dfrac{rx+t}{x^3+x^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}$

39. $\dfrac{3x^3+5rx^2-16tx+32}{x^2(x^2+16)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+16}$

Find the partial fraction decomposition of each rational expression.

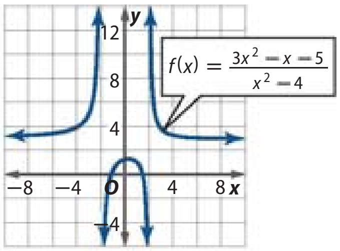
40. $\dfrac{x^3 + 2x - 1}{(x^2 - x - 2)^2}$

41. $\dfrac{x^3 + 4}{(x^2 - 1)(x^2 + 3x + 2)}$

42. $\dfrac{4x^3 + x^2 - 3x + 3}{x(x - 1)^2}$

43. $\dfrac{7x^7 + 2x^6 - 13x^5 + 32x^4 - 19x^3 + 8x^2 - 7x + 2}{x(x - 1)^2(x + 2)(x^2 + 1)}$

44. **MULTIPLE REPRESENTATIONS** In this problem, you will discover the relationship between the partial fraction decomposition of a rational function and its graph. Consider the rational function shown below.



a. **VERBAL** Describe the end behavior and vertical and horizontal asymptotes of the function.

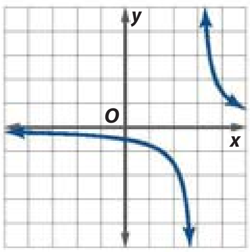
b. **ANALYTICAL** Write the partial fraction decomposition of $f(x)$.

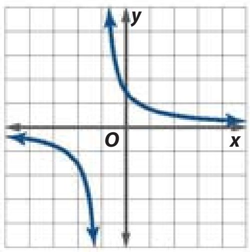
c. **GRAPHICAL** Graph each addend of the partial fraction decomposition you wrote in part **b** as a separate function.

d. **VERBAL** Compare the graphs from part **c** with the graph of $f(x)$ and the analysis you wrote in part **b**.

e. **ANALYTICAL** Make a conjecture as to how the partial fraction decomposition of a function can be used to graph a rational function.

45. **GRAPH ANALYSIS** The rational functions shown make up the partial fraction decomposition of $f(x)$.





Determine which of the four functions listed below could be the original function $f(x)$.

I. $f(x) = \dfrac{6}{x^2 - 2x - 3}$

II. $f(x) = \dfrac{6}{x^2 + 2x - 3}$

III. $f(x) = \dfrac{6}{x^2 + 4x + 3}$

IV. $f(x) = \dfrac{6}{x^2 - 4x + 3}$

H.O.T. Problems Use Higher-Order Thinking Skills

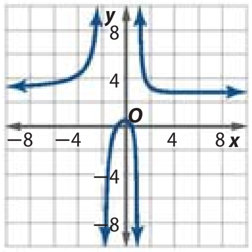
REASONING **Use the partial fraction decompositions of $f(x)$ to explain each of the following.**

46. If $f(x) = \dfrac{-2x^3 - 7x^2 + 13x + 43}{(x - 2)(x + 3)^2}$, explain why $\lim\limits_{x \to \infty} f(x) = -2$.

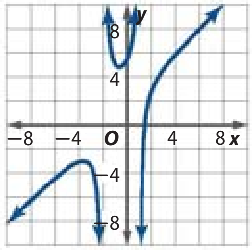
47. If $f(x) = \dfrac{x^2 - x + 1}{(x + 1)^3}$, then what is $\lim\limits_{x \to \infty} f(x)$?

CHALLENGE **Match the graph of each rational function with its equation.**

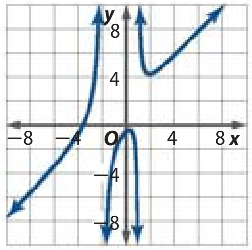
48.



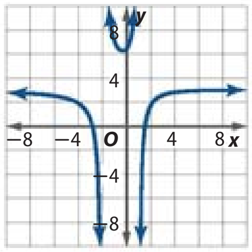
49.



50.



51.



a. $y = x + 2 + \dfrac{1}{x - 1} + \dfrac{-3}{x + 2}$

b. $y = x + 2 + \dfrac{-2}{x - 1} + \dfrac{3}{x + 2}$

c. $y = 3 + \dfrac{1}{x - 1} + \dfrac{-3}{x + 2}$

d. $y = 3 + \dfrac{-2}{x - 1} + \dfrac{3}{x + 2}$

REASONING **Determine whether each of the following statements is *true* or *false*. Explain your reasoning.**

52. If $f(x) = \dfrac{x^3 + 8}{(x^2 - 1)(x - 2)}$, then $\lim\limits_{x \to \infty} f(x) = 8$.

53. The partial fraction decomposition of

$f(x) = \dfrac{-4x^4 + 5x^3 + 27x^2 - 11x - 45}{x(x^2 - 3)^2}$ is

$\dfrac{-5}{x} + \dfrac{4 + x}{(x^2 - 3)} + \dfrac{x^2 + 1}{(x^2 - 3)^2}$.

54. **OPEN ENDED** Write a rational expression of the form $\dfrac{P(x)}{Q(x)}$ in which the partial fraction decomposition contains each of the following in the denominator.

a. nonrepeated linear factors only

b. at least one repeated linear factor

55. **WRITING IN MATH** Describe the steps used to obtain the partial fraction decomposition of a rational expression.

Spiral Review

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists. (Lesson 6-3)

56. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

57. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

58. $p - 2r - 5t = -1$
$p + 2r - 2t = 5$
$4p + r + t = -1$

59. FINANCE For a class project, Jane "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project, she "sold" all of her stock. (Lesson 6-2)

Company	Purchase Price per share ($)	Selling Price per share ($)
utility	54.00	55.20
computer	48.00	58.60
food	60.00	61.10

a. Organize the data in two matrices and use matrix multiplication to find the total amount that Jane spent for the stock.

b. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.

c. How much money did Jane "make" or "lose" in her project?

Simplify each expression. (Lesson 5-1)

60. $\csc \theta \cos \theta \tan \theta$

61. $\sec^2 \theta - 1$

62. $\frac{\tan \theta}{\sin \theta}$

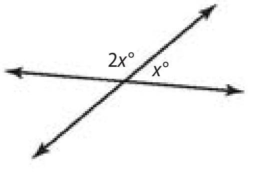
63. MEDICINE Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function. (Lesson 4-4)

a. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

b. How do the periods of the tuning forks compare?

Skills Review for Standardized Tests

64. SAT/ACT In the figure, what is the value of x?



A 40
B 45
C 60
D 75
E 90

65. Decompose $\frac{3p - 1}{p^2 - 1}$ into partial fractions.

F $\frac{2}{p + 1} + \frac{1}{p - 1}$

G $\frac{2}{p - 1} + \frac{1}{p + 1}$

H $\frac{2}{p + 1} - \frac{1}{p - 1}$

J $\frac{2}{p - 1} - \frac{1}{p + 1}$

66. REVIEW A sprinkler waters a circular section of lawn about 20 feet in diameter. The homeowner decides that placing the sprinkler at (7, 5) will maximize the area of grass being watered. Which equation represents the boundary of the area that the sprinkler waters?

A $(x - 7)^2 + (y - 5)^2 = 100$

B $(x + 7)^2 - (y + 5)^2 = 100$

C $(x - 7)^2 - (y + 5)^2 = 100$

D $(x + 7)^2 + (y - 5)^2 = 100$

67. REVIEW Which of the following is the sum of $\frac{x + 2}{x + 3}$ and $\frac{4}{x^2 + x - 6}$?

F $\frac{-3x - 9}{x^2 + x - 6}$

G $\frac{x^2 - 3x - 24}{x^2 + x - 6}$

H $\frac{x^2}{x^2 + x - 6}$

J $\frac{x^2 + x - 1}{x^2 + x - 6}$

LESSON 6-5 Linear Optimization

Then

- You solved systems of linear inequalities. (Lesson 0-4)

Now

1. Use linear programming to solve applications.
2. Recognize situations in which there are no solutions or more than one solution of a linear programming application.

Why?

- In general, businesses strive to minimize costs in order to maximize profits. Factors that create or increase business costs and limit or decrease profits are called *business constraints*.

For a shipping company, one constraint might be the number of hours per day that a trucker can safely drive. For a daycare center, one constraint might be a state regulation restricting the number of children per caregiver for certain age groups.

NewVocabulary
optimization
linear programming
objective function
constraints
feasible solutions
multiple optimal solutions
unbounded

1 Linear Programming

Many applications in business and economics involve **optimization**—the process of finding a minimum value or a maximum value for a specific quantity. When the quantity to be optimized is represented by a linear function, this process is called **linear programming**.

A two-dimensional linear programming problem consists of a linear function to be optimized, called the **objective function**, of the form $f(x, y) = ax + by + c$ and a system of linear inequalities called **constraints**. The solution set of the system of inequalities is the set of possible or **feasible solutions**, which are points of the form (x, y).

Recall from Lesson 0-4 that the solution of a system of linear inequalities is the set of ordered pairs that satisfy each inequality. Graphically, the solution is the intersection of the regions representing the solution sets of the inequalities in the system.

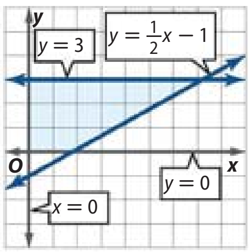
For example, the solution of the system below is the shaded region shown in the graph.

$y \geq \frac{1}{2}x - 1$

$y \leq 3$

$x \geq 0$

$y \geq 0$

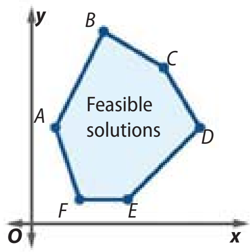


Suppose you were asked to find the maximum value of $f(x, y) = 3x + 5y$ subject to the constraints given by the system above. Because the shaded region representing the set of feasible solutions contains infinitely many points, it would be impossible to evaluate $f(x, y)$ for all of them. Fortunately, the Vertex Theorem provides a strategy for finding the solution, if it exists.

KeyConcept Vertex Theorem for Optimization

Words If a linear programming problem can be optimized, an optimal value will occur at one of the vertices of the region representing the set of feasible solutions.

Example The maximum or minimum value of $f(x, y) = ax + by + c$ over the set of feasible solutions graphed occurs at point A, B, C, D, E, or F.



KeyConcept Linear Programming

To solve a linear programming problem, follow these steps.

Step 1 Graph the region corresponding to the solution of the system of constraints.

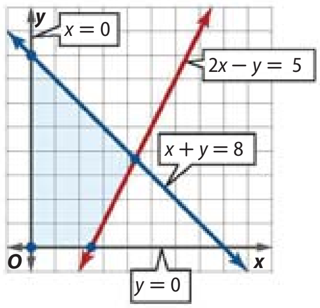
Step 2 Find the coordinates of the vertices of the region formed.

Step 3 Evaluate the objective function at each vertex to determine which *x*- and *y*-values, if any, maximize or minimize the function.

Example 1 Maximize and Minimize an Objective Function

Find the maximum and minimum values of the objective function $f(x, y) = x + 3y$ and for what values of x and y they occur, subject to the following constraints.

$x + y \leq 8$
$2x - y \leq 5$
$x \geq 0$
$y \geq 0$



Begin by graphing the given system of four inequalities. The solution of the system, which makes up the set of feasible solutions for the objective function, is the shaded region, including its boundary segments.

The polygonal region of feasible solutions has four vertices. One vertex is located at (0, 0).

Solve each of the three systems below to find the coordinates of the remaining vertices.

System of Boundary Equations	$x + y = 8$ $2x - y = 5$	$2x - y = 5$ $y = 0$	$x + y = 8$ $x = 0$
Solution (Vertex Point)	$\left(\frac{13}{3}, \frac{11}{3}\right)$	$\left(\frac{5}{2}, 0\right)$	(0, 8)

Find the value of the objective function $f(x, y) = x + 3y$ at each of the four vertices.

$f(0, 0) = 0 + 3(0)$ or 0 ← Minimum value of $f(x, y)$

$f\left(\frac{5}{2}, 0\right) = \frac{5}{2} + 3(0) = \frac{5}{2}$ or $2\frac{1}{2}$

$f\left(\frac{13}{3}, \frac{11}{3}\right) = \frac{13}{3} + 3\left(\frac{11}{3}\right) = \frac{46}{3}$ or $15\frac{1}{3}$

$f(0, 8) = 0 + 3(8)$ or 24 ← Maximum value of $f(x, y)$

So, the maximum value of f is 24 when $x = 0$ and $y = 8$. The minimum value of f is 0 when $x = 0$ and $y = 0$.

GuidedPractice

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints.

1A. $f(x, y) = 2x + 5y$
$x + y \geq -3$
$6x + 3y \leq 24$
$x \geq 0$
$y \geq 0$

1B. $f(x, y) = 5x - 6y$
$y \leq 6$
$y \geq 2x - 2$
$y \geq -3x - 12$

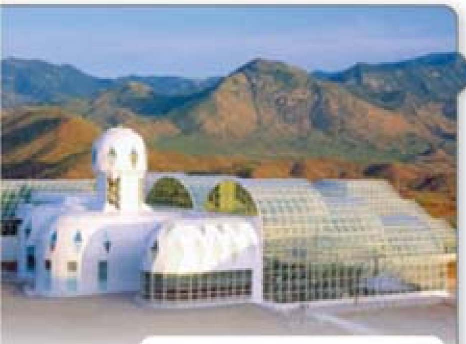
StudyTip

Polygonal Convex Set A bounded set of points on or inside a convex polygon graphed on a coordinate plane is called a *polygonal convex set*.

TechnologyTip

Finding Vertices Recall from Chapter 0 that another way to find a vertex is to calculate the intersection of the boundary lines for the two constraints with a graphing calculator.

[0, 10] scl: 1 by [0, 10] scl: 1



Real-WorldLink

Biosphere 2 in Oracle, Arizona, is a center for research and development of self-sustaining space-colonization technology. The greenhouse consists of 7,200,000 cubic feet of sealed glass, 6500 windows, with a high point of 91 feet.

Source: The University of Arizona

Real-World Example 2 Maximize Profit

BUSINESS A garden center grows only junipers and azaleas in a greenhouse that holds up to 3000 shrubs. Due to labor costs, the number of azaleas grown must be less than or equal to 1200 plus three times the number of junipers. The market demand for azaleas is at least twice that of junipers. The center makes a profit of \$2 per juniper and \$1.50 per azalea.

a. Write an objective function and a list of constraints that model the given situation.

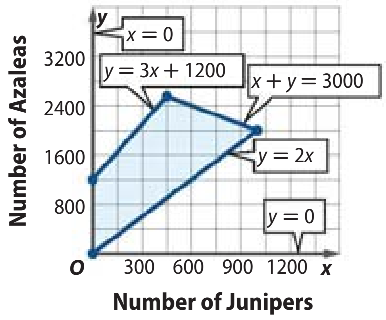
Let x represent the number of junipers produced and y the number of azaleas. The objective function is then given by $f(x, y) = 2x + 1.5y$.

The constraints are given by the following.

$y \geq 2x$	Market demand constraint
$y \leq 3x + 1200$	Production constraint
$x + y \leq 3000$	Greenhouse capacity constraint

Because x and y cannot be negative, additional constraints are that $x \geq 0$ and $y \geq 0$.

b. Sketch a graph of the region determined by the constraints from part a to find how many of each plant the company should grow to maximize profit.



The shaded polygonal region has four vertex points at (0, 0), (0, 1200), (450, 2550), and (1000, 2000). Find the value of $f(x, y) = 2x + 1.5y$ at each of the four vertices.

$f(0, 0) = 2(0) + 1.5(0)$ or 0

$f(0, 1200) = 2(0) + 1.5(1200)$ or 1800

$f(450, 2550) = 2(450) + 1.5(2550)$ or 4725

$f(1000, 2000) = 2(1000) + 1.5(2000)$ or 5000 ← Maximum value of $f(x, y)$

Because f is greatest at (1000, 2000), the garden center should grow 1000 junipers and 2000 azaleas to earn a maximum profit of \$5000.

GuidedPractice

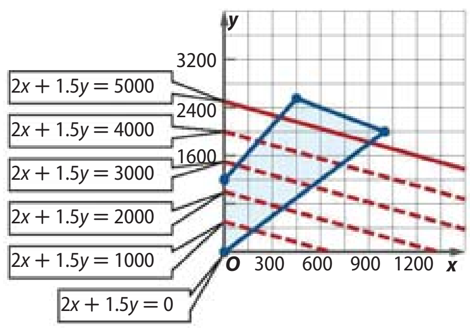
2. MANUFACTURING A lumber mill can produce up to 600 units of product each week. To meet the needs of its regular customers, the mill must produce at least 150 units of lumber and at least 225 units of plywood. The lumber mill makes a profit of \$30 for each unit of lumber and \$45 for each unit of plywood.

A. Write an objective function and a list of constraints that model the given situation.

B. Sketch a graph of the region determined by the constraints to find how many units of each type of wood product the mill should produce to maximize profit.

To better understand why the maximum value of $f(x, y) = 2x + 1.5y$ must occur at a vertex in Example 2, assign f different positive values from 0 to 5000 and then graph the corresponding family of parallel lines.

Notice that the distance of a line in this family from the origin increases as f increases, sweeping across the region of feasible solutions.



Geometrically, to maximize f over the set of feasible solutions, you want the line with the greatest f-value that still intersects the shaded region. From the graph you can see that such a line will intersect the shaded region at one point, the vertex at (1000, 2000).

> **StudyTip**
> **Objective Functions** To find the equation related to the objective function, solve the objective function for y.

2 No or Multiple Optimal Solutions

As with systems of linear equations, linear programming problems can have one, multiple, or no optimal solutions. If the graph of the equation related to the objective function f to be optimized is coincident with one side of the region of feasible solutions, f has **multiple optimal solutions**. In Figure 6.5.1, any point on the segment connecting vertices at (3, 7) and (7, 3) is an optimal solution of f. If the region does not form a polygon, but is instead **unbounded**, f may have no minimum value or no maximum value. In Figure 6.5.2, f has no maximum value.

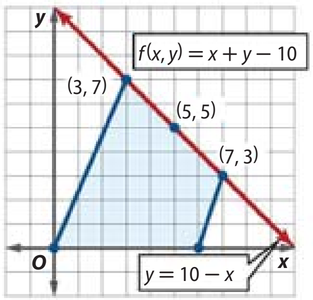


Figure 6.5.1

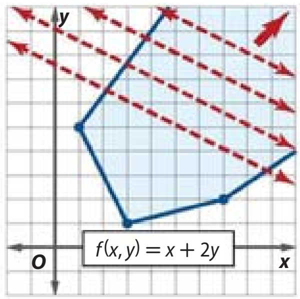
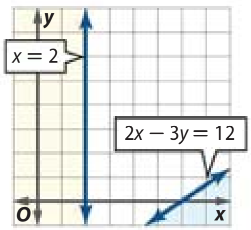


Figure 6.5.2

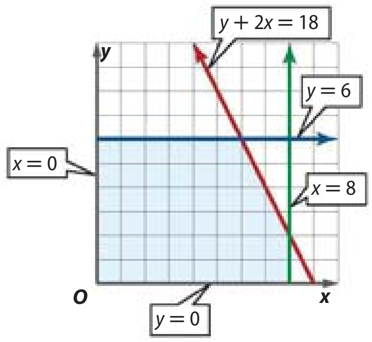
> **StudyTip**
> **Infeasible Linear Programming Problem** The solution of a linear programming problem is said to be *infeasible* if the set of constraints do not define a region with common points. For example, the graph below does not define a region of feasible solutions over which to optimize an objective function.
>
> 
>

Example 3 Optimization at Multiple Points

Find the maximum value of the objective function $f(x, y) = 4x + 2y$ and for what values of x and y it occurs, subject to the following constraints.

$y + 2x \le 18$
$y \le 6$
$x \le 8$
$x \ge 0$
$y \ge 0$

Graph the region bounded by the given constraints. The polygon region of feasible solutions has five vertices at (0, 0), (8, 2), (0, 6), (8, 0), and (6, 6). Find the value of the objective function $f(x, y) = 4x + 2y$ at each vertex.



$f(0, 0) = 4(0) + 2(0)$ or 0

$f(8, 2) = 4(8) + 2(2)$ or 36

$f(0, 6) = 4(0) + 2(6)$ or 12

$f(8, 0) = 4(8) + 2(0)$ or 32

$f(6, 6) = 4(6) + 2(6)$ or 36

Because $f(x, y) = 36$ at (6, 6) and (8, 2), there are multiple points at which f is optimized. An equation of the line through these two vertices is $y = -2x + 18$. Therefore, f has a maximum value of 36 at every point on $y = -2x + 18$ for $6 \le x \le 8$.

GuidedPractice

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y it occurs, subject to the given constraints.

3A. $f(x, y) = 3x + 3y$
$4x + 3y \ge 12$
$y \le 3$
$y \ge 0$
$x \le 4$
$x \ge 0$

3B. $f(x, y) = 4x + 8y$
$x + 2y \le 16$
$y \ge 2$
$x \ge 3$

Real-World Career

Veterinarian Potential veterinarians must graduate with a Doctor of Veterinary Medicine degree from an accredited university. In a recent year, veterinarians held 62,000 jobs in the U.S., where 3 out of 4 were employed in a solo or group practice.

Real-World Example 4 Unbounded Feasible Region

VETERINARY MEDICINE A veterinarian recommends that a new puppy eat a diet that includes at least 1.54 ounces of protein and 0.56 ounce of fat each day. Use the table to determine how much of each dog food should be used in order to satisfy the dietary requirements at the minimum cost.

Dog Food Brand	Protein (oz/cup)	Fat (oz/cup)	Cost per cup ($)
Good Start	0.84	0.21	0.36
Sirius	0.56	0.49	0.22

a. Write an objective function, and list the constraints that model the given situation.

Let x represent the number of cups of *Good Start* eaten and y represent the number of cups of *Sirius* eaten. The objective function is then given by $f(x, y) = 0.36x + 0.22y$.

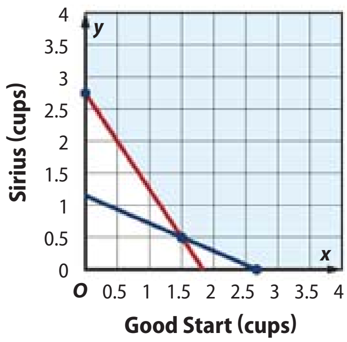
The constraints on required fat and protein are given by

$0.84x + 0.56y \geq 1.54$ Protein constraint

$0.21x + 0.49y \geq 0.56$ Fat constraint

Because x and y cannot be negative, there are also constraints of $x \geq 0$ and $y \geq 0$.

b. Sketch a graph of the region determined by the constraints from part a to find how many cups of each dog food should be used in order to satisfy the dietary requirements at the optimal cost.



The shaded polygonal region has three vertex points at (0, 2.75), (1.5, 0.5), and (2.67, 0).

The optimal cost would be the minimum value of $f(x, y) = 0.36x + 0.22y$. Find the value of the objective function at each vertex.

$f(0, 2.75) = 0.36(0) + 0.22(2.75)$, or 0.605 ← Minimum value of $f(x, y)$

$f(1.5, 0.5) = 0.36(1.5) + 0.22(0.5)$, or 0.65

$f(2.67, 0) = 0.36(2.67) + 0.22(0)$, or 0.9612

Therefore, to meet the veterinarian's requirements at a minimum cost of about $0.61 per cup, the puppy should eat 2.75 cups of only the *Sirius* brand.

Guided Practice

4. **MANAGEMENT** According to the manager of a pizza shop, the productivity in worker-hours of her workers is related to their positions. One worker-hour is the amount of work done by an average employee in one hour. For the next 8-hour shift, she will need two shift leaders, at least two associates, and at least 10 total workers. She will also need to schedule at least 120 worker-hours to meet customer demand during that shift.

Employees Working	Productivity (in worker-hours)	Wage ($)
associate	1.5	7.50
employee	1.0	6.50
shift leader	2.0	9.00

A. Assuming that each employee works an entire 8-hour shift, write an objective function and list the constraints that model the given situation.

B. Sketch a graph of the region determined by the constraints to find how many workers should be scheduled to optimize labor costs.

Exercises

= Step-by-Step Solutions begin on page R29.

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints. (Example 1)

1. $f(x, y) = 3x + y$
   $y \leq 2x + 1$
   $x + 2y \leq 12$
   $1 \leq y \leq 3$

2. $f(x, y) = -x + 4y$
   $y \leq x + 4$
   $y \geq -x + 3$
   $1 \leq x \leq 4$

3. $f(x, y) = x - y$
   $x + 2y \leq 6$
   $2x - y \leq 7$
   $x \geq -2$
   $y \geq -3$

4. $f(x, y) = 3x - 5y$
   $x \geq 0, y \geq 0$
   $x + 2y \leq 6$
   $2y - x \leq 2$
   $x + y \leq 5$

5. $f(x, y) = 3x - 2y$
   $y \leq x + 3$
   $1 \leq x \leq 5$
   $y \geq 2$

6. $f(x, y) = 3y + x$
   $4y \leq x + 8$
   $2y \geq 3x - 6$
   $2x + 2y \geq 4$

7. $f(x, y) = x - 4y$
   $x \geq 2, y \geq 1$
   $x - 2y \geq -4$
   $2x - y \leq 7$
   $x + y \leq 8$

8. $f(x, y) = x - y$
   $3x - 2y \geq -7$
   $x + 6y \geq -9$
   $5x + y \leq 13, x - 3y \geq -7$

9. **MEDICAL OFFICE** Olivia is a receptionist for a medical clinic. One of her tasks is to schedule appointments. She allots 20 minutes for a checkup and 40 minutes for a physical. The doctor can do no more than 6 physicals per day, and the clinic has 7 hours available for appointments. A checkup costs \$55, and a physical costs \$125. (Example 2)
   - **a.** Write an objective function and list the constraints that model the given situation.
   - **b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.
   - **c.** How many of each appointment should Olivia make to maximize income? What is the maximum income?

10. **INCOME** Josh is working part-time to pay for some of his college expenses. Josh delivers pizza for \$5 per hour plus tips, which run about \$8 per hour, and he also tutors in the math lab for \$15 per hour. The math lab is open only 2 hours daily, Monday through Friday, when Josh is available to tutor. Josh can work no more than 20 hours per week due to his class schedule. (Example 2)
    - **a.** Write an objective function and list the constraints that model the given situation.
    - **b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.
    - **c.** How can Josh make the most money, and how much is it?

11. **SMALL BUSINESS** A design company creates Web sites and E-albums. Each Web site requires 10 hours of planning and 4.5 hours of page design. Each family E-album requires 15 hours of planning and 9 hours of page design. There are 70 hours available each week for the staff to plan and 36 hours for page design. (Example 2)
    - **a.** If the profit is \$600 for each Web site and \$700 for each family E-album, write an objective function and list the constraints that model the given situation.
    - **b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.
    - **c.** How much of each product should be produced to achieve maximum profit? What is the profit?

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints. (Examples 3 and 4)

12. $f(x, y) = 4x - 4y$
    $2x + y \geq -7$
    $y \leq x + 2$
    $y \leq 11 - 2x$

13. $f(x, y) = 3x + 6y$
    $y \leq -\frac{1}{2}x + \frac{5}{2}$
    $y \leq 2, y \geq 0$
    $x \leq 3, x \geq 0$

14. $f(x, y) = -3x - 6y$
    $y \leq -\frac{1}{2}x + 5$
    $y \leq 4, y \geq 0$
    $x \leq 6, x \geq 0$

15. $f(x, y) = 6x - 4y$
    $2x + 3y \geq 6$
    $3x - 2y \geq -4$
    $5x + y \geq 15$

16. $f(x, y) = 3x + 4y$
    $y \leq x - 3$
    $y \leq 6 - 2x$
    $2x + y \geq -3$

17. $f(x, y) = 8x + 10y$
    $y \leq -\frac{4}{5}x + 4$
    $y \leq 4, y \geq 0$
    $x \leq 5, x \geq 0$

18. **NUTRITION** Michelle wants to consume more nutrients. She wants to receive at least 40 milligrams of calcium, 600 milligrams of potassium, and 50 milligrams of vitamin C. Michelle's two favorite fruits are apples and bananas. The average nutritional content of both are given. (Example 4)

Fruit	Calcium	Potassium	Vitamin C
apple	9.5 mg	158 mg	9 mg
banana	7.0 mg	467 mg	11 mg

    - **a.** If each apple costs \$0.55 and each banana costs \$0.35, write an objective function. List the constraints that model the given situation.
    - **b.** Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions for the objective function.
    - **c.** Determine the number of each type of fruit that Michelle should eat to minimize her cost while still obtaining her desired nutritional intake.

Find a value of a so that each objective function has maximum values at the indicated vertex, subject to the following constraints.

$$x \geq 0$$
$$y \geq 2$$
$$x + y \leq 9$$
$$-4x + 3y \leq 6$$

19. $f(x, y) = -4x + ay$, $(0, 2)$

20. $f(x, y) = -4x + ay$, $(3, 6)$

21. $f(x, y) = x - ay$, $(3, 6)$

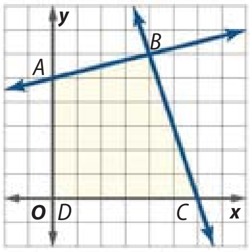
22. $f(x, y) = x - ay$, $(7, 2)$

23. $f(x, y) = ax + 4y$, $(7, 2)$

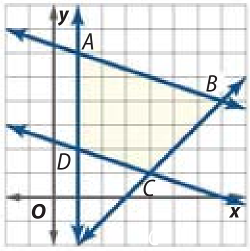
24. $f(x, y) = ax - 3y$, $(0, 2)$

Find an objective function that has a maximum or minimum value at each indicated vertex.

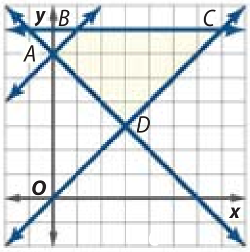
25. minimum at A



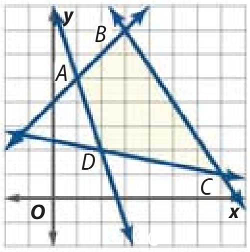
26. maximum at C



27. maximum at B



28. minimum at D



29. **BUSINESS** A batch of Mango Sunrise uses 3 liters of mango juice and 1 liter of strawberry juice. A batch of Oasis Dream uses 2 liters of mango juice and 1 liter of strawberry juice. The store has 40 liters of mango juice and 15 liters of strawberry juice that it wants to use up before the end of the day. The profit on Mango Sunrise is $16 per batch, and the profit on the Oasis Dream is $12 per batch.

a. Write an objective function, and list the constraints that model the given situation.

b. In order to maximize profits, how many batches of each drink should the juice store make?

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints.

30. $f(x, y) = 4x - 8y$
$y \geq x^2 - 8x + 18$
$y \leq -x^2 + 8x - 10$
$y \leq 8 - x$

31. $f(x, y) = -2x + 5y$
$y \geq x^2 + 6x + 3$
$y \leq -x^2 - 4x + 15$
$y \leq x + 9$

Find the area enclosed by the polygonal convex set defined by each system of inequalities.

32. $x \geq 0$
$x \leq 12$
$2x + 6y \leq 84$
$2x - 3y \leq -3$
$8x + 3y \geq 33$

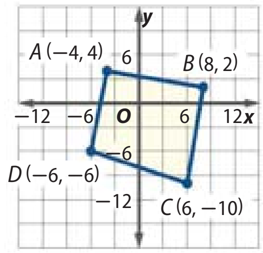
33. $y \leq 9$
$x \geq 2$
$x - y \leq 5$
$2x + y \leq 25$
$3x + 2y \leq 20$

34. $x \geq 0$
$y \geq \frac{1}{2}x$
$3x + 2y \geq 8$
$-3x + 4y \leq 28$
$x + 2y \leq 24$

35. $x \leq 10$
$x + y \leq 14$
$x + 3y \geq 13$
$-x + 5y \leq 40$
$4x + y \geq 8$

H.O.T. Problems Use Higher-Order Thinking Skills

36. **CHALLENGE** Provide a system of inequalities that forms the polygonal convex set shown below.



37. **REASONING** Consider the profit function $P(x, y) = ax + by$. Will P always have a positive maximum value if the feasible region lies entirely within the first quadrant? Explain your reasoning.

38. **CHALLENGE** Find the maximum and minimum values of the objective function $f(x, y) = -6x + 3y$ and for what values of x and y they occur, subject to the given constraints.

$y \geq 4$
$3x + 2y \geq 14$
$-2x + 5y \leq 60$
$-x + y \geq -3$
$-7x + 5y \leq 35$
$2x + y \leq 36$

39. **OPEN ENDED** Linear programming has numerous real-world applications.

a. Write a real-world problem that could be solved using linear programming.

b. Using at least 4 constraints, write an objective function to be maximized or minimized.

c. Sketch a graph of the region determined by the constraints from part **a** to find the set of feasible solutions.

d. Find the solution to the problem.

40. **WRITING IN MATH** Is it possible for a linear programming problem to have no maximum solution and no minimum solution? Explain your reasoning.

Spiral Review

Find the partial fraction decomposition of each rational expression. (Lesson 6-4)

41. $\frac{8y + 7}{y^2 + y - 2}$

42. $\frac{x - 6}{x^2 - 2x}$

43. $\frac{5m - 4}{m^2 - 4}$

44. $\frac{-4y}{3y^2 - 4y + 1}$

45. **ARCADE GAMES** Marcus and Cody purchased game cards to play virtual games at the arcade. Marcus used 47 points from his game card to drive the race car and snowboard simulators four times each. Cody used 48.25 points from his game card to drive the race car simulator five times and the snowboard simulator three times. How many points did each game require per play? (Lesson 6-3)

46. **WAVES** After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds. (Lesson 5-3)

a. If $h = 3$ and $P = 2$, write the equation for the wave. Draw its graph over a 10-second interval.

b. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

47. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity. (Lesson 5-2)

Find the area of each triangle to the nearest tenth. (Lesson 4-7)

48. $\triangle ABC$, if $A = 127°$, $b = 12$ m, and $c = 9$ m

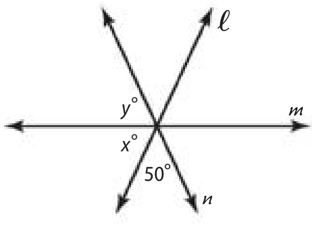
49. $\triangle ABC$, if $a = 7$ yd, $b = 8$ yd, and $C = 44°$

50. $\triangle ABC$, if $A = 50°$, $b = 15$ in., and $c = 10$ in.

51. $\triangle ABC$, if $a = 6$ cm, $B = 135°$, and $c = 3$ cm

Skills Review for Standardized Tests

52. **SAT/ACT** In the figure below, lines ℓ, m, and n intersect in a single point. What is the value of $x + y$?



A 40

B 70

C 90

D 130

E 260

53. The area of a parking lot is 600 square meters. A car requires 6 square meters of space, and a bus requires 30 square meters of space. The attendant can handle no more than 60 vehicles. If it cost \$3 to park a car and \$8 to park a bus, how many of each should the attendant accept to maximize income?

F 20 buses and 0 cars

G 10 buses and 50 cars

H 5 buses and 55 cars

J 0 buses and 60 cars

54. **FREE RESPONSE** Use the two systems of equations to answer each of the following.

A	*B*
$-5x + 2y + 11z = 31$	$x + 2y + 2z = 3$
$2y + 6z = 26$	$3x + 7y + 9z = 30$
$2x - y - 5z = -15$	$-x - 4y - 7z = -37$

a. Write the coefficient matrix for each system. Label the matrices A and B.

b. Find AB and BA if possible.

c. Write the augmented matrix for system A in reduced row-echelon form.

d. Find the determinant of each coefficient matrix. Which matrices are invertible? Explain your reasoning.

e. Find the inverse of matrix B.

f. Use the inverse of B to solve the system.

g. Which systems could you use Cramer's Rule to solve? Explain your reasoning.

Study Guide and Review

Study Guide

KeyConcepts

Multivariable Linear Systems and Row Operations (Lesson 6-1)

- Each of these row operations produces an equivalent augmented matrix.
  - Interchange any two rows.
  - Multiply one row by a nonzero real number.
  - Add a multiple of one row to another row.

Multiplying Matrices (Lesson 6-2)

- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is an $m \times n$ matrix in which
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj}.$$
- I_n is an $n \times n$ matrix consisting of all 1s on its main diagonal and 0s for all other elements.
- The inverse of A is A^{-1} where $AA^{-1} = A^{-1}A = I_n$.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - cb \neq 0$, then $A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The number $ad - cb$ is called the *determinant* of the 2×2 matrix and is denoted by $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solving Systems (Lesson 6-3)

- Suppose $AX = B$, where A is the matrix of coefficients of a linear system, X is the matrix of variables, and B is the matrix of constant terms. If A is invertible, then $AX = B$ has a unique solution given by $X = A^{-1}B$.
- If $\det(A) \neq 0$, then the unique solution of a system is given by $x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, x_3 = \frac{|A_3|}{|A|}, \ldots, x_n = \frac{|A_n|}{|A|}$. If $\det(A) = 0$, then $AX = B$ has no solution or infinitely many solutions.

Partial Fractions (Lesson 6-4)

- If the degree of $f(x)$ is greater than or equal to the degree of $d(x)$, use polynomial long division and the division algorithm to write $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. Then apply partial fraction decomposition to $\frac{r(x)}{d(x)}$.

Linear Optimization (Lesson 6-5)

- The maximum and minimum values of a linear function in x and y are determined by linear programming techniques.
  Step 1. Graph the solution of the system of constraints.
  Step 2. Find the coordinates of the vertices of the region.
  Step 3. Evaluate the objective function at each vertex to find which values maximize or minimize the function.

KeyVocabulary

- **augmented matrix** (p. 366)
- **coefficient matrix** (p. 366)
- **constraint** (p. 405)
- **Cramer's Rule** (p. 390)
- **determinant** (p. 381)
- **feasible solution** (p. 405)
- **Gaussian elimination** (p. 364)
- **identity matrix** (p. 378)
- **inverse** (p. 379)
- **inverse matrix** (p. 379)
- **invertible** (p. 379)
- **linear programming** (p. 405)
- **multiple optimal solutions** (p. 408)
- **multivariable linear system** (p. 364)
- **objective function** (p. 405)
- **optimization** (p. 405)
- **partial fraction** (p. 398)
- **partial fraction decomposition** (p. 398)
- **reduced row-echelon form** (p. 369)
- **row-echelon form** (p. 364)
- **singular matrix** (p. 379)
- **square system** (p. 388)
- **unbounded** (p. 408)

VocabularyCheck

Choose the word or phrase that best completes each sentence.

1. A(n) (augmented matrix, coefficient matrix) is a matrix made up of all the coefficients and constant terms of a linear system.
2. (Gaussian elimination, Elementary row operations) reduce(s) a system of equations to an equivalent, simpler system, making it easier to solve.
3. The result of Gauss-Jordan elimination is a matrix that is in (reduced row-echelon, invertible) form.
4. The product of an $n \times n$ matrix A with the (inverse matrix, identity matrix) is A.
5. The identity matrix I is its own (augmented matrix, inverse matrix).
6. A square matrix that has no inverse is (nonsingular, singular).
7. The (determinant, square system) of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.
8. When solving a square linear system, an alternative to Gaussian elimination is (Cramer's Rule, partial fraction decomposition).
9. A two-dimensional linear programming problem contains (constraints, feasible solutions), which are linear inequalities.
10. If the graph of the objective function to be optimized, f, is coincident with one side of the region of feasible solutions, then there may be (multiple optimal, unbounded) solutions.

Lesson-by-Lesson Review

6-1 Multivariable Linear Systems and Row Operations (pp. 364–374)

Write each system of equations in triangular form using Gaussian elimination. Then solve the system.

11. $3x + 4y = 7$
$2y = -5x + 7$

12. $5x - 3y = 16$
$x + 3y = -4$

13. $x + y + z = 4$
$2x - y - 3z = 4$
$-3x - 4y - 5z = -13$

14. $x + y - z = 5$
$2x - 3y + 5z = -1$
$3x - y + 2z = 10$

15. $2x - 5y = 2z + 11$
$3y + 4z = x - 28$
$3z - x = -18 - 3y$

16. $2x - 3z = y - 1$
$5z - 8 = 3x + 4y$
$x + y + z = 3$

Solve each system of equations.

17. $2x + 2y = 8$
$3x - 8y = -21$

18. $x - 2y = 13$
$-5x - 6y = 15$

19. $x + y = 4$
$x + y + z = 7$
$x - z = -1$

20. $x + y = 1$
$3x - 7y + z = -7$
$4x + 8y + 3z = -9$

21. $3x - y + z = 8$
$2x - 3y = 3z - 13$
$x + z = 6 - y$

22. $x + y = z - 1$
$2x + 2y + z = 13$
$3x - 5y + 4z = 8$

Example 1

Solve the system of equations using Gaussian elimination.
$x + 2y + 3z = 8$
$2x - 4y + z = 2$
$-3x - 6y + 7z = 8$

Write the augmented matrix. Then apply elementary row operations to obtain a row echelon form.

Augmented matrix $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 8 \\ 2 & -4 & 1 & 2 \\ -3 & -6 & 7 & 8 \end{array}\right]$

$-2R_1 + R_2 \rightarrow$, $3R_1 + R_3 \rightarrow$ $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 8 \\ 0 & -8 & -5 & -14 \\ 0 & 0 & 16 & 32 \end{array}\right]$

$\frac{1}{16}R_3 \rightarrow$ $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 8 \\ 0 & -8 & -5 & -14 \\ 0 & 0 & 1 & 2 \end{array}\right]$

You can use substitution to find that $y = 0.5$ and $x = 1$. Therefore, the solution of the system is $x = 1$, $y = 0.5$, and $z = 2$, or the ordered triple (1, 0.5, 2).

6-2 Matrix Multiplication, Inverses, and Determinants (pp. 375–386)

Find *AB* and *BA*, if possible.

23. $A = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 0 & 1 \end{bmatrix}$
$B = \begin{bmatrix} -5 & -4 \\ 1 & 7 \end{bmatrix}$

24. $A = \begin{bmatrix} -2 & 3 \\ -4 & 7 \end{bmatrix}$
$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

25. $A = \begin{bmatrix} 4 & -3 \end{bmatrix}$
$B = \begin{bmatrix} -8 & 5 \\ -7 & -1 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 3 \end{bmatrix}$
$B = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

Find A^{-1}, if it exists. If A^{-1} does not exist, write *singular*.

27. $A = \begin{bmatrix} 2 & 1 \\ -3 & 8 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

29. $A = \begin{bmatrix} 2 & 3 \\ -8 & -12 \end{bmatrix}$

30. $A = \begin{bmatrix} -5 & 2 \\ 3 & 8 \end{bmatrix}$

Example 2

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$. Find A^{-1}, if it exists. If A^{-1} does not exist, write *singular*.

First, write a doubly augmented matrix. Then apply elementary row operations to write the matrix in reduced row-echelon form.

Augmented matrix $\rightarrow$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 2 & 9 & 0 & 1 \end{array}\right]$

$-2R_1 + R_2 \rightarrow$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$

$-4R_2 + R_1 \rightarrow$ $\left[\begin{array}{rr|rr} 1 & 0 & 9 & -4 \\ 0 & 1 & -2 & 1 \end{array}\right]$

Because the system has a solution, $a = 9$, $b = -4$, $c = -2$, and $d = 1$, A is invertible and $A^{-1} = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$.

6-3 Solving Linear Systems Using Inverses and Cramer's Rule (pp. 388–394)

Use an inverse matrix to solve each system of equations, if possible.

31. $2x - 3y = -23$
$3x + 7y = 23$

32. $3x - 6y = 9$
$-5x - 8y = -6$

33. $2x + y = 1$
$x - 3y + z = -4$
$y + 8z = -7$

34. $x + y + z = 1$
$x + y - z = -7$
$y + z = -1$

35. $3y + 5z = 25$
$2x - 7y - 3z = 15$
$x + y - z = -11$

36. $x - 2y - 3z = 0$
$2x - 3y + 4z = 11$
$x - 8y + 2z = -1$

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.

37. $2x - 4y = 30$
$3x + 5y = 12$

38. $2x + 6y = 14$
$x - 3y = 1$

39. $2x + 3y - z = 1$
$x + y - 3z = 12$
$5x - 7y + 2z = 28$

40. $x + 2y + z = -2$
$2x + 2y - 5z = -19$
$3x - 4y + 8z = -1$

41. $-3x - 4y + z = 15$
$x - 5y - z = 3$
$4x - 3y - 2z = -8$

42. $2x + 3y + 4z = 29$
$x - 8y - z = -3$
$2x + y + z = 4$

Example 3

Use an inverse matrix to solve the system of equations, if possible.

$x - y + z = -5$
$2x + 2y - 3z = -27$
$-3x - y + z = 17$

Write the system in matrix form.

$$\underset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ -3 & -1 & 1 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} -5 \\ -27 \\ 17 \end{bmatrix}}$$

Use a graphing calculator to find A^{-1}.

$$A^{-1} = \begin{bmatrix} 0.25 & 0 & -0.25 \\ -1.75 & -1 & -1.25 \\ -1 & -1 & -1 \end{bmatrix}$$

Multiply A^{-1} by B to solve the system.

$$X = \begin{bmatrix} 0.25 & 0 & -0.25 \\ -1.75 & -1 & -1.25 \\ -1 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -27 \\ 17 \end{bmatrix} = \begin{bmatrix} -5.5 \\ 14.5 \\ 15 \end{bmatrix}$$

Therefore, the solution is (−5.5, 14.5, 15).

6-4 Partial Fractions (pp. 398–404)

Find the partial fraction decomposition of each rational expression.

43. $\dfrac{2x}{x^2 - 4}$

44. $\dfrac{7x - 6}{x^2 - x - 6}$

45. $\dfrac{-2x + 9}{x^2 - 11x + 30}$

46. $\dfrac{6x^2 - 4x - 6}{x^3 - 2x^2 - 3x}$

47. $\dfrac{2x^3 - 14x^2 + 2x + 7}{x^2 - 7x}$

48. $\dfrac{2x^4 + 3x^3 + 5x^2 + 3x + 2}{x(x^2 + 1)^2}$

49. $\dfrac{2x^2 + 4}{x^2 - 2x}$

50. $\dfrac{2x^2 - 12x - 20}{x^2 + 4x}$

51. $\dfrac{x^2 + x - 6}{2x^2 - 3x}$

52. $\dfrac{3x^2 - 10x - 20}{2x^2 + 5x}$

Example 4

Find the partial fraction decomposition of $\dfrac{x - 8}{x^2 - 11x + 18}$.

Rewrite the equation as partial fractions with constant numerators, A and B.

$$\frac{x + 12}{x^2 - 11x + 18} = \frac{A}{x - 9} + \frac{B}{x - 2}$$

$$x + 12 = A(x - 2) + B(x - 9)$$

$$x + 12 = Ax - 2A + Bx - 9B$$

$$x + 12 = (A + B)x + (-2A - 9B)$$

Equate the coefficients on the left and right sides of the equation to obtain a system of two equations.

$A + B = 1$

$-2A - 9B = 12$

The solution of the system is $A = 3$ and $B = -2$.

Therefore, $\dfrac{x + 12}{x^2 - 11x + 18} = \dfrac{3}{x - 9} + \dfrac{-2}{x - 2}$.

6-5 Linear Optimization (pp. 405–412)

Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints.

53. $f(x, y) = 2x - y$
$x \geq 0$
$y \leq -x + 7$
$y \geq x + 1$

54. $f(x, y) = 3x + y$
$2x - y \leq 1$
$1 \leq y \leq 9$
$x \geq 1$

55. $f(x, y) = x + y$
$0 \leq x \leq 10$
$x + 2y \geq 8$
$0 \leq y \leq 8$

56. $f(x, y) = 2x - 4y$
$x \geq 3$
$y \geq 3$
$4x + 5y \leq 47$

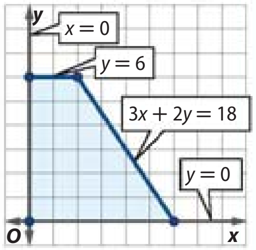
57. $f(x, y) = 4x + 3y$
$3x + y \geq 8$
$2x + y \leq 12$
$y \geq x$

58. $f(x, y) = 2y - 5x$
$2x + y \geq 0$
$x - 5y \leq 0$
$3x + 7y \leq 22$

Example 5

Find the maximum value of the objective function $f(x, y) = 2x + 6y$ and for what values of x and y they occur, subject to the following constraints.
$y \geq 0$, $x \geq 0$, $y \leq 6$, $3x + 2y \leq 18$

Graph the region bounded by the given constraints. Find the value of the objective function $f(x, y) = 2x + 6y$ at each vertex.



$f(0, 0) = 0 + 0$ or 0

$f(0, 6) = 0 + 36$ or 36

$f(6, 0) = 12 + 0$ or 12

$f(2, 6) = 4 + 36$ or 40

So, the maximum value of f is 40 when $x = 2$ and $y = 6$.

Applications and Problem Solving

59. HAMBURGERS The table shows the number of hamburgers, cheeseburgers, and veggie burgers sold at a diner over a 3-hour lunch period. Find the price for each type of burger. (Lesson 6-1)

Hours	Plain	Cheese	Veggie	Total Sales ($)
11 A.M.–12 P.M.	2	8	2	53
12–1 P.M.	7	12	8	119
1–2 P.M.	1	5	7	64

60. GRADING Ms. Hebert decides to base grades on tests, homework, projects and class participation. She assigns a different percentage weight for each category, as shown. Find the final grade for each student to the nearest percent. (Lesson 6-2)

Category	tests	HW	projects	participation
Weight	40%	30%	20%	10%

Category	Serena	Andrew	Corey	Shannon
tests	88	72	78	91
HW	95	90	68	71
projects	80	73	75	85
participation	100	95	100	80

61. SHAVED ICE A shaved ice stand sells 3 flavors: strawberry, pineapple, and cherry. Each flavor sells for $1.25. One day, the stand had $60 in total sales. The stand made $13.75 more in cherry sales than pineapple sales and $16.25 more than strawberry sales. Use Cramer's Rule to determine how many of each flavor was sold. (Lesson 6-3)

62. BIKING On a biking trip, a couple traveled 240 miles on Day 1 and 270 miles on Day 2. The average rate traveled during Day 1 is 5 miles per hour faster than the average rate traveled during Day 2. The total number of hours spent biking is $T = \frac{510r - 1200}{r(r - 5)}$. (Lesson 6-4)

a. Find the partial fraction decomposition of T.

b. Each fraction represents the time spent biking each day. If the couple biked 6 hours longer on Day 2, how many total hours did they bike?

63. RECYCLING A recycling company will collect from private sites if the site produces at least 60 pounds of recycling a week. The company can collect at most 50 pounds of paper and 30 pounds of glass from each site. The company earns $20 for each pound of glass and $25 for each pound of paper. (Lesson 6-5)

a. Write an objective function, and list the constraints that model the given situation.

b. Determine the number of pounds of glass and paper needed to produce the maximum profit.

c. What is the maximum profit?

CHAPTER 6 Practice Test

Write each system of equations in triangular form using Gaussian elimination. Then solve the system.

1. $-3x + y = 4$
$5x - 7y = 20$

2. $x + 4y - 3z = -8$
$5x - 7y + 3z = -4$
$3x - 2y + 4z = 24$

Solve the system of equations.

3. $5x - 6y = 28$
$6x + 5y = -3$

4. $2x - 4y + z = 8$
$3x + 3y + 4z = 20$
$5x + y - 3z = -13$

5. LIBRARY Kristen checked out books, CDs, and DVDs from the library. She checked out a total of 16 items. The total number of CDs and DVDs equaled the number of books. She checked out two more CDs than DVDs.

a. Let b = number of books, c = number of CDs, and d = number of DVDs. Write a system of three linear equations to represent the problem.

b. Solve the system of equations. Interpret your solution.

Find AB and BA, if possible.

6. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & -1 \\ -1 & 6 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & -5 & 4 \\ -2 & 3 & 5 \\ 6 & -3 & 1 \end{bmatrix}$, $B = [2 \quad -1 \quad -8]$

8. GEOMETRY The coordinates of a point (x, y) can be written as a 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

a. Let P be the point $(-3, 4)$. Discuss what effect multiplying A by P has on P.

b. A triangle contains vertices (0, 0), (2, 6), and (8, 3). Create B, a 2×3 matrix to represent the triangle. Find AB. What is the effect on the triangle? Does it agree with your answer to part a?

Find A^{-1}, if it exists. If A^{-1} does not exist, write *singular*.

9. $A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} -3 & -5 \\ -6 & 8 \end{bmatrix}$

Use an inverse matrix to solve each system of equations, if possible.

11. $2x - 3y = -7$
$5x + 2y = 11$

12. $2x + 2y + 5z = -6$
$2x - 3y + 7z = -7$
$x - 5y + 9z = 4$

Use Cramer's Rule to find the solution of each system of linear equations, if a unique solution exists.

13. $3x - 2y = -2$
$4x - 2y = 2$

14. $3x - 2y - 3z = -24$
$3x + 5y + 2z = 7$
$-x + 5y + 3z = 25$

Find the partial fraction decomposition of each rational expression.

15. $\dfrac{4x}{x^2 - 9}$

16. $\dfrac{2x + 10}{x^2 - 4x + 3}$

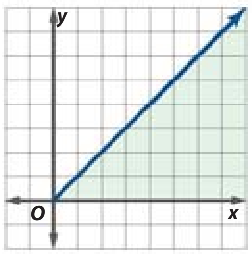
Find the maximum and minimum values of the objective function $f(x, y)$ and for what values of x and y they occur, subject to the given constraints.

17. $f(x, y) = 2x - y$
$x \geq 0$
$y \geq 0$
$y \geq -2x + 8$

18. $f(x, y) = -x + 2y$
$x - 3y \leq 0$
$x \geq 0$
$y \leq 9$

19. PRICING The Harvest Nut Company sells create-your-own trail mixes where customers can choose whatever combinations they want. Colin's favorite mix contains peanuts, dried cranberries, and carob-coated pretzels. The prices for each are shown below. If Colin bought a 5-pound mixture for \$16.80 that contained twice as many pounds of carob-coated pretzels as cranberries, how many pounds of each item did he buy?

20. MULTIPLE CHOICE The graph displays the constraints for an objective function. Which of the following CANNOT be one of the constraints?



A $y \geq 0$

B $x \geq 0$

C $x - y \leq 0$

D $x - y \geq 0$

CHAPTER 6

Connect to AP Calculus
Nonlinear Optimization

Objective

- Approximate solutions to nonlinear optimization problems.

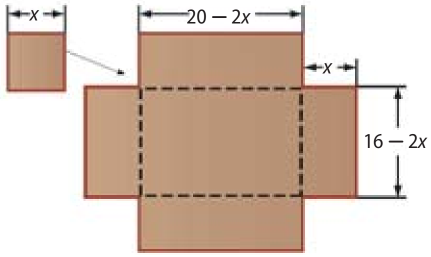
In Lesson 6-5, you learned how to solve optimization problems by using linear programming. The objective function and the system of constraints were represented by linear functions. Unfortunately, not all situations that require optimization can be defined by linear functions.

Advanced optimization problems involving quadratic, cubic, and other nonlinear functions require calculus to find exact solutions. However, we can find good approximations using graphing calculators.

Activity 1 Maximum Volume

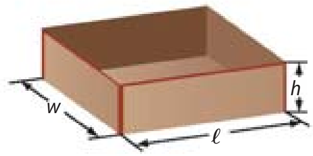
A 16-inch × 20-inch piece of cardboard is made into a box with no top by cutting congruent squares from each corner and folding the sides up. What are the dimensions of the box with the largest possible volume? What is the maximum volume?

Step 1 Sketch a diagram of the situation.



Step 2 Let x represent the side length of one of the squares that is to be removed. Write expressions for the length, width, and height of the box in terms of x.

Step 3 Find an equation for the volume of the box V in terms of x using the dimensions found in Step 2.



Step 4 Use a calculator to graph the equation from Step 3.

Analyze the Results

1. Describe the domain of x. Explain your reasoning.
2. Use your calculator to find the coordinates of the maximum point on your graph. Interpret the meaning of these coordinates.
3. What are the dimensions of the box with the largest possible volume? What is the maximum volume?

The desired outcome and complexity of each optimization problem differs. You can use the following steps to analyze and solve each problem.

KeyConcept Optimization

To solve an optimization problem, review these steps.

Step 1 Sketch a diagram of the situation and label all known and unknown quantities.

Step 2 Determine the quantity that needs to be maximized or minimized. Decide on the values necessary to find the desired quantity and represent each value with a number, a variable, or an expression.

Step 3 Write an equation for the quantity that is to be optimized in terms of one variable.

Step 4 Graph the equation and find either the maximum or minimum value. Determine the allowable domain of the variable.

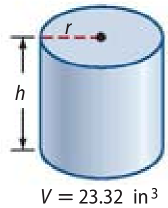
StudyTip

Cylinders Recall that an equation for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder.

Activity 2 Minimum Surface Area

A typical soda can is about 2.5 inches wide and 4.75 inches tall yielding a volume of about 23.32 cubic inches. What would be the dimensions of a soda can if you kept the volume constant but minimized the amount of material used to construct the can?

Step 1 Sketch a diagram of the situation.



Step 2 The quantity to be minimized is surface area. Values for the radius and height of the can are needed. Find an expression for the height h of the can in terms of the radius r using the given volume.

Step 3 Using the expression found in Step 2, write an equation for surface area SA.

Step 4 Use a calculator to graph the equation from Step 3. State the domain of r.

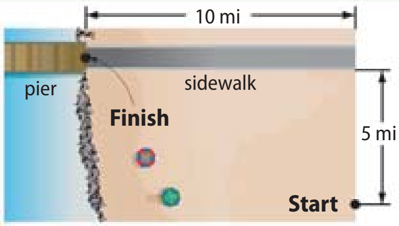
Analyze the Results

4. Find the coordinates of the minimum point. Interpret the meaning of these coordinates.
5. What are the dimensions and surface area of the can with the smallest possible surface area?
6. A right cylinder with no top is to be constructed with a surface area of 6π square inches. What height and radius will maximize the volume of the cylinder? What is the maximum volume?

Minimizing materials is not the only application of optimization.

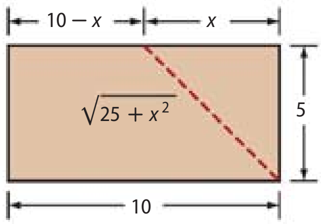
Activity 3 Quickest Path

Participants in a foot race travel over a beach or a sidewalk to a pier as shown. Racers can take any path they choose. If a racer can run 6 miles per hour on the sand and 7.5 miles per hour on the sidewalk, what path will require the shortest amount of time?



Step 1 Sketch a diagram of the situation.

Step 2 To minimize time, write expressions for the distances traveled on each surface at each rate. Let x represent the distance the runner does not run on the sidewalk as shown. Find expressions for the distances traveled on each surface in terms of x.



Step 3 Using the expressions found in Step 2, write an equation for time.

Step 4 Use a calculator to graph the equation from Step 3. State the domain of x.

Analyze the Results

7. Find the coordinates of the minimum point. Interpret the meaning of these coordinates.
8. What path will require the shortest amount of time? How long will it take?
9. Find the average rate of change m at the minimum point of your graph using the difference quotient. What does this value suggest about the line tangent to the graph at this point?
10. Make a conjecture about the rates of change and the tangent lines of graphs at minimum and maximum points. Does your conjecture hold true for the first two activities? Explain.

CHAPTER 7

Conic Sections and Parametric Equations

Then

In **Chapter 6**, you learned how to solve systems of linear equations using matrices.

Now

In **Chapter 7**, you will:

- Analyze, graph, and write equations of parabolas, circles, ellipses, and hyperbolas.
- Use equations to identify types of conic sections.
- Graph rotated conic sections.
- Solve problems related to the motion of projectiles.

Why? ▲

BASEBALL When a baseball is hit, the path of the ball can be represented and traced by parametric equations.

PREREAD Scan the Study Guide and Review and use it to make two or three predictions about what you will learn in Chapter 7.

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