CHAPTER 5

Trigonometric Identities and Equations

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Then

In **Chapter 4**, you learned to graph trigonometric functions and to solve right and oblique triangles.

Now

In **Chapter 5**, you will:

- Use and verify trigonometric identities.
- Solve trigonometric equations.
- Use sum and difference identities to evaluate trigonometric expressions and solve equations.
- Use double-angle, power-reducing, half-angle, and product-sum identities to evaluate trigonometric expressions and solve equations.

Why?

BUSINESS Musicians tune their instruments by listening for a *beat,* which is an interference between two sound waves with slightly different frequencies. The sum of the sound waves can be represented using a trigonometric equation.

PREREAD Using what you know about trigonometric functions, make a prediction of what you will learn in Chapter 5.

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Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

Solve each equation by factoring. (Lesson 0-3)

1. $x^2 + 5x - 24 = 0$

2. $x^2 - 11x + 28 = 0$

3. $2x^2 - 9x - 5 = 0$

4. $15x^2 + 26x + 8 = 0$

5. $2x^3 - 2x^2 - 12x = 0$

6. $12x^3 + 78x^2 - 42x = 0$

7. **ROCKETS** A rocket is projected vertically into the air. Its vertical distance in feet after t seconds is represented by $s(t) = -16t^2 + 192t$. Find the amount of time that the rocket is in the air. (Lesson 0-3)

Find the missing side lengths and angle measures of each triangle. (Lessons 4-1 and 4-7)

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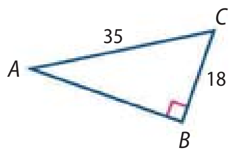

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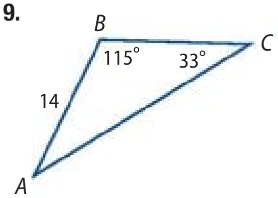

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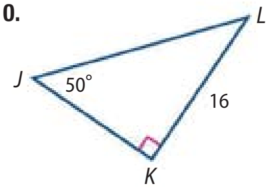

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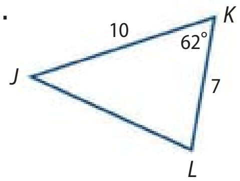


Find the exact value of each expression. (Lesson 4-3)

12. $\cot 420°$

13. $\cos \frac{7\pi}{4}$

14. $\sec \frac{10\pi}{3}$

15. $\tan 480°$

16. $\csc \frac{2\pi}{3}$

17. $\sin 510°$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

NewVocabulary

English		Español
trigonometric identity	p. 312	identidad trigonométrica
reciprocal identity	p. 312	identidad recíproca
quotient identity	p. 312	cociente de identidad
Pythagorean identity	p. 313	Pitágoras identidad
odd-even-identity	p. 314	impar-incluso-de identidad
cofunction	p. 314	co función
verify an identity	p. 320	verificar una identidad
sum identity	p. 337	suma de identidad
reduction identity	p. 340	identidad de reducción
double-angle identity	p. 346	doble-ángulo de la identidad
power-reducing identity	p. 347	poder-reducir la identidad
half-angle identity	p. 348	medio ángulo identidad

ReviewVocabulary

extraneous solution p. 91 solución extraña a solution that does not satisfy the original equation

quadrantal angle p. 243 ángulo cuadranta an angle θ in standard position that has a terminal side that lies on one of the coordinate axes

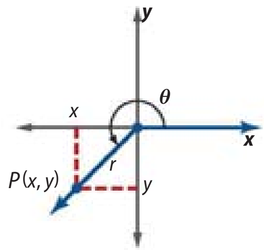
unit circle p. 247 círculo unitario a circle of radius 1 centered at the origin

periodic function p. 250 función periódica a function with range values that repeat at regular intervals

trigonometric functions p. 220 funciones trigonométricas Let θ be any angle in standard position and point $P(x, y)$ be a point on the terminal side of θ. Let r represent the nonzero distance from P to the origin or $|r| = \sqrt{x^2 + y^2} \neq 0$. Then the trigonometric functions of θ are as follows.

$\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$ $\tan \theta = \frac{y}{x}, x \neq 0$

$\csc \theta = \frac{r}{y}, y \neq 0$ $\sec \theta = \frac{r}{x}, x \neq 0$ $\cot \theta = \frac{x}{y}, y \neq 0$



LESSON 5-1

Trigonometric Identities

:: **Then**

- You found trigonometric values using the unit circle. (Lesson 4-3)

:: **Now**

1. Identify and use basic trigonometric identities to find trigonometric values.
2. Use basic trigonometric identities to simplify and rewrite trigonometric expressions.

:: **Why?**

- Many physics and engineering applications, such as determining the path of an aircraft, involve trigonometric functions. These functions are made more flexible if you can change the trigonometric expressions involved from one form to an equivalent but more convenient form. You can do this by using trigonometric identities.

NewVocabulary
identity
trigonometric identity
cofunction

1 Basic Trigonometric Identities

An equation is an **identity** if the left side is equal to the right side for all values of the variable for which both sides are defined. Consider the equations below.

$\frac{x^2 - 9}{x - 3} = x + 3$ — This *is* an identity since both sides of the equation are defined and equal for all x such that $x \neq 3$.

$\sin x = 1 - \cos x$ — This *is not* an identity. Both sides of this equation are defined and equal for certain values, such as when $x = 0$, but not for other values for which both sides are defined, such as when $x = \frac{\pi}{4}$.

Trigonometric identities are identities that involve trigonometric functions. You already know a few basic trigonometric identities. The reciprocal and quotient identities below follow directly from the definitions of the six trigonometric functions introduced in Lesson 4-1.

KeyConcept Reciprocal and Quotient Identities

Reciprocal Identities			Quotient Identities
$\sin \theta = \frac{1}{\csc \theta}$	$\cos \theta = \frac{1}{\sec \theta}$	$\tan \theta = \frac{1}{\cot \theta}$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$
$\csc \theta = \frac{1}{\sin \theta}$	$\sec \theta = \frac{1}{\cos \theta}$	$\cot \theta = \frac{1}{\tan \theta}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$

You can use these basic trigonometric identities to find trigonometric values. As with any fraction, the denominator cannot equal zero.

Example 1 Use Reciprocal and Quotient Identities

a. If $\csc \theta = \frac{7}{4}$, find $\sin \theta$.

$\sin \theta = \frac{1}{\csc \theta}$ — Reciprocal Identity

$= \frac{1}{\frac{7}{4}}$ — $\csc \theta = \frac{7}{4}$

$= \frac{4}{7}$ — Divide.

b. If $\cot x = \frac{2}{5\sqrt{5}}$ and $\sin x = \frac{\sqrt{5}}{3}$, find $\cos x$.

$\cot x = \frac{\cos x}{\sin x}$ — Quotient Identity

$\frac{2}{5\sqrt{5}} = \frac{\cos x}{\frac{\sqrt{5}}{3}}$ — $\cot x = \frac{2}{5\sqrt{5}}$; $\sin x = \frac{\sqrt{5}}{3}$

$\frac{2}{5\sqrt{5}} \cdot \frac{\sqrt{5}}{3} = \cos x$ — Multiply each side by $\frac{\sqrt{5}}{3}$.

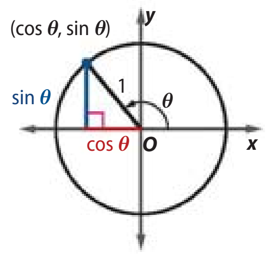
$\frac{2}{15} = \cos x$ — Simplify.

GuidedPractice

1A. If $\sec x = \frac{5}{3}$, find $\cos x$.

1B. If $\csc \beta = \frac{25}{7}$ and $\sec \beta = \frac{25}{24}$, find $\tan \beta$.

Recall from Lesson 4-3 that trigonometric functions can be defined on a unit circle as shown. Notice that for any angle θ, sine and cosine are the directed lengths of the legs of a right triangle with hypotenuse 1. We can apply the Pythagorean Theorem to this right triangle to establish another basic trigonometric identity.



$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$ Pythagorean Theorem

$\sin^2\theta + \cos^2\theta = 1$ Simplify.

While the signs of these directed lengths may change depending on the quadrant in which the triangle lies, notice that because these lengths are squared, the equation above holds true for any value of θ. This equation is one of three **Pythagorean identities**.

ReadingMath

Powers of Trigonometric Functions $\sin^2\theta$ is read as *sine squared theta* and interpreted as the square of the quantity $\sin\theta$.

KeyConcept Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$

You will prove the remaining two Pythagorean Identities in Exercises 69 and 70.

Notice the shorthand notation used to represent powers of trigonometric functions: $\sin^2\theta = (\sin\theta)^2$, $\cos^2\theta = (\cos\theta)^2$, $\tan^2\theta = (\tan\theta)^2$, and so on.

Example 2 Use Pythagorean Identities

If $\tan\theta = -8$ and $\sin\theta > 0$, find $\sin\theta$ and $\cos\theta$.

Use the Pythagorean Identity that involves $\tan\theta$.

$\tan^2\theta + 1 = \sec^2\theta$ Pythagorean Identity

$(-8)^2 + 1 = \sec^2\theta$ $\tan\theta = -8$

$65 = \sec^2\theta$ Simplify.

$\pm\sqrt{65} = \sec\theta$ Take the square root of each side.

$\pm\sqrt{65} = \frac{1}{\cos\theta}$ Reciprocal Identity

$\pm\frac{\sqrt{65}}{65} = \cos\theta$ Solve for $\cos\theta$.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is negative and $\sin\theta$ is positive, $\cos\theta$ must be negative. So, $\cos\theta = -\frac{\sqrt{65}}{65}$. You can then use this quotient identity again to find $\sin\theta$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ Quotient Identity

$-8 = \frac{\sin\theta}{-\frac{\sqrt{65}}{65}}$ $\tan\theta = -8$ and $\cos\theta = -\frac{\sqrt{65}}{65}$

$\frac{8\sqrt{65}}{65} = \sin\theta$ Multiply each side by $-\frac{\sqrt{65}}{65}$.

CHECK $\sin^2\theta + \cos^2\theta = 1$ Pythagorean Identity

$\left(\frac{8\sqrt{65}}{65}\right)^2 + \left(-\frac{\sqrt{65}}{65}\right)^2 \stackrel{?}{=} 1$ $\sin\theta = \frac{8\sqrt{65}}{65}$ and $\cos\theta = -\frac{\sqrt{65}}{65}$

$\frac{64}{65} + \frac{1}{65} = 1$ ✓   Simplify.

StudyTip

Checking Answers It is beneficial to confirm your answers using a different identity than the ones you used to solve the problem, as in Example 2, so that you do not make the same mistake twice.

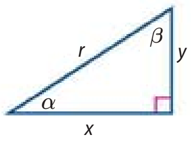
GuidedPractice

Find the value of each expression using the given information.

2A. $\csc\theta$ and $\tan\theta$; $\cot\theta = -3$, $\cos\theta < 0$

2B. $\cot x$ and $\sec x$; $\sin x = \frac{1}{6}$, $\cos x > 0$

Another set of basic trigonometric identities involve cofunctions. A trigonometric function f is a **cofunction** of another trigonometric function g if $f(\alpha) = g(\beta)$ when α and β are complementary angles. In the right triangle shown, angles α and β are complementary angles. Using the right triangle ratios, you can show that the following statements are true.



$$\sin \alpha = \cos \beta = \cos (90° - \alpha) = \frac{y}{r}$$

$$\tan \alpha = \cot \beta = \cot (90° - \alpha) = \frac{y}{x}$$

$$\sec \alpha = \csc \beta = \csc (90° - \alpha) = \frac{r}{y}$$

From these statements, we can write the following cofunction identities, which are valid for all real numbers, not just acute angle measures.

StudyTip

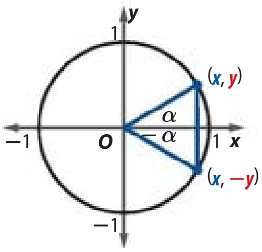
Writing Cofunction Identities Each of the cofunction identities can also be written in terms of degrees. For example, $\sin \theta = \cos (90° - \theta)$.

KeyConcept Cofunction Identities

$\sin \theta = \cos \left(\frac{\pi}{2} - \theta\right)$	$\tan \theta = \cot \left(\frac{\pi}{2} - \theta\right)$	$\sec \theta = \csc \left(\frac{\pi}{2} - \theta\right)$
$\cos \theta = \sin \left(\frac{\pi}{2} - \theta\right)$	$\cot \theta = \tan \left(\frac{\pi}{2} - \theta\right)$	$\csc \theta = \sec \left(\frac{\pi}{2} - \theta\right)$

You will prove these identities for any angle in Lesson 5-3.

You have also seen that each of the basic trigonometric functions—sine, cosine, tangent, cosecant, secant, and cotangent—is either odd or even. Using the unit circle, you can show that the following statements are true.



$\sin \alpha = y \qquad \sin (-\alpha) = -y$

$\cos \alpha = x \qquad \cos (-\alpha) = x$

Recall from Lesson 1-2 that a function f is even if for every x in the domain of f, $f(-x) = f(x)$ and odd if for every x in the domain of f, $f(-x) = -f(x)$. These relationships lead to the following odd-even identities.

KeyConcept Odd-Even Identities

$\sin (-\theta) = -\sin \theta$	$\cos (-\theta) = \cos \theta$	$\tan (-\theta) = -\tan \theta$
$\csc (-\theta) = -\csc \theta$	$\sec (-\theta) = \sec \theta$	$\cot (-\theta) = -\cot \theta$

You can use cofunction and odd-even identities to find trigonometric values.

Example 3 Use Cofunction and Odd-Even Identities

If $\tan \theta = 1.28$, find $\cot \left(\theta - \frac{\pi}{2}\right)$.

$\cot \left(\theta - \frac{\pi}{2}\right) = \cot \left[-\left(\frac{\pi}{2} - \theta\right)\right]$ Factor.

$= -\cot \left(\frac{\pi}{2} - \theta\right)$ Odd-Even Identity

$= -\tan \theta$ Cofunction Identity

$= -1.28$ $\tan \theta = 1.28$

GuidedPractice

3. If $\sin x = -0.37$, find $\cos \left(x - \frac{\pi}{2}\right)$.

2 Simplify and Rewrite Trigonometric Expressions

To simplify a trigonometric expression, start by rewriting it in terms of one trigonometric function or in terms of sine and cosine only.

Example 4 Simplify by Rewriting Using Only Sine and Cosine

Simplify $\csc \theta \sec \theta - \cot \theta$.

Solve Algebraically

$$\csc \theta \sec \theta - \cot \theta = \frac{1}{\sin \theta} \cdot \frac{1}{\cos \theta} - \frac{\cos \theta}{\sin \theta}$$ Rewrite in terms of sine and cosine using Reciprocal and Quotient Identities.

$$= \frac{1}{\sin \theta \cos \theta} - \frac{\cos \theta}{\sin \theta}$$ Multiply.

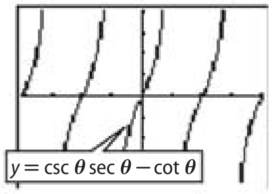
$$= \frac{1}{\sin \theta \cos \theta} - \frac{\cos^2 \theta}{\sin \theta \cos \theta}$$ Rewrite fractions using a common denominator.

$$= \frac{1 - \cos^2 \theta}{\sin \theta \cos \theta}$$ Subtract.

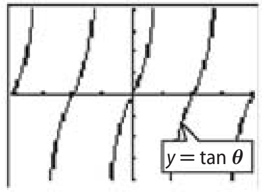
$$= \frac{\sin^2 \theta}{\sin \theta \cos \theta}$$ Pythagorean Identity

$$= \frac{\sin \theta}{\cos \theta} \text{ or } \tan \theta$$ Divide the numerator and denominator by $\sin \theta$.

Support Graphically The graphs of $y = \csc \theta \sec \theta - \cot \theta$ and $y = \tan \theta$ appear to be identical.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 2]$ scl: 0.5



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 2]$ scl: 0.5

Guided Practice

4. Simplify $\sec x - \tan x \sin x$.

Technology Tip

Graphing Reciprocal Functions When using a calculator to graph a reciprocal function, such as $y = \csc x$, you can enter the reciprocal of the function.

Some trigonometric expressions can be simplified by applying identities and factoring.

Example 5 Simplify by Factoring

Simplify $\sin^2 x \cos x - \sin\left(\frac{\pi}{2} - x\right)$.

Solve Algebraically

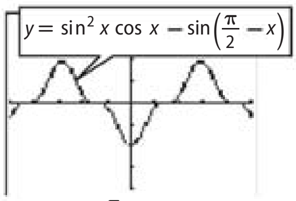
$$\sin^2 x \cos x - \sin\left(\frac{\pi}{2} - x\right) = \sin^2 x \cos x - \cos x$$ Cofunction Identity

$$= -\cos x\,(-\sin^2 x + 1)$$ Factor $-\cos x$ from each term.

$$= -\cos x\,(1 - \sin^2 x)$$ Commutative Property

$$= -\cos x\,(\cos^2 x) \text{ or } -\cos^3 x$$ Pythagorean Identity

Support Graphically The graphs below appear to be identical.



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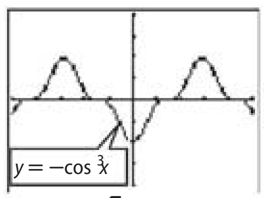


$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 2]$ scl: 0.5

Guided Practice

5. Simplify $-\csc\left(\frac{\pi}{2} - x\right) - \tan^2 x \sec x$.

Watch Out!

Graphing While the graphical approach shown in Examples 4 and 5 can lend support to the equality of two expressions, it cannot be used to prove that two expressions are equal. It is impossible to show that the graphs are identical over their entire domain using only the portion of the graph shown on your calculator.

You can simplify some trigonometric expressions by combining fractions.

Example 6 Simplify by Combining Fractions

Simplify $\frac{\sin x \cos x}{1 - \sin x} - \frac{1 + \sin x}{\cos x}$.

$$\frac{\sin x \cos x}{1 - \sin x} - \frac{1 + \sin x}{\cos x} = \frac{\sin x \cos x (\cos x)}{(1 - \sin x)(\cos x)} - \frac{(1 + \sin x)(1 - \sin x)}{(\cos x)(1 - \sin x)}$$ Common denominator

$$= \frac{\sin x \cos^2 x}{\cos x - \sin x \cos x} - \frac{1 - \sin^2 x}{\cos x - \sin x \cos x}$$ Multiply.

$$= \frac{\sin x \cos^2 x}{\cos x - \sin x \cos x} - \frac{\cos^2 x}{\cos x - \sin x \cos x}$$ Pythagorean Identity

$$= \frac{\sin x \cos^2 x - \cos^2 x}{\cos x - \sin x \cos x}$$ Subtract.

$$= \frac{(\cos^2 x)(\sin x - 1)}{(-\cos x)(\sin x - 1)}$$ Factor the numerator and denominator.

$$= -\cos x$$ Divide out common factors.

GuidedPractice

Simplify each expression.

6A. $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x}$

6B. $\frac{\csc x}{1 + \sec x} + \frac{\csc x}{1 - \sec x}$

In calculus, you will sometimes need to rewrite a trigonometric expression so it does not involve a fraction. When the denominator is of the form $1 \pm u$ or $u \pm 1$, you can sometimes do so by multiplying the numerator and denominator by the conjugate of the denominator and applying a Pythagorean identity.

ReviewVocabulary

conjugate a binomial factor which when multiplied by the original binomial factor has a product that is the difference of two squares (Lesson 0-3)

Example 7 Rewrite to Eliminate Fractions

Rewrite $\frac{1}{1 + \cos x}$ as an expression that does not involve a fraction.

$$\frac{1}{1 + \cos x} = \frac{1}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x}$$ Multiply numerator and denominator by the conjugate of $1 + \cos x$, which is $1 - \cos x$.

$$= \frac{1 - \cos x}{1 - \cos^2 x}$$ Multiply.

$$= \frac{1 - \cos x}{\sin^2 x}$$ Pythagorean Identity

$$= \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$ Write as the difference of two fractions.

$$= \frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$ Factor.

$$= \csc^2 x - \cot x \csc x$$ Reciprocal and Quotient Identities

GuidedPractice

Rewrite as an expression that does not involve a fraction.

7A. $\frac{\cos^2 x}{1 - \sin x}$

7B. $\frac{4}{\sec x + \tan x}$

Exercises

= Step-by-Step Solutions begin on page R29.

Find the value of each expression using the given information. (Example 1)

1. If $\cot \theta = \frac{5}{7}$, find $\tan \theta$.

2. If $\cos x = \frac{2}{3}$, find $\sec x$.

3. If $\tan \alpha = \frac{1}{5}$, find $\cot \alpha$.

4. If $\sin \beta = -\frac{5}{6}$, find $\csc \beta$.

5. If $\cos x = \frac{1}{6}$ and $\sin x = \frac{\sqrt{35}}{6}$, find $\cot x$.

6. If $\sec \varphi = 2$ and $\tan \varphi = \sqrt{3}$, find $\sin \varphi$.

7. If $\csc \alpha = \frac{7}{3}$ and $\cot \alpha = \frac{2\sqrt{10}}{3}$, find $\sec \alpha$.

8. If $\sec \theta = 8$ and $\tan \theta = 3\sqrt{7}$, find $\csc \theta$.

Find the value of each expression using the given information. (Example 2)

9. $\sec \theta$ and $\cos \theta$; $\tan \theta = -5$, $\cos \theta > 0$

10. $\cot \theta$ and $\sec \theta$; $\sin \theta = \frac{1}{3}$, $\tan \theta < 0$

11. $\tan \theta$ and $\sin \theta$; $\sec \theta = 4$, $\sin \theta > 0$

12. $\sin \theta$ and $\cot \theta$; $\cos \theta = \frac{2}{5}$, $\sin \theta < 0$

13. $\cos \theta$ and $\tan \theta$; $\csc \theta = \frac{8}{3}$, $\tan \theta > 0$

14. $\sin \theta$ and $\cos \theta$; $\cot \theta = 8$, $\csc \theta < 0$

15. $\cot \theta$ and $\sin \theta$; $\sec \theta = -\frac{9}{2}$, $\sin \theta > 0$

16. $\tan \theta$ and $\csc \theta$; $\cos \theta = -\frac{1}{4}$, $\sin \theta < 0$

Find the value of each expression using the given information. (Example 3)

17. If $\csc \theta = -1.24$, find $\sec\left(\theta - \frac{\pi}{2}\right)$.

18. If $\cos x = 0.61$, find $\sin\left(x - \frac{\pi}{2}\right)$.

19. If $\tan \theta = -1.52$, find $\cot\left(\theta - \frac{\pi}{2}\right)$.

20. If $\sin \theta = 0.18$, find $\cos\left(\theta - \frac{\pi}{2}\right)$.

21. If $\cot x = 1.35$, find $\tan\left(x - \frac{\pi}{2}\right)$.

Simplify each expression. (Examples 4 and 5)

22. $\csc x \sec x - \tan x$

23. $\csc x - \cos x \cot x$

24. $\sec x \cot x - \sin x$

25. $\frac{\tan x + \sin x \sec x}{\csc x \tan x}$

26. $\frac{1 - \sin^2 x}{\csc^2 x - 1}$

27. $\frac{\csc x \cos x + \cot x}{\sec x \cot x}$

28. $\frac{\sec x \csc x - \tan x}{\sec x \csc x}$

29. $\frac{\sec^2 x}{\cot^2 x + 1}$

30. $\cot x - \csc^2 x \cot x$

31. $\cot x - \cos^3 x \csc x$

Simplify each expression. (Example 6)

32. $\frac{\cos x}{\sec x + 1} + \frac{\cos x}{\sec x - 1}$

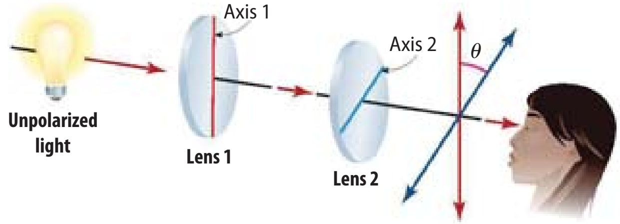
33. $\frac{1 - \cos x}{\tan x} + \frac{\sin x}{1 + \cos x}$

34. $\frac{1}{\sec x + 1} + \frac{1}{\sec x - 1}$

35. $\frac{\cos x \cot x}{\sec x + \tan x} + \frac{\sin x}{\sec x - \tan x}$

36. $\frac{\sin x}{\csc x + 1} + \frac{\sin x}{\csc x - 1}$

37. **SUNGLASSES** Many sunglasses are made with polarized lenses, which reduce the intensity of light. The intensity of light emerging from a system of two polarizing lenses I can be calculated by $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where I_0 is the intensity of light entering the system of lenses and θ is the angle of the axis of the second lens in relation to that of the first lens. (Example 6)



a. Simplify the formula for the intensity of light emerging from a system of two polarized lenses.

b. If a pair of sunglasses contains a system of two polarizing lenses with axes at 30° to one another, what proportion of the intensity of light entering the sunglasses emerges?

Rewrite as an expression that does not involve a fraction. (Example 7)

38. $\frac{\sin x}{\csc x - \cot x}$

39. $\frac{\csc x}{1 - \sin x}$

40. $\frac{\cot x}{\sec x - \tan x}$

41. $\frac{\cot x}{1 + \sin x}$

42. $\frac{3 \tan x}{1 - \cos x}$

43. $\frac{2 \sin x}{\cot x + \csc x}$

44. $\frac{\sin x}{1 - \sec x}$

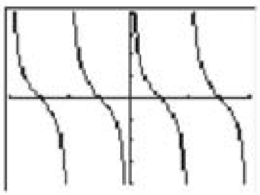
45. $\frac{\cot^2 x \cos x}{\csc x - 1}$

46. $\frac{5}{\sec x + 1}$

47. $\frac{\sin x \tan x}{\cos x + 1}$

Determine whether each parent trigonometric function shown is odd or even. Explain your reasoning.

48.



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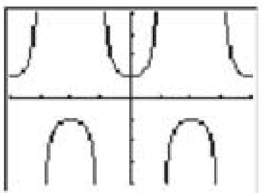
49.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

50. **SOCCER** When a soccer ball is kicked from the ground, its height y and horizontal displacement x are related by $y = \frac{-gx^2}{2{v_0}^2 \cos^2 \theta} + \frac{x \sin \theta}{\cos \theta}$, where v_0 is the initial velocity of the ball, θ is the angle at which it was kicked, and g is the acceleration due to gravity. Rewrite this equation so that $\tan \theta$ is the only trigonometric function that appears in the equation.

Write each expression in terms of a single trigonometric function.

51. $\tan x - \csc x \sec x$

52. $\cos x + \tan x \sin x$

53. $\csc x \tan^2 x - \sec^2 x \csc x$

54. $\sec x \csc x - \cos x \csc x$

55. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the verification of trigonometric identities. Consider the functions shown.

i. $y_1 = \tan x + 1$
$y_2 = \sec x \cos x - \sin x \sec x$

ii. $y_3 = \tan x \sec x - \sin x$
$y_4 = \sin x \tan^2 x$

a. **TABULAR** Copy and complete the table below, without graphing the functions.

x	-2π	$-\pi$	0	π	2π
y_1					
y_2					
y_3					
y_4					

b. **GRAPHICAL** Graph each function on a graphing calculator.

c. **VERBAL** Make a conjecture about the relationship between y_1 and y_2. Repeat for y_3 and y_4.

d. **ANALYTICAL** Are the conjectures that you made in part **c** valid for the entire domain of each function? Explain your reasoning.

Rewrite each expression as a single logarithm and simplify the answer.

56. $\ln |\sin x| - \ln |\cos x|$

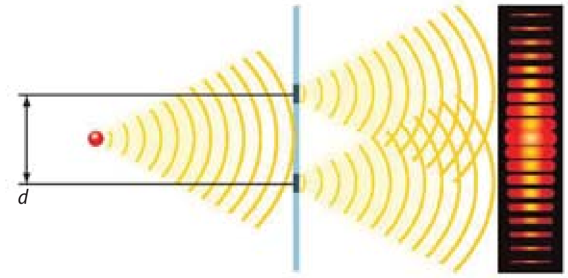
57. $\ln |\sec x| - \ln |\cos x|$

58. $\ln (\cot^2 x + 1) + \ln |\sec x|$

59. $\ln (\sec^2 x - \tan^2 x) - \ln (1 - \cos^2 x)$

60. **ELECTRICITY** A current in a wire in a magnetic field causes a force to act on the wire. The strength of the magnetic field can be determined using the formula $B = \frac{F \csc \theta}{I\ell}$, where F is the force on the wire, I is the current in the wire, ℓ is the length of the wire, and θ is the angle the wire makes with the magnetic field. Some physics books give the formula as $F = I\ell B \sin \theta$. Show that the two formulas are equivalent.

61. **LIGHT WAVES** When light shines through two narrow slits, a series of light and dark fringes appear. The angle θ, in radians, locating the mth fringe can be calculated by $\sin \theta = \frac{m\lambda}{d}$, where d is the distance between the two slits, and λ is the wavelength of light.

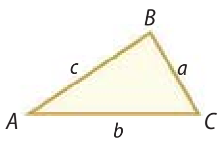


a. Rewrite the formula in terms of $\csc \theta$.

b. Determine the angle locating the 100th fringe when light having a wavelength of 550 nanometers is shined through double slits spaced 0.5 millimeters apart.

H.O.T. Problems Use Higher-Order Thinking Skills

62. **PROOF** Prove that the area of the triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$. (*Hint*: The area of an oblique triangle is $A = \frac{1}{2} bc \sin A$.)



63. **ERROR ANALYSIS** Jenelle and Chloe are simplifying $\frac{1 - \sin^2 x}{\sin^2 x - \cos^2 x}$. Jenelle thinks that the expression simplifies to $\frac{\cos^2 x}{1 - 2\cos^2 x}$, and Chloe thinks that it simplifies to $\csc^2 x - \tan^2 x$. Is either of them correct? Explain your reasoning.

CHALLENGE **Write each of the basic trigonometric functions in terms of the following functions.**

64. $\sin x$ **65.** $\cos x$ **66.** $\tan x$

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

67. $\csc^2 x \tan x = \csc x \sec x$ is true for all real numbers.

68. The odd-even identities can be used to prove that the graphs of $y = \cos x$ and $y = \sec x$ are symmetric with respect to the y-axis.

PROOF **Prove each Pythagorean identity.**

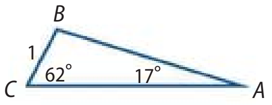
69 $\tan^2 \theta + 1 = \sec^2 \theta$ **70.** $\cot^2 \theta + 1 = \csc^2 \theta$

71. **PREWRITE** Use a chart or a table to help you organize the major trigonometric identities found in Lesson 5-1.

Spiral Review

Solve each triangle. Round to the nearest tenth, if necessary. (Lesson 4-7)

72.



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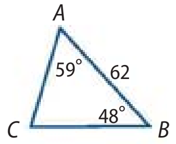

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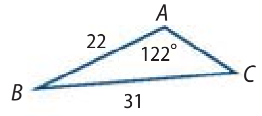

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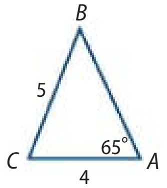

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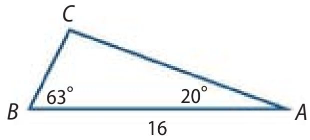

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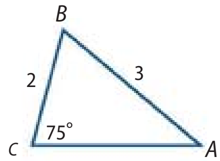


Find the exact value of each expression, if it exists. (Lesson 4-6)

78. $\cot\left(\sin^{-1}\frac{7}{9}\right)$

79. $\tan(\arctan 3)$

80. $\cos\left[\arccos\left(-\frac{1}{2}\right)\right]$

81. $\cos\left(\frac{\pi}{2} - \cos^{-1}\frac{\sqrt{2}}{2}\right)$

82. $\cos^{-1}\left(\sin^{-1}\frac{\pi}{2}\right)$

83. $\sin\left(\cos^{-1}\frac{3}{5}\right)$

84. **ANTHROPOLOGY** *Allometry* is the study of the relationship between the size of an organism and the size of any of its parts. A researcher decided to test for an allometry between the size of the human head compared to the human body as a person ages. The data in the table represent the average American male. (Lesson 3-5)

a. Find a quadratic model relating these data by linearizing the data and finding the linear regression equation.

b. Use the model for the linearized data to find a model for the original data.

c. Use your model to predict the height of an American male whose head circumference is 24 inches.

Growth of the Average American Male (0–3 years of age)

Head Circumference (in.)	Height (in.)
14.1	19.5
18.0	26.4
18.3	29.7
18.7	32.3
19.1	34.4
19.4	36.2
19.6	37.7

Source: National Center for Health Statistics

Let $U = \{0, 1, 2, 3, 4, 5\}$, $A = \{6, 9\}$, $B = \{6, 9, 10\}$, $C = \{0, 1, 6, 9, 11\}$, $D = \{2, 5, 11\}$. Determine whether each statement is *true* or *false*. Explain your reasoning. (Lesson 0-1)

85. $A \subset B$

86. $D \subset U$

Skills Review for Standardized Tests

87. **SAT/ACT** If $x > 0$, then

$$\frac{x^2 - 1}{x + 1} + \frac{(x + 1)^2 - 1}{x + 2} + \frac{(x + 2)^2 - 1}{x + 3} =$$

A $(x + 1)^2$

B $(x - 1)^2$

C $3x - 1$

D $3x$

E $3(x - 1)^2$

88. **REVIEW** If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$

F $\frac{1}{m^2}$

G $\frac{1 - m^2}{m}$

H $\frac{m\sqrt{1 - m^2}}{1 - m^2}$

J $\frac{m}{1 - m^2}$

89. Which of the following is equivalent to $\frac{1 - \sin^2\theta}{1 - \cos^2\theta} \cdot \tan\theta$?

A $\tan\theta$

B $\cot\theta$

C $\sin\theta$

D $\cos\theta$

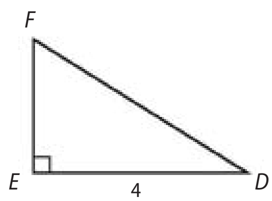
90. **REVIEW** Refer to the figure. If $\cos D = 0.8$, what is the length of $\overline{DF}$?

F 5

G 4

H 3.2

J $\frac{4}{5}$



LESSON 5-2 Verifying Trigonometric Identities

Then

- You simplified trigonometric expressions. (Lesson 5-1)

Now

- **1** Verify trigonometric identities.
- **2** Determine whether equations are identities.

Why?

- Two fireworks travel at the same speed v. The fireworks technician wants to explode one firework higher than another by adjusting the angle θ of the path each rocket makes with the ground. To calculate the maximum height h of each rocket, the formula $h = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta}$ could be used, but would $h = \frac{v^2 \sin^2 \theta}{2g}$ give the same result?

Steve Allen/Getty Images

NewVocabulary

verify an identity

1 Verify Trigonometric Identities

In Lesson 5-1, you used trigonometric identities to rewrite expressions in equivalent and sometimes more useful forms. Once verified, these *new* identities can also be used to solve problems or to rewrite other trigonometric expressions.

To **verify an identity** means to *prove* that both sides of the equation are equal for all values of the variable for which both sides are defined. This is done by transforming the expression on one side of the identity into the expression on the other side through a sequence of intermediate expressions that are each equivalent to the first. As with other types of proofs, each step is justified by a reason, usually another verified trigonometric identity or an algebraic operation.

You will find that it is often easier to start the verification of a trigonometric identity by beginning on the side with the more complicated expression and working toward the less complicated expression.

Example 1 Verify a Trigonometric Identity

Verify that $\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$.

The left-hand side of this identity is more complicated, so start with that expression first.

$\frac{\csc^2 x - 1}{\csc^2 x} = \frac{\cot^2 x}{\csc^2 x}$	Pythagorean Identity
$= \cot^2 x \sin^2 x$	Reciprocal Identity
$= \left(\frac{\cos^2 x}{\sin^2 x}\right)\sin^2 x$	Quotient Identity
$= \cos^2 x$ ✓	Pythagorean Identity

Notice that the verification ends with the expression on the other side of the identity.

GuidedPractice

Verify each identity.

1A. $\sec^2 \theta \cot^2 \theta - 1 = \cot^2 \theta$

1B. $\tan^2 \alpha = \sec \alpha \csc \alpha \tan \alpha - 1$

There is usually more than one way to verify an identity. For example, the identity in Example 1 can also be verified as follows.

$\frac{\csc^2 x - 1}{\csc^2 x} = \frac{\csc^2 x}{\csc^2 x} - \frac{1}{\csc^2 x}$	Write as the difference of two fractions.
$= 1 - \sin^2 x$	Simplify and apply a Reciprocal Identity.
$= \cos^2 x$	Pythagorean Identity

When there are multiple fractions with different denominators in an expression, you can find a common denominator to reduce the expression to one fraction.

Example 2 Verify a Trigonometric Identity by Combining Fractions

Verify that $2 \csc x = \dfrac{1}{\csc x + \cot x} + \dfrac{1}{\csc x - \cot x}$.

The right-hand side of this identity is more complicated, so start there, rewriting each fraction using the common denominator $(\csc x + \cot x)(\csc x - \cot x)$.

$\dfrac{1}{\csc x + \cot x} + \dfrac{1}{\csc x - \cot x}$ — Start with the right-hand side of the identity.

$= \dfrac{\csc x - \cot x}{(\csc x + \cot x)(\csc x - \cot x)} + \dfrac{\csc x + \cot x}{(\csc x + \cot x)(\csc x - \cot x)}$ — Common denominator

$= \dfrac{2 \csc x}{(\csc x + \cot x)(\csc x - \cot x)}$ — Add.

$= \dfrac{2 \csc x}{\csc^2 x - \cot^2 x}$ — Multiply.

$= 2 \csc x$ ✓ — Pythagorean Identity

GuidedPractice

2. Verify that $\dfrac{\cos \alpha}{1 + \sin \alpha} + \dfrac{\cos \alpha}{1 - \sin \alpha} = 2 \sec \alpha$.

To eliminate a fraction in which the denominator is of the form $1 \pm u$ or $u \pm 1$, remember to try multiplying the numerator and denominator by the conjugate of the denominator. Then you can potentially apply a Pythagorean Identity.

Example 3 Verify a Trigonometric Identity by Multiplying

Verify that $\dfrac{\sin \alpha}{1 - \cos \alpha} = \csc \alpha + \cot \alpha$.

Because the left-hand side of this identity involves a fraction, it is slightly more complicated than the right side. So, start with the left side.

$\dfrac{\sin \alpha}{1 - \cos \alpha} = \dfrac{\sin \alpha}{1 - \cos \alpha} \cdot \dfrac{1 + \cos \alpha}{1 + \cos \alpha}$ — Multiply numerator and denominator by the conjugate of $1 - \cos \alpha$, which is $1 + \cos \alpha$.

$= \dfrac{\sin \alpha (1 + \cos \alpha)}{1 - \cos^2 \alpha}$ — Multiply.

$= \dfrac{\sin \alpha (1 + \cos \alpha)}{\sin^2 \alpha}$ — Pythagorean Identity

$= \dfrac{1 + \cos \alpha}{\sin \alpha}$ — Divide out the common factor of $\sin \alpha$.

$= \dfrac{1}{\sin \alpha} + \dfrac{\cos \alpha}{\sin \alpha}$ — Write as the sum of two fractions.

$= \csc \alpha + \cot \alpha$ ✓ — Reciprocal and Quotient Identities

StudyTip

Alternate Method You do not always have to start with the more complicated side of the equation. If you start with the right-hand side in Example 3, you can still prove the identity.

$\csc \alpha + \cot \alpha$

$= \dfrac{1}{\sin \alpha} + \dfrac{\cos \alpha}{\sin \alpha}$

$= \dfrac{1 + \cos \alpha}{\sin \alpha}$

$= \dfrac{1 + \cos \alpha}{\sin \alpha} \cdot \dfrac{1 - \cos \alpha}{1 - \cos \alpha}$

$= \dfrac{\sin \alpha}{1 - \cos \alpha}$ ✓

GuidedPractice

3. Verify that $\dfrac{\tan x}{\sec x + 1} = \csc x - \cot x$.

Until an identity has been verified, you cannot assume that both sides of the equation are equal. Therefore, you cannot use the properties of equality to perform algebraic operations on each side of an identity, such as adding the same quantity to each side of the equation.

When the more complicated expression in an identity involves powers, try factoring.

Example 4 Verify a Trigonometric Identity by Factoring

Verify that $\cot \theta \sec \theta \csc^2 \theta - \cot^3 \theta \sec \theta = \csc \theta$.

$\cot \theta \sec \theta \csc^2 \theta - \cot^3 \theta \sec \theta$ — Start with the left-hand side of the identity.

$= \cot \theta \sec \theta (\csc^2 \theta - \cot^2 \theta)$ — Factor.

$= \cot \theta \sec \theta$ — Pythagorean Identity

$= \dfrac{\cos \theta}{\sin \theta} \cdot \dfrac{1}{\cos \theta}$ — Reciprocal and Quotient Identities

$= \dfrac{1}{\sin \theta}$ — Multiply.

$= \csc \theta$ ✓ — Reciprocal Identity

GuidedPractice

4. Verify that $\sin^2 x \tan^2 x \csc^2 x + \cos^2 x \tan^2 x \csc^2 x = \sec^2 x$.

It is sometimes helpful to work each side of an identity separately to obtain a common intermediate expression.

StudyTip

Additional Steps When verifying an identity, the number of steps that are needed to justify the verification may be obvious. However, if it is unclear, it is usually safer to include too many steps, rather than too few.

Example 5 Verify an Identity by Working Each Side Separately

Verify that $\dfrac{\tan^2 x}{1 + \sec x} = \dfrac{1 - \cos x}{\cos x}$.

Both sides look complicated, but the left-hand side is slightly more complicated since its denominator involves two terms. So, start with the expression on the left.

$\dfrac{\tan^2 x}{1 + \sec x} = \dfrac{\sec^2 x - 1}{1 + \sec x}$ — Pythagorean Identity

$= \dfrac{(\sec x - 1)(\sec x + 1)}{1 + \sec x}$ — Factor.

$= \sec x - 1$ — Divide out common factor of $\sec x + 1$.

From here, it is unclear how to transform $\sec x - 1$ into $\dfrac{1 - \cos x}{\cos x}$, so start with the right-hand side and work to transform it into the intermediate form $\sec x - 1$.

$\dfrac{1 - \cos x}{\cos x} = \dfrac{1}{\cos x} - \dfrac{\cos x}{\cos x}$ — Write as the difference of two fractions.

$= \sec x - 1$ — Use the Quotient Identity and simplify.

To complete the proof, work backward to connect the two parts of the proof.

$\dfrac{\tan^2 x}{1 + \sec x} = \dfrac{\sec^2 x - 1}{1 + \sec x}$ — Pythagorean Identity

$= \dfrac{(\sec x - 1)(\sec x + 1)}{1 + \sec x}$ — Factor.

$= \sec x - 1$ — Divide out common factor of $\sec x + 1$.

$= \dfrac{1}{\cos x} - \dfrac{\cos x}{\cos x}$ — Use the Quotient Identity and write 1 as $\dfrac{\cos x}{\cos x}$.

$= \dfrac{1 - \cos x}{\cos x}$ ✓ — Combine fractions.

GuidedPractice

5. Verify that $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$.

ConceptSummary Strategies for Verifying Trigonometric Identities

- Start with the more complicated side of the identity and work to transform it into the simpler side, keeping the other side of the identity in mind as your goal.
- Use reciprocal, quotient, Pythagorean, and other basic trigonometric identities.
- Use algebraic operations such as combining fractions, rewriting fractions as sums or differences, multiplying expressions, or factoring expressions.
- Convert a denominator of the form $1 \pm u$ or $u \pm 1$ to a single term using its conjugate and a Pythagorean Identity.
- Work each side separately to reach a common intermediate expression.
- If no other strategy presents itself, try converting the entire expression to one involving only sines and cosines.

2 Identifying Identities and Nonidentities

You can use a graphing calculator to investigate whether an equation may be an identity by graphing the functions related to each side of the equation.

WatchOut!

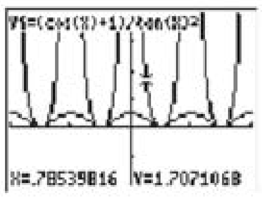
Using a Graph You can use a graphing calculator to help confirm a nonidentity, but you cannot use a graphing calculator to *prove* that an equation is an identity. You must provide algebraic verification of an identity.

Example 6 Determine Whether an Equation is an Identity

Use a graphing calculator to test whether each equation is an identity. If it appears to be an identity, verify it. If not, find a value for which both sides are defined but not equal.

a. $\dfrac{\cos\beta + 1}{\tan^2\beta} = \dfrac{\cos\beta}{\sec\beta + 1}$

The graphs of the related functions do not coincide for all values of x for which the both functions are defined. When $x = \frac{\pi}{4}$, Y1 ≈ 1.7 but Y2 ≈ 0.3. The equation is not an identity.

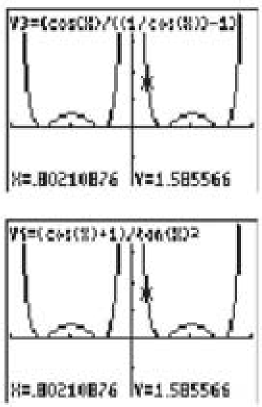


[−2π, 2π] scl: π by [−1, 3] scl: 1

[−2π, 2π] scl: π by [−1, 3] scl: 1

b. $\dfrac{\cos\beta + 1}{\tan^2\beta} = \dfrac{\cos\beta}{\sec\beta - 1}$

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.



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$\dfrac{\cos\beta}{\sec\beta - 1} = \dfrac{\cos\beta}{\sec\beta - 1} \cdot \dfrac{\sec\beta + 1}{\sec\beta + 1}$ Multiply numerator and denominator by the conjugate of $\sec\beta - 1$.

$= \dfrac{\cos\beta\sec\beta + \cos\beta}{\sec^2\beta - 1}$ Multiply.

$= \dfrac{\cos\beta\left(\frac{1}{\cos\beta}\right) + \cos\beta}{\sec^2\beta - 1}$ Reciprocal Identity

$= \dfrac{1 + \cos\beta}{\sec^2\beta - 1}$ Simplify.

$= \dfrac{\cos\beta + 1}{\tan^2\beta}$ ✓ Commutative Property and Pythagorean Identity

GuidedPractice

6A. $\csc\theta = \dfrac{\cot\theta\tan^2\theta + \cot\theta}{\sec\theta}$

6B. $\dfrac{\cos x + 1}{\sec^2 x} = \dfrac{\cos x}{\sec x - 1}$

Exercises

= Step-by-Step Solutions begin on page R29.

Verify each identity. (Examples 1–3)

1. $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

2. $\sec^2\theta(1-\cos^2\theta) = \tan^2\theta$

3. $\sin\theta - \sin\theta\cos^2\theta = \sin^3\theta$

4. $\csc\theta - \cos\theta\cot\theta = \sin\theta$

5. $\cot^2\theta\csc^2\theta - \cot^2\theta = \cot^4\theta$

6. $\tan\theta\csc^2\theta - \tan\theta = \cot\theta$

7. $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$

8. $\dfrac{\sin\theta}{1-\cos\theta} + \dfrac{1-\cos\theta}{\sin\theta} = 2\csc\theta$

9. $\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

10. $\dfrac{\sin\theta}{1-\cot\theta} + \dfrac{\cos\theta}{1-\tan\theta} = \sin\theta + \cos\theta$

11. $\dfrac{1}{1-\tan^2\theta} + \dfrac{1}{1-\cot^2\theta} = 1$

12. $\dfrac{1}{\csc\theta+1} + \dfrac{1}{\csc\theta-1} = 2\sec^2\theta\sin\theta$

13. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = 1$

14. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

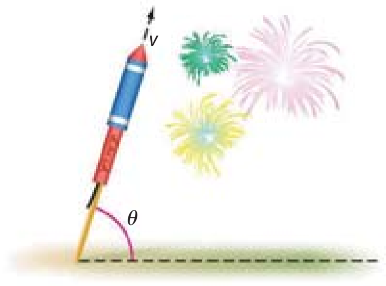
15. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

16. $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

17. $\csc^4\theta - \cot^4\theta = 2\cot^2\theta + 1$

18. $\dfrac{\csc^2\theta + 2\csc\theta - 3}{\csc^2\theta - 1} = \dfrac{\csc\theta+3}{\csc\theta+1}$

19. **FIREWORKS** If a rocket is launched from ground level, the maximum height that it reaches is given by $h = \dfrac{v^2\sin^2\theta}{2g}$, where θ is the angle between the ground and the initial path of the rocket, v is the rocket's initial speed, and g is the acceleration due to gravity, 9.8 meters per second squared. (Example 3)



a. Verify that $\dfrac{v^2\sin^2\theta}{2g} = \dfrac{v^2\tan^2\theta}{2g\sec^2\theta}$.

b. Suppose a second rocket is fired at an angle of 80° from the ground with an initial speed of 110 meters per second. Find the maximum height of the rocket.

Verify each identity. (Examples 4 and 5)

20. $(\csc\theta + \cot\theta)(1-\cos\theta) = \sin\theta$

21. $\sin^2\theta\tan^2\theta = \tan^2\theta - \sin^2\theta$

22. $\dfrac{1-\tan^2\theta}{1-\cot^2\theta} = \dfrac{\cos^2\theta - 1}{\cos^2\theta}$

23. $\dfrac{1+\csc\theta}{\sec\theta} = \cos\theta + \cot\theta$

24. $(\csc\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

25. $\dfrac{1+\tan^2\theta}{1-\tan^2\theta} = \dfrac{1}{2\cos^2\theta - 1}$

26. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$

27. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

28. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta$

29. $(\csc\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

30. $\dfrac{1+\tan\theta}{\sin\theta+\cos\theta} = \sec\theta$

31. $\dfrac{2+\csc\theta\sec\theta}{\csc\theta\sec\theta} = (\sin\theta + \cos\theta)^2$

32. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using $z = 2p\cos\theta$, where z is the combined power of the prisms, p is the power of the individual prisms, and θ is the angle between the two prisms. Verify that $2p\cos\theta = 2p(1-\sin^2\theta)\sec\theta$. (Example 4)

33. **PHOTOGRAPHY** The amount of light passing through a polarization filter can be modeled using $I = I_m\cos^2\theta$, where I is the amount of light passing through the filter, I_m is the amount of light shined on the filter, and θ is the angle of rotation between the light source and the filter. Verify that $I_m\cos^2\theta = I_m - \dfrac{I_m}{\cot^2\theta + 1}$. (Example 4)

GRAPHING CALCULATOR **Test whether each equation is an identity by graphing. If it appears to be an identity, verify it. If not, find a value for which both sides are defined but not equal.** (Example 6)

34. $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{1+\cot x}{1-\cot x}$

35. $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$

36. $\sec^2 x - 2\sec x\tan x + \tan^2 x = \dfrac{1-\cos x}{1+\cos x}$

37. $\dfrac{\cot^2 x - 1}{1+\cot^2 x} = 1 - 2\sin^2 x$

38. $\dfrac{\tan x - \sec x}{\tan x + \sec x} = \dfrac{\tan^2 x - 1}{\sec^2 x}$

39. $\cos^2 x - \sin^2 x = \dfrac{\cot x - \tan x}{\tan x + \cot x}$

Verify each identity.

40. $\sqrt{\frac{\sin x \tan x}{\sec x}} = |\sin x|$

41. $\sqrt{\frac{\sec x - 1}{\sec x + 1}} = \left|\frac{\sec x - 1}{\tan x}\right|$

42. $\ln|\csc x + \cot x| + \ln|\csc x - \cot x| = 0$

43. $\ln|\cot x| + \ln|\tan x \cos x| = \ln|\cos x|$

Verify each identity.

44. $\sec^2\theta + \tan^2\theta = \sec^4\theta - \tan^4\theta$

45. $-2\cos^2\theta = \sin^4\theta - \cos^4\theta - 1$

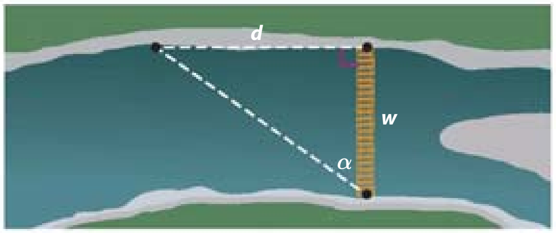
46. $\sec^2\theta\sin^2\theta = \sec^4\theta - (\tan^4\theta + \sec^2\theta)$

47. $3\sec^2\theta\tan^2\theta + 1 = \sec^6\theta - \tan^6\theta$

48. $\sec^4 x = 1 + 2\tan^2 x + \tan^4 x$

49. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$

50. **ENVIRONMENT** A biologist studying pollution situates a net across a river and positions instruments at two different stations on the river bank to collect samples. In the diagram shown, d is the distance between the stations and w is width of the river.



a. Determine an equation in terms of tangent α that can be used to find the distance between the stations.

b. Verify that $d = \frac{w\cos(90° - \alpha)}{\cos\alpha}$.

c. Complete the table shown for $d = 40$ feet.

w	20	40	60	80	100	120
α						

d. If $\alpha > 60°$ or $\alpha < 20°$, the instruments will not function properly. Use the table from part **c** to determine whether sites in which the width of the river is 5, 35, or 140 feet could be used for the experiment.

HYPERBOLIC FUNCTIONS **The *hyperbolic trigonometric functions* are defined in the following ways.**

$\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\quad$ $\operatorname{csch} x = \frac{1}{\sinh x}, x \neq 0$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ $\quad$ $\operatorname{sech} x = \frac{1}{\cosh x}$

$\tanh x = \frac{\sinh x}{\cosh x}$ $\quad$ $\coth x = \frac{1}{\tanh x}, x \neq 0$

Verify each identity using the functions shown above.

51. $\cosh^2 x - \sinh^2 x = 1$

52. $\sinh(-x) = -\sinh x$

53. $\operatorname{sech}^2 x = 1 - \tanh^2 x$

54. $\cosh(-x) = \cosh x$

GRAPHING CALCULATOR **Graph each side of each equation. If the equation appears to be an identity, verify it algebraically.**

55. $\frac{\sec x}{\cos x} - \frac{\tan x \sec x}{\csc x} = 1$

56. $\sec x - \cos^2 x \csc x = \tan x \sec x$

57. $(\tan x + \sec x)(1 - \sin x) = \cos x$

58. $\frac{\sec x \cos x}{\cot^2 x} - \frac{1}{\tan^2 x - \sin^2 x \tan^2 x} = -1$

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate methods used to solve trigonometric equations. Consider $1 = 2\sin x$.

a. **NUMERICAL** Isolate the trigonometric function in the equation so that $\sin x$ is the only expression on one side of the equation.

b. **GRAPHICAL** Graph the left and right sides of the equation you found in part **a** on the same graph over $[0, 2\pi)$. Locate any points of intersection and express the values in terms of radians.

c. **GEOMETRIC** Use the unit circle to verify the answers you found in part **b**.

d. **GRAPHICAL** Graph the left and right sides of the equation you found in part **a** on the same graph over $-2\pi < x < 2\pi$. Locate any points of intersection and express the values in terms of radians.

e. **VERBAL** Make a conjecture as to the solutions of $1 = 2\sin x$. Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

60. **REASONING** Can substitution be used to determine whether an equation is an identity? Explain your reasoning.

61 **CHALLENGE** Verify that the area A of a triangle is given by

$$A = \frac{a^2\sin\beta\sin\gamma}{2\sin(\beta + \gamma)}$$

where a, b, and c represent the sides of the triangle and α, β, and γ are the respective opposite angles.

62. **WRITING IN MATH** Use the properties of logarithms to explain why the sum of the natural logarithms of the six basic trigonometric functions for any angle θ is 0.

63. **OPEN ENDED** Create identities for $\sec x$ and $\csc x$ in terms of two or more of the other basic trigonometric functions.

64. **REASONING** If two angles α and β are complementary, is $\cos^2\alpha + \cos^2\beta = 1$? Explain your reasoning. Justify your answers.

65. **WRITING IN MATH** Explain how you would verify a trigonometric identity in which both sides of the equation are equally complex.

Spiral Review

Simplify each expression. (Lesson 5-1)

66. $\cos\theta \csc\theta$

67. $\tan\theta \cot\theta$

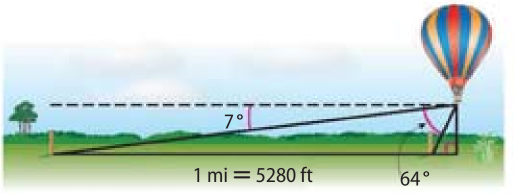
68. $\sin\theta \cot\theta$

69. $\frac{\cos\theta \csc\theta}{\tan\theta}$

70. $\frac{\sin\theta \csc\theta}{\cot\theta}$

71. $\frac{1-\cos^2\theta}{\sin^2\theta}$

72. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts, the angles of depression are 64° and 7°. How high is the balloon to the nearest foot? (Lesson 4-7)



Locate the vertical asymptotes, and sketch the graph of each function. (Lesson 4-5)

73. $y = \frac{1}{4}\tan x$

74. $y = \csc 2x$

75. $y = \frac{1}{2}\sec 3x$

Write each degree measure in radians as a multiple of π and each radian measure in degrees. (Lesson 4-2)

76. 660°

77. 570°

78. 158°

79. $\frac{29\pi}{4}$

80. $\frac{17\pi}{6}$

81. 9

Solve each inequality. (Lesson 2-6)

82. $x^2 - 3x - 18 > 0$

83. $x^2 + 3x - 28 < 0$

84. $x^2 - 4x \leq 5$

85. $x^2 + 2x \geq 24$

86. $-x^2 - x + 12 \geq 0$

87. $-x^2 - 6x + 7 \leq 0$

88. FOOD The manager of a bakery is randomly checking slices of cake prepared by employees to ensure that the correct amount of flavor is in each slice. Each 12-ounce slice should contain half chocolate and half vanilla flavored cream. The amount of chocolate by which each slice varies can be represented by $g(x) = \frac{1}{2}|x - 12|$. Describe the transformations in the function. Then graph the function. (Lesson 1-5)

Skills Review for Standardized Tests

89. SAT/ACT

a, b, a, b, b, a, b, b, b, a, b, b, b, b, a, …

If the sequence continues in this manner, how many *b*s are there between the 44th and 47th appearances of the letter *a*?

A 91 **C** 138 **E** 230
B 135 **D** 182

90. Which expression can be used to form an identity with $\frac{\sec\theta + \csc\theta}{1 + \tan\theta}$ when $\tan\theta \neq -1$?

F $\sin\theta$
G $\cos\theta$
H $\tan\theta$
J $\csc\theta$

91. REVIEW Which of the following is not equivalent to $\cos\theta$ when $0 < \theta < \frac{\pi}{2}$?

A $\frac{\cos\theta}{\cos^2\theta + \sin^2\theta}$ **C** $\cot\theta \sin\theta$
B $\frac{1-\sin^2\theta}{\cos\theta}$ **D** $\tan\theta \csc\theta$

92. REVIEW Which of the following is equivalent to $\sin\theta + \cot\theta \cos\theta$?

F $2\sin\theta$
G $\frac{1}{\sin\theta}$
H $\cos^2\theta$
J $\frac{\sin\theta + \cos\theta}{\sin^2\theta}$

LESSON 5-3

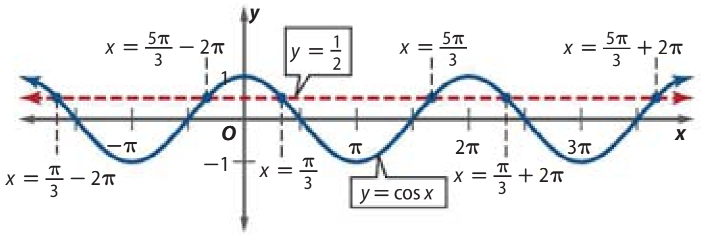
Solving Trigonometric Equations

:: Then	:: Now	:: Why?
● You verified trigonometric identities. (Lesson 5-2)	● 1 Solve trigonometric equations using algebraic techniques. 2 Solve trigonometric equations using basic identities.	● A baseball leaves a bat at a launch angle θ and returns to its initial batted height after a distance of d meters. To find the velocity v_0 of the ball as it leaves the bat, you can solve the trigonometric equation $d = \frac{2v_0^2 \sin\theta\cos\theta}{9.8}$.

1 Use Algebraic Techniques to Solve

In Lesson 5-2, you verified trigonometric equations called identities that are true for all values of the variable for which both sides ar e defined. In this lesson we will consider *conditional* trigonometric equations, which may be true for certain values of the variable but false for others.

Consider the graphs of both sides of the conditional trigonometric equation $\cos x = \frac{1}{2}$.



The graph shows that $\cos x = \frac{1}{2}$ has two solutions on the interval $[0, 2\pi)$, $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Since $y = \cos x$ has a period of 2π, $\cos x = \frac{1}{2}$ has infinitely many solutions on the interval $(-\infty, \infty)$. Additional solutions are found by adding integer multiples of the period, so we express all solutions by writing

$$x = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

To solve a trigonometric equation that involves only one trigonometric expression, begin by isolating this expression.

Example 1 Solve by Isolating Trigonometric Expressions

Solve $2 \tan x - \sqrt{3} = \tan x$.

$2 \tan x - \sqrt{3} = \tan x$	Original equation
$\tan x - \sqrt{3} = 0$	Subtract $\tan x$ from each side to isolate the trigonometric expression.
$\tan x = \sqrt{3}$	Add $\sqrt{3}$ to each side.

The period of tangent is π, so you only need to find solutions on the interval $[0, \pi)$. The only solution on this interval is $x = \frac{\pi}{3}$. The solutions on the interval $(-\infty, \infty)$ are then found by adding integer multiples π. Therefore, the general form of the solutions is

$$x = \frac{\pi}{3} + n\pi, \quad \text{where } n \text{ is an integer.}$$

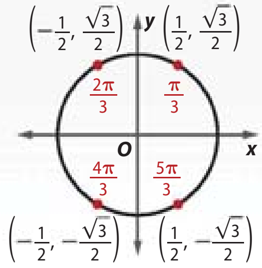
GuidedPractice

1. Solve $4 \sin x = 2 \sin x + \sqrt{2}$.

Tom Carter/PhotoEdit

StudyTip

Find Solutions Using the Unit Circle Since sine corresponds to the y-coordinate on the unit circle, you can find the solutions of $\sin x = \pm\frac{\sqrt{3}}{2}$ on the interval $[0, 2\pi]$ using the unit circle, as shown.



Any angle coterminal with these angles will also be a solution of the equation.

Example 2 Solve by Taking the Square Root of Each Side

Solve $4 \sin^2 x + 1 = 4$.

$4 \sin^2 x + 1 = 4$	Original equation
$4 \sin^2 x = 3$	Subtract 1 from each side.
$\sin^2 x = \frac{3}{4}$	Divide each side by 4.
$\sin x = \pm\frac{\sqrt{3}}{2}$	Take the square root of each side.

On the interval $[0, 2\pi)$, $\sin x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$ and $\sin x = -\frac{\sqrt{3}}{2}$ when $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$. Since sine has a period of 2π, the solutions on the interval $(-\infty, \infty)$ have the general form $x = \frac{\pi}{3} + 2n\pi$, $x = \frac{2\pi}{3} + 2n\pi$, $x = \frac{4\pi}{3} + 2n\pi$, and $x = \frac{5\pi}{3} + 2n\pi$, where n is an integer.

GuidedPractice

2. Solve $3 \cot^2 x + 4 = 7$.

When trigonometric functions cannot be combined on one side of an equation, try factoring and applying the Zero Product Property. If the equation has quadratic form, factor if possible. If not possible, apply the Quadratic Formula.

Watch Out!

Dividing by Trigonometric Factors Do not divide out the $\cos x$ in Example 3a. If you were to do this, notice that you might conclude that the equation had no solutions, when in fact, it has two on the interval $[0, 2\pi)$.

Example 3 Solve by Factoring

Find all solutions of each equation on the interval $[0, 2\pi)$.

a. $\cos x \sin x = 3 \cos x$

$\cos x \sin \theta = 3 \cos x$			Original equation
$\cos x \sin x - 3 \cos x = 0$			Isolate the trigonometric expression.
$\cos x(\sin x - 3) = 0$			Factor.
$\cos x = 0$	or	$\sin x - 3 = 0$	Zero Product Property
$x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$		$\sin x = 3$	Solve for x on $[0, 2\pi]$.

The equation $\sin x = 3$ has no solution since the maximum value the sine function can attain is 1. Therefore, on the interval $[0, 2\pi)$, the solutions of the original equation are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

b. $\cos^4 x + \cos^2 x - 2 = 0$

$\cos^4 x + \cos^2 x - 2 = 0$			Original equation
$(\cos^2 x)^2 + \cos^2 x - 2 = 0$			Write in quadratic form.
$(\cos^2 x + 2)(\cos^2 x - 1) = 0$			Factor.
$\cos^2 x + 2 = 0$	or	$\cos^2 x - 1 = 0$	Zero Product Property
$\cos^2 x = -2$		$\cos^2 x = 1$	Solve for $\cos^2 x$.
$\cos x = \pm\sqrt{-2}$		$\cos x = \pm\sqrt{1}$ or ± 1	Take the square root of each side.

The equation $\cos x = \pm\sqrt{-2}$ has no real solutions. On the interval $[0, 2\pi)$, the equation $\cos x = \pm 1$ has solutions 0 and π.

GuidedPractice

3A. $2 \sin x \cos x = \sqrt{2} \cos x$

3B. $4 \cos^2 x + 2 \cos x - 2\sqrt{2} \cos x = \sqrt{2}$

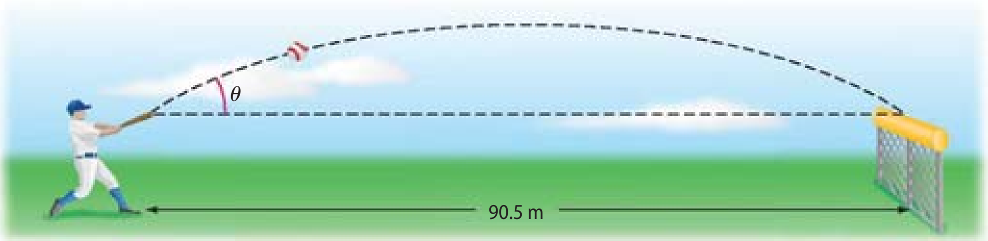
Some trigonometric equations involve functions of multiple angles, such as $\cos 2x = \frac{1}{2}$. To solve these equations, first solve for the multiple angle.

StudyTip

Exact Versus Approximate Solutions When solving trigonometric equations that are not in a real-world context, write your answers using exact values rather than decimal approximations. For example, the general solutions of the equation $\tan x = 2$ should be expressed as $x = \tan^{-1} 2 + n\pi$ or $x = \arctan 2 + n\pi$.

Real-World Example 4 Trigonometric Functions of Multiple Angles

BASEBALL A baseball leaves a bat with an initial speed of 30 meters per second and clears a fence 90.5 meters away. The height of the fence is the same height as the initial height of the batted ball. If the distance the ball traveled is given by $d = \frac{v_0^2 \sin 2\theta}{9.8}$, where 9.8 is in meters per second squared, find the interval of possible launch angles of the ball.



$d = \frac{v_0^2 \sin 2\theta}{9.8}$	Original formula
$90.5 = \frac{30^2 \sin 2\theta}{9.8}$	$d = 90.5$ and $v_0 = 30$
$90.5 = \frac{900 \sin 2\theta}{9.8}$	Simplify.
$886.9 = 900 \sin 2\theta$	Multiply each side by 9.8.
$\frac{886.9}{900} = \sin 2\theta$	Divide each side by 900.
$\sin^{-1} \frac{886.9}{900} = 2\theta$	Definition of inverse sine

Recall from Lesson 4-6 that the range of the inverse sine function is restricted to acute angles of θ in the interval $[-90°, 90°]$. Since we are finding the inverse sine of 2θ instead of θ, we need to consider angles in the interval $[-2(90°), 2(90°)]$ or $[-180°, 180°]$. Use your calculator to find the acute angle and the reference angle relationship $\sin(180° - \theta) = \sin \theta$ to find the obtuse angle.

$\sin^{-1} \frac{886.9}{900} = 2\theta$	Definition of inverse sine
$80.2°$ or $99.8° = 2\theta$	$\sin^{-1} \frac{886.9}{900} \approx 80.2°$ and $\sin(180° - 80.2°) = \sin 99.8°$
$40.1°$ or $49.9° = \theta$	Divide by 2.

The interval is $[40.1°, 49.9°]$. The ball will clear the fence if the angle is between 40.1° and 49.9°.

CHECK Substitute the angle measures into the original equation to confirm the solution.

$d = \frac{v_0^2 \sin 2\theta}{9.8}$	Original formula	$d = \frac{v_0^2 \sin 2\theta}{9.8}$
$90.5 \stackrel{?}{=} \frac{30^2 \sin(2 \cdot 40.1°)}{9.8}$	$\theta = 40.1°$ or $\theta = 49.9°$	$90.5 \stackrel{?}{=} \frac{30^2 \sin(2 \cdot 49.9°)}{9.8}$
$90.5 \approx 90.497$ ✓	Use a calculator.	$90.5 \approx 90.497$ ✓

StudyTip

Optimal Angle Ignoring wind resistance and other factors, a baseball will travel the greatest distance when it is hit at a 45° angle. This is because $\sin 2(45) = 1$, which maximizes the distance formula in the example.

GuidedPractice

4. BASEBALL Find the interval of possible launch angles required to clear the fence if:

A. the initial speed was increased to 35 meters per second.

B. the initial speed remained the same, but the distance to the fence was 80 meters.

2 Use Trigonometric Identities to Solve

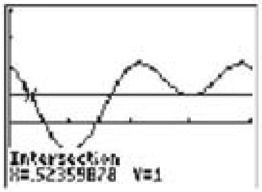
You can use trigonometric identities along with algebraic methods to solve trigonometric equations.

Example 5 Solve by Rewriting Using a Single Trigonometric Function

Find all solutions of $2\cos^2 x - \sin x - 1 = 0$ on the interval $[0, 2\pi)$.

$2\cos^2 x - \sin x - 1 = 0$	Original equation
$2(1 - \sin^2 x) - \sin x - 1 = 0$	Pythagorean Identity
$2 - 2\sin^2 x - \sin x - 1 = 0$	Multiply.
$-2\sin^2 x - \sin x + 1 = 0$	Simplify.
$-1(2\sin x - 1)(\sin x + 1) = 0$	Factor.
$2\sin x - 1 = 0$ or $\sin x + 1 = 0$	Zero Product Property
$\sin x = \frac{1}{2}$ $\quad$ $\sin x = -1$	Solve for $\sin x$.
$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ $\quad$ $x = \frac{3\pi}{2}$	Solve for x on $[0, 2\pi]$.

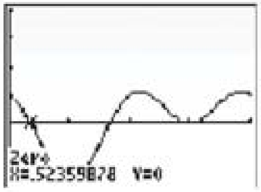
CHECK The graphs of $Y_1 = 2\cos^2 x - \sin x$ and $Y_2 = 1$ intersect at $\frac{\pi}{6}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$ on the interval $[0, 2\pi]$ as shown ✔



$[0, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 4]$ scl: 1

StudyTip

Alternate Method An alternate way to check Example 5 is to graph $y = 2\cos^2 x - \sin x - 1$. It has zeros $\frac{\pi}{6}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$ on the interval $[0, 2\pi)$ as shown.



$[0, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 4]$ scl: 1

GuidedPractice

Find all solutions of each equation on the interval $[0, 2\pi)$.

5A. $1 - \cos x = 2\sin^2 x$

5B. $\cot^2 x \csc^2 x + 2\csc^2 x - \cot^2 x = 2$

Sometimes you can obtain an equation in one trigonometric function by squaring each side, but this technique may produce extraneous solutions.

Example 6 Solve by Squaring

Find all solutions of $\csc x - \cot x = 1$ on the interval $[0, 2\pi]$.

$\csc x - \cot x = 1$	Original equation
$\csc x = 1 + \cot x$	Add $\cot x$ to each side.
$(\csc x)^2 = (1 + \cot x)^2$	Square each side.
$\csc^2 x = 1 + 2\cot x + \cot^2 x$	Multiply.
$1 + \cot^2 x = 1 + 2\cot x + \cot^2 x$	Pythagorean Identity
$0 = 2\cot x$	Subtract $1 + \cot^2 x$ from each side.
$0 = \cot x$	Divide each side by 2.
$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$	Solve for x on $[0, 2\pi]$.

CHECK

$\csc x - \cot x = 1$	Original equation	$\csc x - \cot x = 1$
$\csc \frac{\pi}{2} - \cot \frac{\pi}{2} \stackrel{?}{=} 1$	Substitute.	$\csc \frac{3\pi}{2} - \cot \frac{3\pi}{2} \stackrel{?}{=} 1$
$1 - 0 = 1$ ✔	Simplify.	$-1 - 0 \neq 1$ ✘

Therefore, the only valid solution is $\frac{\pi}{2}$ on the interval $[0, 2\pi]$.

GuidedPractice

6A. $\sec x + 1 = \tan x$

6B. $\cos x = \sin x - 1$

Exercises

= Step-by-Step Solutions begin on page R29.

Solve each equation for all values of x. (Examples 1 and 2)

1. $5 \sin x + 2 = \sin x$

2. $5 = \sec^2 x + 3$

3. $2 = 4 \cos^2 x + 1$

4. $4 \tan x - 7 = 3 \tan x - 6$

5. $9 + \cot^2 x = 12$

6. $2 - 10 \sec x = 4 - 9 \sec x$

7. $3 \csc x = 2 \csc x + \sqrt{2}$

8. $11 = 3 \csc^2 x + 7$

9. $6 \tan^2 x - 2 = 4$

10. $9 + \sin^2 x = 10$

11. $7 \cot x - \sqrt{3} = 4 \cot x$

12. $7 \cos x = 5 \cos x + \sqrt{3}$

Find all solutions of each equation on $[0, 2\pi]$. (Example 3)

13. $\sin^4 x + 2 \sin^2 x - 3 = 0$

14. $-2 \sin x = -\sin x \cos x$

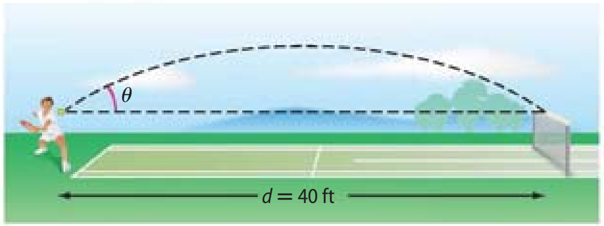
15. $4 \cot x = \cot x \sin^2 x$

16. $\csc^2 x - \csc x + 9 = 11$

17. $\cos^3 x + \cos^2 x - \cos x = 1$

18. $2 \sin^2 x = \sin x + 1$

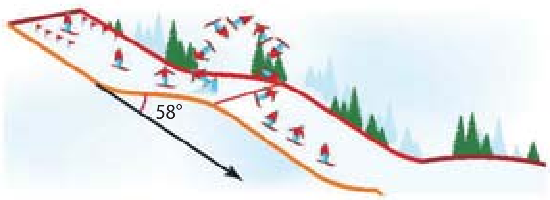
19. **TENNIS** A tennis ball leaves a racquet and heads toward a net 40 feet away. The height of the net is the same height as the initial height of the tennis ball. (Example 4)



a. If the ball is hit at 50 feet per second, neglecting air resistance, use $d = \frac{1}{32}v_0^2 \sin 2\theta$ to find the interval of possible angles of the ball needed to clear the net.

b. Find θ if the initial velocity remained the same but the distance to the net was 50 feet.

20. **SKIING** In the Olympic aerial skiing competition, skiers speed down a slope that launches them into the air, as shown. The maximum height a skier can reach is given by $h_{peak} = \frac{v_0^2 \sin^2 \theta}{2g}$, where g is the acceleration due to gravity or 9.8 meters per second squared. (Example 4)



a. If a skier obtains a height of 5 meters above the end of the ramp, what was the skier's initial speed?

b. Use your answer from part **a** to determine how long it took the skier to reach the maximum height if $t_{peak} = \frac{v_0 \sin \theta}{g}$.

Find all solutions of each equation on the interval $[0, 2\pi]$. (Examples 5 and 6)

21. $1 = \cot^2 x + \csc x$

22. $\sec x = \tan x + 1$

23. $\tan^2 x = 1 - \sec x$

24. $\csc x + \cot x = 1$

25. $2 - 2 \cos^2 x = \sin x + 1$

26. $\cos x - 4 = \sin x - 4$

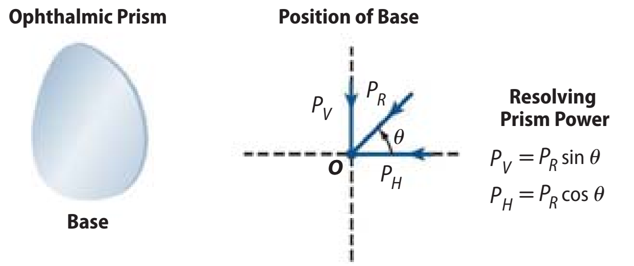
27. $3 \sin x = 3 - 3 \cos x$

28. $\cot^2 x \csc^2 x - \cot^2 x = 9$

29. $\sec^2 x - 1 + \tan x - \sqrt{3} \tan x = \sqrt{3}$

30. $\sec^2 x \tan^2 x + 3 \sec^2 x - 2 \tan^2 x = 3$

31. **OPTOMETRY** Optometrists sometimes join two oblique or tilted prisms to correct vision. The resultant refractive power P_R of joining two oblique prisms can be calculated by first resolving each prism into its horizontal and vertical components, P_H and P_V.



Using the equations above, determine the values of θ for which P_V and P_H are equivalent.

Find all solutions of each equation on the interval $[0, 2\pi]$.

32. $\frac{\tan^2 x}{\sec x} + \cos x = 2$

33. $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = -4$

34. $\frac{\sin x + \cos x}{\tan x} + \frac{1 - \sin x}{\sin x} = \cos x$

35. $\cot x \cos x + 1 = \frac{1}{\sec x - 1} + \frac{\sin x}{\tan^2 x}$

GRAPHING CALCULATOR **Solve each equation on the interval $[0, 2\pi]$ by graphing. Round to the nearest hundredth.**

36. $3 \cos 2x = e^x + 1$

37. $\sin \pi x + \cos \pi x = 3x$

38. $x^2 = 2 \cos x + x$

39. $x \log x + 5x \cos x = -2$

40. **METEOROLOGY** The average daily temperature in degrees Fahrenheit for a city can be modeled by $t = 8.05 \cos\left(\frac{\pi}{6}x - \pi\right) + 66.95$, where x is a function of time, $x = 1$ represents January 15, $x = 2$ represents February 15, and so on.

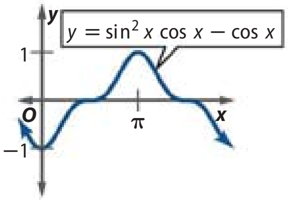
a. Use a graphing calculator to estimate the temperature on January 31.

b. Approximate the number of months that the average daily temperature is greater than 70° throughout the entire month.

c. Estimate the highest temperature of the year and the month in which it occurs.

Find the x-intercepts of each graph on the interval $[0, 2\pi]$.

41.



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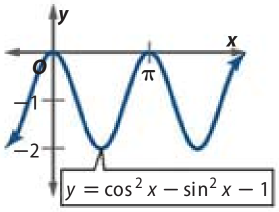

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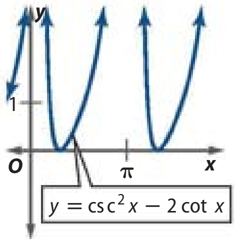

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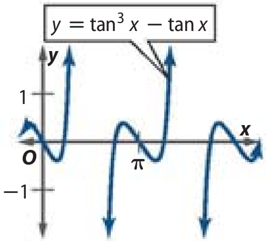


Find all solutions of each equation on the interval $[0, 4\pi]$.

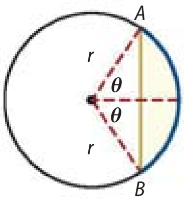
45. $4 \tan x = 2 \sec^2 x$

46. $2 \sin^2 x + 1 = -3 \sin x$

47. $\csc x \cot^2 x = \csc x$

48. $\sec x + 5 = 2 \sec x + 3$

49. **GEOMETRY** Consider the circle below.



a. The length s of arc AB is given by $s = r(2\theta)$ where $0 \leq \theta \leq \pi$. When $s = 18$ and $AB = 14$, the radius is $r = \frac{7}{\sin \theta}$. Use a graphing calculator to find the measure of 2θ in radians.

b. The area of the shaded region is given by $A = \frac{r^2(\theta - \sin \theta)}{2}$. Use a graphing calculator to find the radian measure of θ if the radius is 5 inches and the area is 36 square inches. Round to the nearest hundredth.

Solve each inequality on the interval $[0, 2\pi)$.

50. $1 > 2 \sin x$

51. $0 < 2 \cos x - \sqrt{2}$

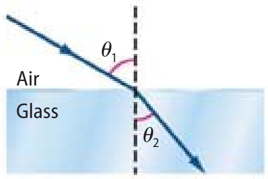
52. $\cos\left(\frac{\pi}{2} - x\right) \geq \frac{\sqrt{3}}{2}$

53. $\sin\left(x - \frac{\pi}{2}\right) \leq \tan x \cot x$

54. $\cos x \leq -\frac{\sqrt{3}}{2}$

55. $\sqrt{2} \sin x - 1 < 0$

56. **REFRACTION** When light travels from one transparent medium to another it bends or *refracts*, as shown.



Refraction is described by $n_2 \sin \theta_1 = n_1 \sin \theta_2$, where n_1 is the index of refraction of the medium the light is entering, n_2 is the index of refraction of the medium the light is exiting, θ_1 is the angle of incidence, and θ_2 is the angle of refraction.

a. Find θ_2 for each material shown if the angle of incidence is 40° and the index of refraction for air is 1.00.

b. If the angle of incidence is doubled to 80°, will the resulting angles of refraction be twice as large as those found in part **a**?

Material	Index of Refraction
glass	1.52
ice	1.31
plastic	1.50
water	1.33

H.O.T. Problems Use Higher-Order Thinking Skills

57. **ERROR ANALYSIS** Vijay and Alicia are solving $\tan^2 x - \tan x + \sqrt{3} = \sqrt{3} \tan x$. Vijay thinks that the solutions are $x = \frac{\pi}{4} + n\pi$, $x = \frac{5\pi}{4} + n\pi$, $x = \frac{\pi}{3} + n\pi$, and $x = \frac{4\pi}{3} + n\pi$. Alicia thinks that the solutions are $x = \frac{\pi}{4} + n\pi$ and $x = \frac{\pi}{3} + n\pi$. Is either of them correct? Explain your reasoning.

CHALLENGE **Solve each equation on the interval $[0, 2\pi]$.**

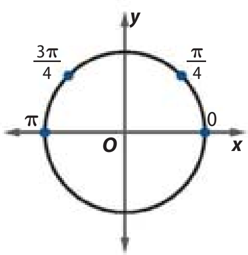
58. $16 \sin^5 x + 2 \sin x = 12 \sin^3 x$

59. $4 \cos^2 x - 4 \sin^2 x \cos^2 x + 3 \sin^2 x = 3$

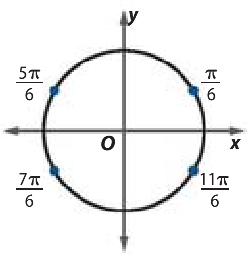
60. **REASONING** Are the solutions of $\csc x = \sqrt{2}$ and $\cot^2 x + 1 = 2$ equivalent? If so, verify your answer algebraically. If not, explain your reasoning.

OPEN ENDED **Write a trigonometric equation that has each of the following solutions.**

61.



62.



63. **WRITING IN MATH** Explain the difference in the techniques that are used when solving equations and verifying identities.

Spiral Review

Verify each identity. (Lesson 5-2)

64. $\dfrac{1+\sin\theta}{\sin\theta} = \dfrac{\cot^2\theta}{\csc\theta - 1}$

65. $\dfrac{1+\tan\theta}{1+\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

66. $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta} = 1$

Find the value of each expression using the given information. (Lesson 5-1)

67. $\tan\theta$; $\sin\theta = \frac{1}{2}$, $\tan\theta > 0$

68. $\csc\theta$, $\cos\theta = -\frac{3}{5}$, $\csc\theta < 0$

69. $\sec\theta$; $\tan\theta = -1$, $\sin\theta < 0$

70. **POPULATION** The population of a certain species of deer can be modeled by $p = 30{,}000 + 20{,}000\cos\left(\frac{\pi}{10}t\right)$, where p is the population and t is the time in years. (Lesson 4-4)

a. What is the amplitude of the function? What does it represent?

b. What is the period of the function? What does it represent?

c. Graph the function.

Given $f(x) = 2x^2 - 5x + 3$ and $g(x) = 6x + 4$, find each of the following. (Lesson 1-6)

71. $(f - g)(x)$

72. $(f \cdot g)(x)$

73. $\left(\frac{f}{g}\right)(x)$

74. **BUSINESS** A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.

If the mean of these data is 9, what is the population standard deviation for these data? Round to the nearest tenth. (Lesson 0-8)

Month	Number of Additional Employees Needed
August	5
September	14
October	6
November	8
December	12

Skills Review for Standardized Tests

75. SAT/ACT For all positive values of m and n, if $\dfrac{3x}{m - nx} = 2$, then $x =$

A $\dfrac{2m - 2n}{3}$

B $\dfrac{3 + 2n}{2m}$

C $\dfrac{2m - 3}{2n}$

D $\dfrac{2m}{3 + 2n}$

E $\dfrac{3}{2m - 2n}$

76. If $\cos x = -0.45$, what is $\sin\left(x - \frac{\pi}{2}\right)$?

F -0.55

G -0.45

H 0.45

J 0.55

77. Which of the following is *not* a solution of $0 = \sin\theta + \cos\theta\tan^2\theta$?

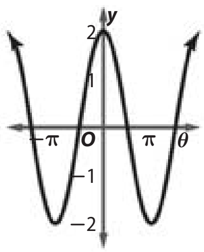
A $\dfrac{3\pi}{4}$

B $\dfrac{7\pi}{4}$

C 2π

D $\dfrac{5\pi}{2}$

78. **REVIEW** The graph of $y = 2\cos\theta$ is shown. Which is a solution for $2\cos\theta = 1$?



F $\dfrac{8\pi}{3}$

G $\dfrac{10\pi}{3}$

H $\dfrac{13\pi}{3}$

J $\dfrac{15\pi}{3}$

EXTEND 5-3

Graphing Technology Lab
Solving Trigonometric Inequalities

Objective

- Use a graphing calculator to solve trigonometric inequalities.

You can use a graphing calculator to solve trigonometric inequalities. Graph each inequality. Then locate the end points of each intersection in the graph to find the intervals within which the inequality is true.

Activity 1 Graph a Trigonometric Inequality

Graph and solve $\sin 2x \geq \cos x$.

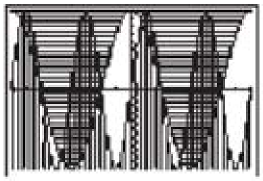
Step 1 Replace each side of this inequality with y to form the new inequalities.

$$\sin 2x \geq Y1 \text{ and } Y2 \geq \cos x$$

Step 2 Graph each inequality. Make each inequality symbol by scrolling to the left of the equal sign and selecting ENTER until the shaded triangles are flashing. The triangle above represents *greater than*, and the triangle below represents *less than* (Figure 5.3.1). In the MODE menu, select RADIAN.

Step 3 Graph the equations in the appropriate window. Use the domain and range of each trigonometric function as a guide (Figure 5.3.2).

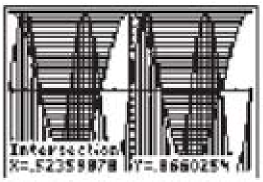
Figure 5.3.1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{4}$ by $[-1, 1]$ scl: 0.1

Figure 5.3.2

Step 4 The darkly shaded area indicates the intersection of the graphs and the solution of the system of inequalities. Use CALC: intersect to locate these intersections. Move the cursor over the intersection and select ENTER 3 times.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{4}$ by $[-1, 1]$ scl: 0.1

Step 5 The first intersection is when $y = 0.866$ or $\frac{\sqrt{3}}{2}$. Since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, the intersection is at $x = \frac{\pi}{6}$. The next intersection is at $x = \frac{\pi}{2}$. Therefore, one of the solution intervals is $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. Another interval is $\left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$. There are an infinite number of intervals, so the solutions for all values of x are $\left[\frac{\pi}{6} + 2n\pi, \frac{\pi}{2} + 2n\pi\right]$ and $\left[\frac{5\pi}{6} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right]$.

Exercises

Graph and solve each inequality.

1. $\sin 3x < 2 \cos x$
2. $3 \cos x \geq 0.5 \sin 2x$
3. $\sec x < 2 \cos x$
4. $\csc 2x > \sin 8x$
5. $2 \tan 2x < 3 \sin 2x$
6. $\tan x \geq \cos x$

CHAPTER 5

Mid-Chapter Quiz

Lessons 5-1 through 5-3

Find the value of each expression using the given information. (Lesson 5-1)

1. $\sin\theta$ and $\cos\theta$, $\cot\theta = 4$, $\cos\theta > 0$

2. $\sec\theta$ and $\sin\theta$, $\tan\theta = -\frac{2}{3}$, $\csc\theta > 0$

3. $\tan\theta$ and $\csc\theta$, $\cos\theta = \frac{1}{4}$, $\sin\theta > 0$

Simplify each expression. (Lesson 5-1)

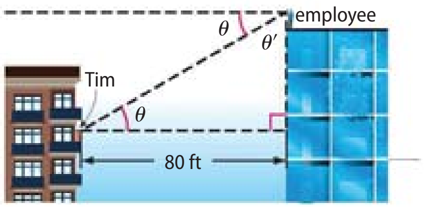
4. $\frac{\sin(-x)}{\tan(-x)}$

5. $\frac{\sec^2 x}{\cot^2 x + 1}$

6. $\frac{\sin(90° - x)}{\cot^2(90° - x) + 1}$

7. $\frac{\sin x}{1 + \sec x}$

8. ANGLE OF DEPRESSION From his apartment window, Tim can see the top of the bank building across the street at an angle of elevation of θ, as shown below. (Lesson 5-1)



a. If a bank employee looks down at Tim's apartment from the top of the bank, what identity could be used to conclude that $\sin\theta = \cos\theta'$?

b. If Tim looks down at a lower window of the bank with an angle of depression of 35°, how far below his apartment is the bank window?

9. MULTIPLE CHOICE Which of the following is not equal to $\csc\theta$? (Lesson 5-1)

A $\sec(90° - \theta)$

B $\sqrt{\cot^2\theta + 1}$

C $\frac{1}{\sin\theta}$

D $\frac{1}{\sin(90° - \theta)}$

Verify each identity. (Lesson 5-2)

10. $\frac{\cos\theta}{1 + \sin\theta} - \frac{\cos\theta}{1 - \sin\theta} = -2\tan\theta$

11. $\csc^2\theta - \sin^2\theta - \cos^2\theta - \cot^2\theta = 0$

12. $\sin\theta + \frac{\cos\theta}{\tan\theta} = \csc\theta$

13. $\frac{\cos\theta}{1 + \sin\theta} = \sec\theta - \tan\theta$

14. $\frac{\csc\theta}{\sin\theta} + \frac{\cot\theta}{\cos\theta} = \cot^2\theta + \csc\theta + 1$

15. $\frac{1 + \sin\theta}{\sin\theta} + \frac{\sin\theta}{1 - \sin\theta} = \frac{\csc\theta}{1 - \sin\theta}$

Find all solutions of each equation on the interval $[0, 2\pi]$. (Lesson 5-3)

16. $4\sec\theta + 2\sqrt{3} = \sec\theta$

17. $2\tan\theta + 4 = \tan\theta + 5$

18. $4\cos^2\theta + 2 = 3$

19. $\cos\theta - 1 = \sin\theta$

20. MULTIPLE CHOICE Which of the following is the solution set for $\cos\theta\tan\theta - \sin^2\theta = 0$? (Lesson 5-3)

F $\frac{\pi}{2}n$, where n is an integer

G $\frac{\pi}{2} + n\pi$, where n is an integer

H $\pi + 2n\pi$, where n is an integer

J $n\pi$, where n is an integer

Solve each equation for all values of θ. (Lesson 5-3)

21. $3\sin^2\theta + 6 = 2\sin^2\theta + 7$

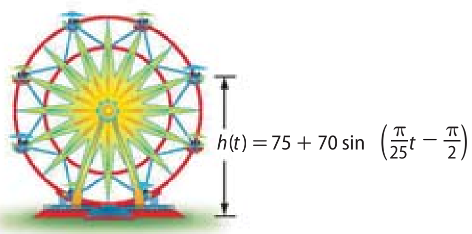
22. $\sin\theta + \cos\theta = 0$

23. $\sec\theta + \tan\theta = 0$

24. $3 - 3\cos^2\theta = 1 + \sin^2\theta$

25. PROJECTILE MOTION The distance d that a kickball travels in feet with an acceleration of 32 feet per second squared is given by $d = \frac{v_0^2 \sin 2\theta}{32}$, where v_0 is the object's initial speed and θ is the angle at which the object is launched. If the ball is kicked with an initial speed of 82 feet per second, and the ball travels 185 feet, find the possible launch angle(s) of the ball. (Lesson 5-3)

26. FERRIS WHEEL The height h of a rider in feet on a Ferris wheel after t seconds is shown below. (Lesson 5-3)



a. If the Ferris wheel begins at $t = 0$, what is the initial height of a rider?

b. When will the rider first reach the maximum height of 145 feet?

LESSON 5-4 Sum and Difference Identities

Then

- You found values of trigonometric functions by using the unit circle. (Lesson 4-3)

Now

1. Use sum and difference identities to evaluate trigonometric functions.
2. Use sum and difference identities to solve trigonometric equations.

Why?

- When the picture on a television screen is blurry or a radio station will not tune in properly, the problem is too often due to *interference*. Interference results when waves pass through the same space at the same time. You can use trigonometric identities to determine the type of interference that is taking place.

NewVocabulary
reduction identity

1 Evaluate Trigonometric Functions

In Lesson 5-1, you used basic identities involving only one variable. In this lesson, we will consider identities involving two variables. One of these is the *cosine of a difference* identity.

Let points A, B, C, and D be located on the unit circle, α and β be angles on the interval $[0, 2\pi]$, and $\alpha > \beta$ as shown in Figure 5.4.1. Because each point is located on the unit circle, $x_1^2 + y_1^2 = 1$, $x_2^2 + y_2^2 = 1$, and $x_3^2 + y_3^2 = 1$. Notice also that the measure of arc $CD = \alpha - (\alpha - \beta)$ or β and the measure of arc $AB = \beta$.

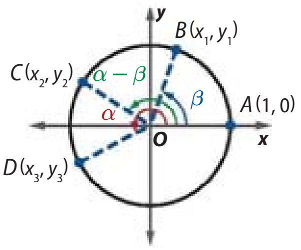


Figure 5.4.1

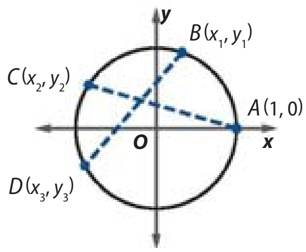


Figure 5.4.2

Since arcs AB and CD have the same measure, chords AC and BD shown in Figure 5.4.2 are congruent.

$AC = BD$	Chords AC and BD are congruent.
$\sqrt{(x_2 - 1)^2 + (y_2 - 0)^2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$	Distance Formula
$(x_2 - 1)^2 + (y_2 - 0)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2$	Square each side.
$x_2^2 - 2x_2 + 1 + y_2^2 = x_3^2 - 2x_3x_1 + x_1^2 + y_3^2 - 2y_3y_1 + y_1^2$	Square each binomial.
$(x_2^2 + y_2^2) - 2x_2 + 1 = (x_1^2 + y_1^2) + (x_3^2 + y_3^2) - 2x_3x_1 - 2y_3y_1$	Group similar squared terms.
$1 - 2x_2 + 1 = 1 + 1 - 2x_3x_1 - 2y_3y_1$	Substitution
$2 - 2x_2 = 2 - 2x_3x_1 - 2y_3y_1$	Add.
$-2x_2 = -2x_3x_1 - 2y_3y_1$	Subtract 2 from each side.
$x_2 = x_3x_1 + y_3y_1$	Divide each side by −2.

In Figure 5.4.1, notice that by the unit circle definitions for cosine and sine, $x_1 = \cos \beta$, $x_2 = \cos(\alpha - \beta)$, $x_3 = \cos \alpha$, $y_1 = \sin \beta$, and $y_3 = \sin \alpha$. Substituting, $x_2 = x_3x_1 + y_3y_1$ becomes

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta. \qquad \text{Cosine Difference Identity}$$

We can now obtain the identity for the cosine of a sum.

$\cos[\alpha + (-\beta)] = \cos \alpha \cos \beta + \sin \alpha \sin \beta$	$\alpha - \beta = \alpha + (-\beta)$
$\cos[\alpha + (-\beta)] = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta)$	$\cos(-\beta) = \cos(\beta)$, $\sin \beta = -\sin(-\beta)$
$\cos[\alpha + \theta] = \cos \alpha \cos \theta - \sin \alpha \sin \theta$	Let $(-\beta) = \theta$

These two cosine identities can be used to establish each of the other sum and difference identities listed below.

KeyConcept Sum and Difference Identities

Sum Identities	Difference Identities
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$	$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

You will prove the sine and tangent sum and difference identities in Exercises 57–60.

By writing angle measures as the sums or differences of special angle measures, you can use these sum and difference identities to find exact values of trigonometric functions of angles that are less common.

Example 1 Evaluate a Trigonometric Expression

Find the exact value of each trigonometric expression.

a. $\sin 15°$

Write 15° as the sum or difference of angle measures with sines that you know.

$\sin 15° = \sin(45° - 30°)$ — $45° - 30° = 15°$

$= \sin 45° \cos 30° - \sin 30° \cos 45°$ — Sine Difference Identity

$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2}$ — $\sin 30° = \frac{1}{2}$, $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$

$= \dfrac{\sqrt{6}}{4} - \dfrac{\sqrt{2}}{4}$ — Multiply.

$= \dfrac{\sqrt{6} - \sqrt{2}}{4}$ — Combine the fractions.

> **StudyTip**
> **Check Your Answer** You can check your answers by using a graphing calculator. In Example 1a, $\sin 15° \approx 0.259$ and $\frac{\sqrt{6} - \sqrt{2}}{4} \approx 0.259$. Make sure that the calculator is in the correct mode.

b. $\tan \frac{7\pi}{12}$

$\tan \dfrac{7\pi}{12} = \tan\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$ — $\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

$= \dfrac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$ — Tangent Sum Identity

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}(1)}$ — $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan\frac{\pi}{4} = 1$

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}}$ — Simplify.

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \dfrac{1 + \sqrt{3}}{1 + \sqrt{3}}$ — Rationalize the denominator.

$= \dfrac{\sqrt{3} + 1 + 3 + \sqrt{3}}{1 - \sqrt{3} + \sqrt{3} - 3}$ — Multiply.

$= \dfrac{4 + 2\sqrt{3}}{-2}$ — Simplify.

$= -2 - \sqrt{3}$ — Simplify.

GuidedPractice

1A. $\cos 15°$

1B. $\sin \frac{5\pi}{12}$

Sum and difference identities are often used to solve real-world problems.

Real-WorldCareer

Lineworker Lineworkers are responsible for the building and upkeep of electric power transmission and distribution facilities. The term also applies to tradeworkers who install and maintain telephone, cable TV, and fiber optic lines.

Real-World Example 2 Use a Sum or Difference Identity

ELECTRICITY **An alternating current i in amperes in a certain circuit can be found after t seconds using $i = 3\ (\sin 165)t$, where 165 is a degree measure.**

a. Rewrite the formula in terms of the sum of two angle measures.

$i = 3\ (\sin 165)t$ — Original equation

$= 3\ [\sin (120 + 45)]t$ — $120 + 45 = 165$

b. Use a sine sum identity to find the exact current after 1 second.

$i = 3\ [\sin (120 + 45)]t$ — Rewritten equation

$= 3 \sin (120 + 45)$ — $t = 1$

$= 3[(\sin 120)(\cos 45) + (\cos 120)(\sin 45)]$ — Sine Sum Identity

$= 3\left[\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$ — Substitute.

$= 3\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)$ — Multiply.

$= \frac{3\sqrt{6} - 3\sqrt{2}}{4}$ — Simplify.

The exact current after 1 second is $\frac{3\sqrt{6} - 3\sqrt{2}}{4}$ amperes.

GuidedPractice

2. **ELECTRICITY** An alternating current i in amperes in another circuit can be found after t seconds using $i = 2\ (\sin 285)t$, where 285 is a degree measure.

A. Rewrite the formula in terms of the difference of two angle measures.

B. Use a sine difference identity to find the exact current after 1 second.

If a trigonometric expression has the form of a sum or difference identity, you can use the identity to find an exact value or to simplify an expression by rewriting the expression as a function of a single angle.

Example 3 Rewrite as a Single Trigonometric Expression

a. Find the exact value of $\frac{\tan 32° + \tan 13°}{1 - \tan 32° \tan 13°}$.

$\frac{\tan 32° + \tan 13°}{1 - \tan 32° \tan 13°} = \tan (32° + 13°)$ — Tangent Sum Identity

$= \tan 45° \text{ or } 1$ — Simplify.

b. Simplify $\sin x \sin 3x - \cos x \cos 3x$.

$\sin x \sin 3x - \cos x \cos 3x = -(\cos x \cos 3x - \sin x \sin 3x)$ — Distributive and Commutative Properties

$= -\cos (x + 3x) \text{ or } -\cos 4x$ — Tangent Sum Identity

GuidedPractice

3A. Find the exact value of $\cos \frac{7\pi}{8} \cos \frac{5\pi}{24} + \sin \frac{7\pi}{8} \sin \frac{5\pi}{24}$.

3B. Simplify $\frac{\tan 6x - \tan 7x}{1 + \tan 6x \tan 7x}$.

Sum and difference identities can be used to rewrite trigonometric expressions as algebraic expressions.

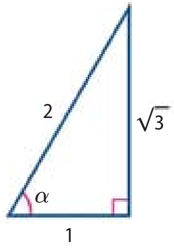
Example 4 Write as an Algebraic Expression

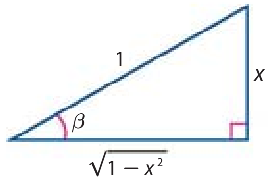
Write $\sin(\arctan\sqrt{3} + \arcsin x)$ as an algebraic expression of x that does not involve trigonometric functions.

Applying the Sine of a Sum Identity, we find that

$\sin(\arctan\sqrt{3} + \arcsin x) = \sin(\arctan\sqrt{3})\cos(\arcsin x) + \cos(\arctan\sqrt{3})\sin(\arcsin x)$.

If we let $\alpha = \arctan\sqrt{3}$ and $\beta = \arcsin x$, then $\tan\alpha = \sqrt{3}$ and $\sin\beta = x$. Sketch one right triangle with an acute angle α and another with an acute angle β. Label the sides such that $\tan\alpha = \sqrt{3}$ and the $\sin\beta = x$. Then use the Pythagorean Theorem to express the length of each third side.





Using these triangles, we find that $\sin(\arctan\sqrt{3}) = \sin\alpha$ or $\frac{\sqrt{3}}{2}$, $\cos(\arctan\sqrt{3}) = \cos\alpha$ or $\frac{1}{2}$, $\cos(\arcsin x) = \cos\beta$ or $\sqrt{1-x^2}$, and $\sin(\arcsin x) = \sin\beta$ or x.

Now apply substitution and simplify.

$$\begin{aligned}\sin(\arctan\sqrt{3} + \arcsin x) &= \sin(\arctan\sqrt{3})\cos(\arcsin x) + \cos(\arctan\sqrt{3})\sin(\arcsin x)\\ &= \frac{\sqrt{3}}{2}\left(\sqrt{1-x^2}\right) + \frac{1}{2}x\\ &= \frac{\sqrt{3-3x^2}}{2} + \frac{x}{2} \text{ or } \frac{x+\sqrt{3-3x^2}}{2}\end{aligned}$$

GuidedPractice

Write each trigonometric expression as an algebraic expression.

4A. $\cos(\arcsin 2x + \arccos x)$

4B. $\sin\left(\arctan x - \arccos\frac{1}{2}\right)$

Sum and difference identities can be used to verify other identities.

ReadingMath

Cofunction Identities The "co" in cofunction stands for "complement." Therefore, sine and cosine, tangent and cotangent, and secant and cosecant are all complementary functions or cofunctions.

Example 5 Verify Cofunction Identities

Verify $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$ — Sine Difference Identity

$= 1(\cos x) - 0(\sin x)$ — $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$

$= \cos x$ ✔ — Multiply.

GuidedPractice

Verify each cofunction identity using a difference identity.

5A. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

5B. $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

Sum and difference identities can be used to rewrite trigonometric expressions in which one of the angles is a multiple of 90° or $\frac{\pi}{2}$ radians. The resulting identity is called a **reduction identity** because it *reduces* the complexity of the expression.

Example 6 Verify Reduction Identities

Verify each reduction identity.

a. $\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos\theta$

$\sin\left(\theta + \frac{3\pi}{2}\right) = \sin\theta\cos\frac{3\pi}{2} + \cos\theta\sin\frac{3\pi}{2}$ — Sine Sum Formula

$= \sin\theta(0) + \cos\theta(-1)$ — $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$

$= -\cos\theta$ ✔ — Simplify.

b. $\tan(x - 180°) = \tan x$

$\tan(x - 180)° = \frac{\tan x - \tan 180°}{1 + \tan x \tan 180°}$ — Tangent Sum Formula

$= \frac{\tan x - 0}{1 + \tan x(0)}$ — $\tan 180° = 0$

$= \tan x$ ✔ — Simplify.

GuidedPractice

Verify each cofunction identity.

6A. $\cos(360° - \theta) = \cos\theta$

6B. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

2 Solve Trigonometric Equations

You can solve trigonometric equations using the sum and difference identities and the same techniques that you used in Lesson 5-3.

Example 7 Solve a Trigonometric Equation

Find the solutions of $\cos\left(\frac{\pi}{3} + x\right) + \cos\left(\frac{\pi}{3} - x\right) = \frac{1}{2}$ on the interval $[0, 2\pi]$.

$\cos\left(\frac{\pi}{3} + x\right) + \cos\left(\frac{\pi}{3} - x\right) = \frac{1}{2}$ — Original equation

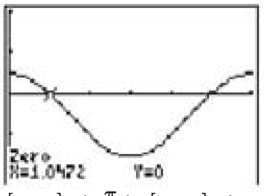
$\cos\frac{\pi}{3}\cos x - \sin\frac{\pi}{3}\sin x + \cos\frac{\pi}{3}\cos x + \sin\frac{\pi}{3}\sin x = \frac{1}{2}$ — Cosine Sum Identities

$\frac{1}{2}(\cos x) - \frac{\sqrt{3}}{2}(\sin x) + \frac{1}{2}(\cos x) + \frac{\sqrt{3}}{2}(\sin x) = \frac{1}{2}$ — Substitute.

$\cos x = \frac{1}{2}$ — Simplify.

On the interval $[0, 2\pi]$, $\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

CHECK The graph of $y = \cos\left(\frac{\pi}{3} + x\right) + \cos\left(\frac{\pi}{3} - x\right) - \frac{1}{2}$ has zeros at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ on the interval $[0, 2\pi]$. ✔



$[0, 2\pi]$ scl: $\frac{\pi}{3}$ by $[-2, 2]$ scl: 1

TechnologyTip

Viewing Window When checking your answer on a graphing calculator, remember that one period for $y = \sin x$ or $y = \cos x$ is 2π and the amplitude is 1. This will help you define the viewing window.

GuidedPractice

7. Find the solutions of $\cos(x + \pi) - \sin(x - \pi) = 0$ on the interval $[0, 2\pi]$.

= Step-by-Step Solutions begin on page R29.

Find the exact value of each trigonometric expression. (Example 1)

1. $\cos 75°$

2. $\sin(-210°)$

3. $\sin \frac{11\pi}{12}$

4. $\cos \frac{17\pi}{12}$

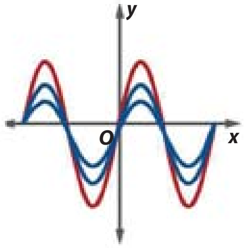
5. $\tan \frac{23\pi}{12}$

6. $\tan \frac{\pi}{12}$

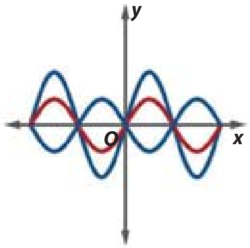
7. **VOLTAGE** Analysis of the voltage in a hairdryer involves terms of the form $\sin(nwt - 90°)$, where n is a positive integer, w is the frequency of the voltage, and t is time. Use an identity to simplify this expression. (Example 2)

8. **BROADCASTING** When the sum of the amplitudes of two waves is greater than that of the component waves, the result is *constructive interference*. When the component waves combine to have a smaller amplitude, *destructive interference* occurs.

Constructive Interference



Destructive Interference



Consider two signals modeled by $y = 10 \sin(2t + 30°)$ and $y = 10 \sin(2t + 210°)$. (Example 2)

a. Find the sum of the two functions.

b. What type of interference results when the signals modeled by the two equations are combined?

9. **WEATHER** The monthly high temperatures for Minneapolis can be modeled by $f(x) = 31.65 \sin\left(\frac{\pi}{6}x - 2.09\right) + 52.35$, where x represents the months in which January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by $g(x) = 31.65 \sin\left(\frac{\pi}{6}x - 2.09\right) + 32.95$. (Example 2)

a. Write a new function $h(x)$ by adding the two functions and dividing the result by 2.

b. What does the function you wrote in part **a** represent?

10. **TECHNOLOGY** A blind mobility aid uses the same idea as a bat's sonar to enable people who are visually impaired to detect objects around them. The sound wave emitted by the device for a certain patient can be modeled by $b = 30(\sin 195°)t$, where t is time in seconds and b is air pressure in pascals. (Example 2)

a. Rewrite the formula in terms of the difference of two angle measures.

b. What is the pressure after 1 second?

Find the exact value of each expression. (Example 3)

11. $\dfrac{\tan 43° - \tan 13°}{1 + \tan 43° \tan 13°}$

12. $\cos \frac{5\pi}{12} \cos \frac{\pi}{4} + \sin \frac{5\pi}{12} \sin \frac{\pi}{4}$

13. $\sin 15° \cos 75° + \cos 15° \sin 75°$

14. $\sin \frac{\pi}{3} \cos \frac{\pi}{12} - \cos \frac{\pi}{3} \sin \frac{\pi}{12}$

15. $\cos 40° \cos 20° - \sin 40° \sin 20°$

16. $\dfrac{\tan 48° + \tan 12°}{1 - \tan 48° \tan 12°}$

Simplify each expression. (Example 3)

17. $\dfrac{\tan 2\theta - \tan \theta}{1 + \tan 2\theta \tan \theta}$

18. $\cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x$

19. $\sin 3y \cos y + \cos 3y \sin y$

20. $\cos 2x \sin x - \sin 2x \cos x$

21. $\cos x \cos 2x + \sin x \sin 2x$

22. $\dfrac{\tan 5\theta + \tan \theta}{\tan 5\theta \tan \theta - 1}$

23. **SCIENCE** An electric circuit contains a capacitor, an inductor, and a resistor. The voltage drop across the inductor is given by $V_L = IwL \cos\left(wt + \frac{\pi}{2}\right)$, where I is the peak current, w is the frequency, L is the inductance, and t is time. Use the cosine sum identity to express V_L as a function of $\sin wt$. (Example 3)

Write each trigonometric expression as an algebraic expression. (Example 4)

24. $\sin(\arcsin x + \arccos x)$

25. $\cos(\sin^{-1} x + \cos^{-1} 2x)$

26. $\cos\left(\sin^{-1} x - \tan^{-1} \frac{\sqrt{3}}{3}\right)$

27. $\sin\left(\sin^{-1} \frac{\sqrt{3}}{2} - \tan^{-1} x\right)$

28. $\cos\left(\arctan \sqrt{3} - \arccos x\right)$

29. $\tan(\cos^{-1} x + \tan^{-1} x)$

30. $\tan\left(\sin^{-1} \frac{1}{2} - \cos^{-1} x\right)$

31. $\tan\left(\sin^{-1} x + \frac{\pi}{4}\right)$

Verify each cofunction identity using one or more difference identities. (Example 5)

32. $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

33. $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

34. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

Verify each reduction identity. (Example 6)

35. $\cos(\pi - \theta) = -\cos\theta$

36. $\cos(2\pi + \theta) = \cos\theta$

37. $\sin(\pi - \theta) = \sin\theta$

38. $\sin(90° + \theta) = \cos\theta$

39. $\cos(270° - \theta) = -\sin\theta$

Find the solution to each expression on the interval $[0, 2\pi]$. (Example 7)

40. $\cos\left(\frac{\pi}{2} + x\right) - \sin\left(\frac{\pi}{2} + x\right) = 0$

41. $\cos(\pi + x) + \cos(\pi + x) = 1$

42. $\cos\left(\frac{\pi}{6} + x\right) + \sin\left(\frac{\pi}{3} + x\right) = 0$

43. $\sin\left(\frac{\pi}{6} + x\right) + \sin\left(\frac{\pi}{6} - x\right) = \frac{1}{2}$

44. $\sin\left(\frac{3\pi}{2} + x\right) + \sin\left(\frac{3\pi}{2} + x\right) = -2$

45. $\tan(\pi + x) + \tan(\pi + x) = 2$

Verify each identity.

46. $\tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y}$

47. $\cot\alpha - \tan\beta = \frac{\cos(\alpha + \beta)}{\sin\alpha\cos\beta}$

48. $\frac{(\tan u - \tan v)}{(\tan u + \tan v)} = \frac{\sin(u - v)}{\sin(u + v)}$

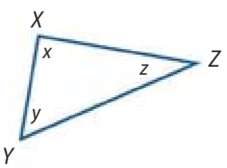
49. $2\sin a\cos b = \sin(a + b) + \sin(a - b)$

GRAPHING CALCULATOR **Graph each function and make a conjecture based on the graph. Verify your conjecture algebraically.**

50. $y = \frac{1}{2}[\sin(x + 2\pi) + \sin(x - 2\pi)]$

51. $y = \cos^2\left(x + \frac{\pi}{4}\right) + \cos^2\left(x - \frac{\pi}{4}\right)$

PROOF **Consider $\triangle XYZ$. Prove each identity.** **(*Hint*: $x + y + z = \pi$)**



52. $\cos(x + y) = -\cos z$

53. $\sin z = \sin x\cos y + \cos x\sin y$

54. $\tan x + \tan y + \tan z = \tan x\tan y\tan z$

55. **CALCULUS** The difference quotient is given by $\frac{f(x + h) - f(x)}{h}$.

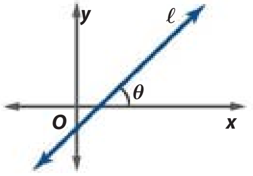
a. Let $f(x) = \sin x$. Write and expand an expression for the difference quotient.

b. Set your answer from part **a** equal to y. Use a graphing calculator to graph the function for the following values of h: 2, 1, 0.1, and 0.01.

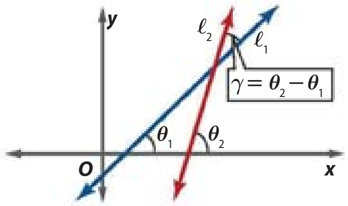
c. What function does the graph in part **b** resemble as h approaches zero?

56. **ANGLE OF INCLINATION** The *angle of inclination* θ of a line is the angle formed between the positive x-axis and the line, where $0° < \theta < 180°$.

a. Prove that the slope m of line ℓ shown at the right is given by $m = \tan\theta$.



b. Consider lines ℓ_1 and ℓ_2 below with slopes m_1 and m_2, respectively. Derive a formula for the angle γ formed by the two lines.



c. Use the formula you found in part **b** to find the angle formed by $y = \frac{\sqrt{3}}{3}x$ and $y = x$.

H.O.T. Problems Use Higher-Order Thinking Skills

PROOF **Verify each identity.**

57. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

58. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

59. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

60. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

61. **REASONING** Use the sum identity for sine to derive an identity for $\sin(x + y + z)$ in terms of sines and cosines.

CHALLENGE **If $\sin x = -\frac{2}{3}$ and $\cos y = \frac{1}{3}$, find each of the following if x is in Quadrant IV and y is in Quadrant I.**

62. $\cos(x + y)$ **63.** $\sin(x - y)$ **64.** $\tan(x + y)$

65. **REASONING** Consider $\sin 3x\cos 2x = \cos 3x\sin 2x$.

a. Find the solutions of the equation over $[0, 2\pi]$ algebraically.

b. Support your answer graphically.

PROOF **Prove each difference quotient identity.**

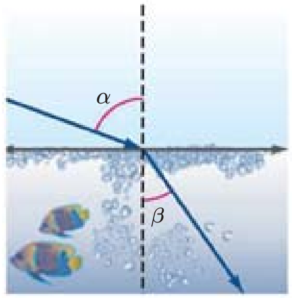
66. $\frac{\sin(x + h) - \sin x}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\frac{\sin h}{h}$

67. $\frac{\cos(x + h) - \cos x}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\frac{\sin h}{h}$

68. **WRITING IN MATH** Can a tangent sum or difference identity be used to solve any tangent reduction formula? Explain your reasoning.

Spiral Review

69. PHYSICS According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water? (Lesson 5-3)



Verify each identity. (Lesson 5-2)

70. $\dfrac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$

71. $\dfrac{\sec \theta}{\sin \theta} - \dfrac{\sin \theta}{\cos \theta} = \cot \theta$

Find the exact value of each expression, if it exists. (Lesson 4-6)

72. $\sin^{-1}(-1)$

73. $\tan^{-1}\sqrt{3}$

74. $\tan\left(\arcsin \frac{3}{5}\right)$

75. MONEY Suppose you deposit a principal amount of P dollars in a bank account that pays compound interest. If the annual interest rate is r (expressed as a decimal) and the bank makes interest payments n times every year, the amount of money A you would have after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$. (Lesson 3-1)

a. If the principal, interest rate, and number of interest payments are known, what type of function is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$? Explain your reasoning.

b. Write an equation giving the amount of money you would have after t years if you deposit \$1000 into an account paying 4% annual interest compounded quarterly (four times per year).

c. Find the account balance after 20 years.

List all possible rational zeros of each function. Then determine which, if any, are zeros. (Lesson 2-4)

76. $p(x) = x^4 + x^3 - 11x - 5x + 30$

77. $d(x) = 2x^4 - x^3 - 6x^2 + 5x - 1$

78. $f(x) = x^3 - 2x^2 - 5x - 6$

Skills Review for Standardized Tests

79. SAT/ACT There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

A 4

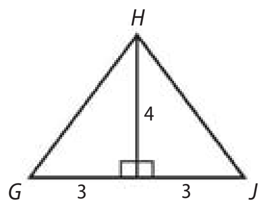
B 6

C 8

D 12

E 16

80. REVIEW Refer to the figure below. Which equation could be used to find $m\angle G$?



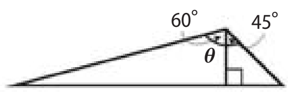
F $\sin G = \frac{3}{4}$

G $\cos G = \frac{3}{4}$

H $\cot G = \frac{3}{4}$

I $\tan G = \frac{3}{4}$

81. Find the exact value of $\sin \theta$.



A $\dfrac{\sqrt{2} + \sqrt{6}}{4}$

B $\dfrac{\sqrt{2} - \sqrt{6}}{4}$

C $\dfrac{2 + \sqrt{3}}{4}$

D $\dfrac{2 - \sqrt{3}}{4}$

82. REVIEW Which of the following is equivalent to $\dfrac{\cos \theta (\cot^2 \theta + 1)}{\csc \theta}$?

F $\tan \theta$

G $\cot \theta$

H $\sec \theta$

J $\csc \theta$

EXTEND 5-4

Graphing Technology Lab
Reduction Identities

Objective

- Use TI-Nspire technology and quadrantal angles to reduce identities.

Another reduction identity involves the sum or difference of the measures of an angle and a quadrantal angle. This can be illustrated by comparing graphs of functions on the unit circle with TI-Nspire technology.

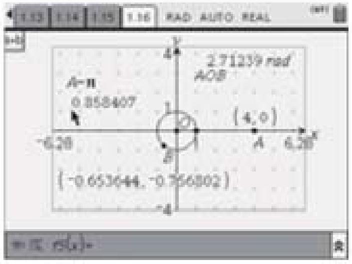
Activity 1 Use the Unit Circle

Use the unit circle to explore a reduction identity graphically.

Step 1 Add a Graphs page. Select Zoom-Trig from the Window menu, and select Show Grid from the View menu. From the File menu under Tools, choose Document Settings, set the Display Digits to Float 2, and confirm that the angle measure is in radians.

Step 2 Select Points & Lines and then Point from the menu. Place a point at (1, 0). Next, select Shapes, and then Circle from the menu. To construct a circle centered at the origin through (1, 0), click on the screen and define the center point at the origin. Move the cursor away from the center, and the circle will appear. Stop when you get to a radius of 1 and (1, 0) lies on the circle.

Step 3 Place a point to the right of the circle on the *x*-axis, and label it *A*. Choose Actions and then Coordinates and Equations from the menu, and then double-click the point to display its coordinates. From the Construction menu, choose Measurement transfer. Select the *x*-coordinate of *A*, the circle, and the point at (1, 0). Label the point created on the circle as *B*, and display its coordinates.

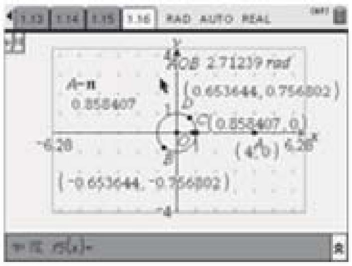


Step 4 With *O* as the origin, calculate and label the measure of ∠*AOB*. Select Text from the Actions menu to write the expression *a* − π. Then select Calculate from the Actions menu to calculate the difference of the *x*-coordinate of *A* and π.

Step 5 Move *A* along the *x*-axis, and observe the effect on the measure of ∠*AOB*.

Location of *A*	*m*∠AOB (radians)
(4, 0)	2.7124
(3, 0)	3.0708
(2, 0)	2.5708
(5, 0)	2.2123
(−2, 0)	0.5708

Step 6 From the Construction menu, choose Measurement transfer. Select the *x*-axis and the value of *a* − π. Label the point as *C* and display its coordinates. Using Measurement transfer again, select the *x*-coordinate of *C*, the circle, and the point at (1, 0). Label the point as *D* and display its coordinates.



Analyze the Results

1A. In Step 5, how are the location of *A* and the measure of ∠*AOB* related?

1B. Consider the locations of points *B* and *D*. What reduction identity or identities does this relationship suggest are true?

1C. **MAKE A CONJECTURE** If you change the expression *a* − π to *a* + π, what reduction identities do you think would result?

StudyTip

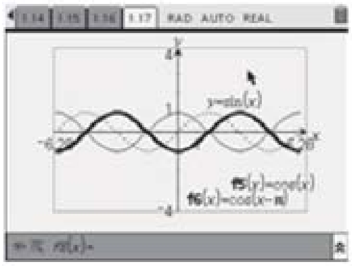
Angle Measure The TI-Nspire only measures angles between 0 and π.

Activity 2 Use Graphs

Use graphs to identify equal trigonometric functions.

Step 1 Open a new Graphs page. Select Zoom-Trig from the Window menu.

Step 2 Graph $f(x) = \cos x$, $f(x) = \cos(x - \pi)$, and $f(x) = \sin x$. Using the Attributes feature from the Actions menu, make the line weight of $f(x) = \cos(x - \pi)$ medium and the line style of $f(x) = \sin x$ dotted.



Step 3 Use translations, reflections, or dilations to transform $f(x) = \sin x$ so that the graph coincides with $f(x) = \cos x$. Select the graph and drag it over $f(x) = \cos x$. As you move the graph, its function will change on the screen.

Step 4 Use translations, reflections, or dilations to transform $f(x) = \cos(x - \pi)$ so that the graph coincides with the other two graphs. Again, as you move the graph, its function will change on the screen.

Analyze the Results

2A. Write the identity that results from your transformation of $f(x) = \sin x$ in Step 3. Graph the functions to confirm your identity.

2B. Write the identity that results from your alteration of $f(x) = \cos(x - \pi)$ in Step 3. Graph the functions to confirm your identity.

2C. **MAKE A CONJECTURE** What does the reflection of a graph suggest for the purpose of developing an identity? a translation?

Exercises

Use the unit circle to write an identity relating the given expressions. Verify your identity by graphing.

1. $\cos(90° - x)$, $\sin x$

2. $\cos\left(\frac{3\pi}{2} - x\right)$, $\sin x$

Insert the trigonometric function that completes each identity.

3. $\cos x =$ ________ $\left(x - \frac{3\pi}{2}\right)$

4. $\cot x =$ ________ $(x + 90°)$

5. $\sec x =$ ________ $(x - 180°)$

6. $\csc x =$ ________ $\left(x + \frac{\pi}{2}\right)$

Use transformations to find the value of *a* for each expression.

7. $\sin ax = 2 \sin x \cos x$

8. $\cos 4ax = \cos^2 x - \sin^2 x$

9. $a \sin^2 x = 1 - \cos 2x$

10. $1 + \cos 6ax = 2 \cos^2 x$

LESSON 5-5 Multiple-Angle and Product-to-Sum Identities

Then

- You proved and used sum and difference identities. (Lesson 5-4)

Now

1. Use double-angle, power-reducing, and half-angle identities to evaluate trigonometric expressions and solve trigonometric equations.
2. Use product-to-sum identities to evaluate trigonometric expressions and solve trigonometric equations.

Why?

- The speed at which a plane travels can be described by a *mach number*, a ratio of the plane's speed to the speed of sound. Exceeding the speed of sound produces a shock wave in the shape of a cone behind the plane. The angle θ at the vertex of this cone is related to the mach number M describing the plane's speed by the half-angle equation $\sin \frac{\theta}{2} = \frac{1}{M}$.

1 Use Multiple-Angle Identities

By letting α and β both equal θ in each of the angle sum identities you learned in the previous lesson, you can derive the following double-angle identities.

KeyConcept Double-Angle Identities

$\sin 2\theta = 2 \sin \theta \cos \theta$	$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
$\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$	$\cos 2\theta = 2 \cos^2 \theta - 1$
	$\cos 2\theta = 1 - 2 \sin^2 \theta$

Proof Double-Angle Identity for Sine

$\sin 2\theta = \sin(\theta + \theta)$	$2\theta = \theta + \theta$
$= \sin \theta \cos \theta + \cos \theta \sin \theta$	Sine Sum Identity where $\alpha = \beta = \theta$
$= 2 \sin \theta \cos \theta$	Simplify.

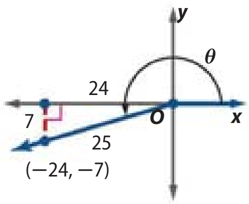
You will prove the double-angle identities for cosine and tangent in Exercises 63–65.

Example 1 Evaluate Expressions Involving Double Angles

If $\sin \theta = -\frac{7}{25}$ on the interval $\left(\pi, \frac{3\pi}{2}\right)$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Since $\sin \theta = \frac{y}{r} = -\frac{7}{25}$ on the interval $\left(\pi, \frac{3\pi}{2}\right)$, one point on the terminal side of θ has y-coordinate -7 and a distance of 25 units from the origin, as shown. The x-coordinate of this point is therefore $-\sqrt{25^2 - 7^2}$ or -24. Using this point, we find that

$$\cos \theta = \frac{x}{r} \text{ or } -\frac{24}{25} \quad \text{and} \quad \tan \theta = \frac{y}{x} \text{ or } \frac{7}{25}.$$



Use these values and the double-angle identities for sine and cosine to find $\sin 2\theta$ and $\cos 2\theta$. Then find $\tan 2\theta$ using either the tangent double-angle identity or the definition of tangent.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$= 2\left(-\frac{7}{25}\right)\left(-\frac{24}{25}\right) \text{ or } \frac{336}{625}$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$
$$= 2\left(-\frac{24}{25}\right)^2 - 1 \text{ or } \frac{527}{625}$$

Method 1 $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$

$$= \frac{2\left(\frac{7}{24}\right)}{1 - \left(\frac{7}{24}\right)^2} \text{ or } \frac{336}{527}$$

Method 2 $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta}$

$$= \frac{\frac{336}{625}}{\frac{527}{625}} \text{ or } \frac{336}{527}$$

GuidedPractice

1. If $\cos \theta = \frac{3}{5}$ on the interval $\left(0, \frac{\pi}{2}\right)$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

StudyTip

More than One Identity Notice that there are three identities associated with $\cos 2\theta$. While there are other identities that could also be associated with $\sin 2\theta$ and $\tan 2\theta$, those associated with $\cos 2\theta$ are worth memorizing because they are more commonly used.

Example 2 Solve an Equation Using a Double-Angle Identity

Solve $\sin 2\theta - \sin \theta = 0$ on the interval $[0, 2\pi]$.

Use the sine double-angle identity to rewrite the equation as a function of a single angle.

$\sin 2\theta - \sin \theta = 0$	Original equation
$2 \sin \theta \cos \theta - \sin \theta = 0$	Sine Double-Angle Identity
$\sin \theta (2 \cos \theta - 1) = 0$	Factor.
$\sin \theta = 0$ or $2 \cos \theta - 1 = 0$	Zero Product Property
$\theta = 0$ or π ; $\cos \theta = \frac{1}{2}$	Therefore, $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

The solutions on the interval $[0, 2\pi]$ are $\theta = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

GuidedPractice **Solve each equation on the interval $[0, 2\pi]$.**

2A. $\cos 2\alpha = -\sin^2 \alpha$

2B. $\tan 2\beta = 2 \tan \beta$

Math HistoryLink

François Viète (1540–1603)
Born in a village in western France, Viète was called to Paris to decipher messages for King Henri III. Extremely skilled at manipulating equations, he used double-angle identities for sine and cosine to derive triple-, quadruple-, and quintuple-angle identities.

SPL/Photo Researchers, Inc.

The double angle identities can be used to derive the power-reducing identities below. These identities make calculus-related manipulations of functions like $y = \cos^2 x$ much easier.

KeyConcept Power-Reducing Identities

$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$ $\quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ $\quad \tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$

Proof Power-Reducing Identity for Sine

$\frac{1 - \cos 2\theta}{2} = \frac{1 - (1 - 2\sin^2 \theta)}{2}$	Cosine Double-Angle Identity
$= \frac{2 \sin^2 \theta}{2}$	Subtract.
$= \sin^2 \theta$	Simplify.

You will prove the power-reducing identities for cosine and tangent in Exercises 82 and 83.

Example 3 Use an Identity to Reduce a Power

Rewrite $\sin^4 x$ in terms with no power greater than 1.

$\sin^4 x = (\sin^2 x)^2$	$(\sin^2 x)^2 = \sin^4 x$
$= \left(\frac{1 - \cos 2x}{2}\right)^2$	Sine Power-Reducing Identity
$= \frac{1 - 2\cos 2x + \cos^2 2x}{4}$	Multiply.
$= \frac{1 - 2\cos 2x + \frac{1 + \cos 4x}{2}}{4}$	Cosine Power-Reducing Identity
$= \frac{2 - 4\cos 2x + 1 + \cos 4x}{8}$	Common denominator
$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$	Factor.

GuidedPractice

Rewrite each expression in terms with no power greater than 1.

3A. $\cos^4 x$

3B. $\sin^3 \theta$

Example 4 Solve an Equation Using a Power-Reducing Identity

Solve $\cos^2 x - \cos 2x = \frac{1}{2}$.

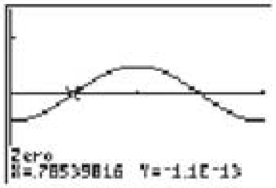
Solve Algebraically

$\cos^2 x - \cos 2x = \frac{1}{2}$	Original equation
$\frac{1 + \cos 2x}{2} - \cos 2x = \frac{1}{2}$	Cosine Power-Reducing Identity
$1 + \cos 2x - 2\cos 2x = 1$	Multiply each side by 2.
$\cos 2x - 2\cos 2x = 0$	Subtract 1 from each side.
$-\cos 2x = 0$	Subtract like terms.
$\cos 2x = 0$	Multiply each side by −1.
$2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$	Solutions for double angle in $[0, 2\pi]$
$x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$	Divide each solution by 2.

The graph of $y = \cos 2x$ has a period of π, so the solutions are $x = \frac{\pi}{4} + n\pi$ or $\frac{3\pi}{4} + n\pi, n \in \mathbb{Z}$.

Support Graphically

The graph of $y = \cos^2 x - \cos 2x - \frac{1}{2}$ has zeros at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ on the interval $[0, \pi]$. ✔



$[0, \pi]$ scl: $\frac{\pi}{4}$ by $[-1.5, 1.5]$ scl: 1

GuidedPractice **Solve each equation.**

4A. $\cos^4 \alpha - \sin^4 \alpha = \frac{1}{2}$

4B. $\sin^2 3\beta = \sin^2 \beta$

By replacing θ with $\frac{\theta}{2}$ in each of the power-reducing identities, you can derive each of the following half-angle identities. The sign of each identity that involves the ± symbol is determined by checking the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

WatchOut!

Determining Signs To determine which sign is appropriate when using a half-angle identity, check the quadrant in which $\frac{\theta}{2}$ lies, *not* the quadrant in which θ lies.

KeyConcept Half-Angle Identities

$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$

$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$

$\tan \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$

$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$

$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$

Proof Half-Angle Identity for Cosine

$\pm\sqrt{\frac{1 + \cos\theta}{2}} = \pm\sqrt{\frac{1 + \cos\left(2 \cdot \frac{\theta}{2}\right)}{2}}$	Rewrite θ as $2 \cdot \frac{\theta}{2}$.
$= \pm\sqrt{\frac{1 + \cos 2x}{2}}$	Substitute $x = \frac{\theta}{2}$.
$= \pm\sqrt{\cos^2 x}$	Cosine Power-Reducing Identity
$= \cos x$	Simplify.
$= \cos \frac{\theta}{2}$	Substitute.

You will prove the half-angle identities for sine and tangent in Exercises 66–68.

Example 5 Evaluate an Expression Involving a Half Angle

Find the exact value of cos 112.5°.

Notice that 112.5° is half of 225°. Therefore, apply the half-angle identity for cosine, noting that since 112.5° lies in Quadrant II, its cosine is negative.

$\cos 112.5° = \cos \frac{225°}{2}$	$112.5° = \frac{225°}{2}$
$= -\sqrt{\frac{1 + \cos 225°}{2}}$	Cosine Half-Angle Identity (Quadrant II angle)
$= -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$	$\cos 225° = -\frac{\sqrt{2}}{2}$
$= -\sqrt{\frac{2 - \sqrt{2}}{4}}$	Subtract and then divide.
$= -\frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}}$ or $-\frac{\sqrt{2 - \sqrt{2}}}{2}$	Quotient Property of Square Roots

CHECK Use a calculator to support your assertion that $\cos 112.5° = -\frac{\sqrt{2 - \sqrt{2}}}{2}$.

$\cos 112.5° \approx -0.3826834324$ and $-\frac{\sqrt{2 - \sqrt{2}}}{2} \approx -0.3826834324$ ✓

GuidedPractice

Find the exact value of each expression.

5A. sin 75°

5B. $\tan \frac{7\pi}{12}$

StudyTip

Tangent Half-Angle Identities When evaluating the tangent function for half-angle values, it is usually easiest to use the form of the tangent half-angle identity $\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$ since its denominator has only one term.

Recall that you can use sum and difference identities to solve equations. Half-angle identities can also be used to solve equations.

Example 6 Solve an Equation Using a Half-Angle Identity

Solve $\sin^2 x = 2\cos^2 \frac{x}{2}$ on the interval $[0, 2\pi]$.

$\sin^2 x = 2\cos^2 \frac{x}{2}$	Original equation
$\sin^2 x = 2\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2$	Cosine Half-Angle Identity
$\sin^2 x = 2\left(\frac{1 + \cos x}{2}\right)$	Simplify.
$\sin^2 x = 1 + \cos x$	Multiply.
$1 - \cos^2 x = 1 + \cos x$	Pythagorean Identity
$-\cos^2 x - \cos x = 0$	Subtract 1 from each side.
$\cos x(-\cos x - 1) = 0$	Factor.

$\cos x = 0$ or $-\cos x - 1 = 0$ — Zero Product Property

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ — $\cos x = -1$; therefore, $x = \pi$. — Solutions in $[0, 2\pi]$

The solutions on the interval $[0, 2\pi]$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or π.

GuidedPractice

Solve each equation on the interval $[0, 2\pi]$.

6A. $2\sin^2 \frac{x}{2} + \cos x = 1 + \sin x$

6B. $8\tan \frac{x}{2} + 8\cos x \tan \frac{x}{2} = 1$

2 Use Product-to-Sum Identities

To work with functions such as $y = \cos 5x \sin 3x$ in calculus, you will need to apply one of the following product-to-sum identities.

KeyConcept Product-to-Sum Identities

$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

StudyTip

Proofs Remember to work the more complicated side first when proving these identities.

Proof Product-to-Sum Identity for $\sin \alpha \cos \beta$

$\frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$	More complicated side of identity
$= \frac{1}{2}(\sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta)$	Sum and Difference Identities
$= \frac{1}{2}(2 \sin \alpha \cos \beta)$	Combine like terms.
$= \sin \alpha \cos \beta$	Multiply.

You will prove the remaining three product-to-sum identities in Exercises 84–86.

Example 7 Use an Identity to Write a Product as a Sum or Difference

Rewrite $\cos 5x \sin 3x$ as a sum or difference.

$\cos 5x \sin 3x = \frac{1}{2}[\sin(5x + 3x) - \sin(5x - 3x)]$	Product-to-Sum Identity
$= \frac{1}{2}(\sin 8x - \sin 2x)$	Simplify.
$= \frac{1}{2}\sin 8x - \frac{1}{2}\sin 2x$	Distributive Property

GuidedPractice **Rewrite each product as a sum or difference.**

7A. $\sin 4\theta \cos \theta$

7B. $\sin 7x \sin 6x$

These product-to-sum identities have corresponding sum-to-product identities.

KeyConcept Sum-to-Product Identities

$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$

$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$

$\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$

$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$

Proof Sum-to-Product Identity for $\sin \alpha + \sin \beta$

$2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$	
$= 2 \sin x \cos y$	Substitute $x = \frac{\alpha + \beta}{2}$ and $y = \frac{\alpha - \beta}{2}$.
$= 2\left\{\frac{1}{2}[\sin(x + y) + \sin(x - y)]\right\}$	Product-to-Sum Identity
$= \sin\left(\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2}\right) + \sin\left(\frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2}\right)$	Substitute and simplify.
$= \sin\left(\frac{2\alpha}{2}\right) + \sin\left(\frac{2\beta}{2}\right)$	Combine fractions.
$= \sin \alpha + \sin \beta$	Simplify.

You will prove the remaining three sum-to-product identities in Exercises 87–89.

Example 8 Use a Product-to-Sum or Sum-to-Product Identity

Find the exact value of $\sin \frac{5\pi}{12} + \sin \frac{\pi}{12}$.

$$\sin \frac{5\pi}{12} + \sin \frac{\pi}{12} = 2 \sin \left(\frac{\frac{5\pi}{12} + \frac{\pi}{12}}{2}\right) \cos \left(\frac{\frac{5\pi}{12} - \frac{\pi}{12}}{2}\right) \quad \text{Sum-to-Product Identity}$$

$$= 2 \sin \frac{\pi}{4} \cos \frac{\pi}{6} \quad \text{Simplify.}$$

$$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \quad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{6}}{2} \quad \text{Simplify.}$$

GuidedPractice

Find the exact value of each expression.

8A. $3 \cos 37.5° \cos 187.5°$

8B. $\cos \frac{7\pi}{12} - \cos \frac{\pi}{12}$

You can also use sum-to-product identities to solve some trigonometric equations.

Example 9 Solve an Equation Using a Sum-to-Product Identity

Solve $\cos 4x + \cos 2x = 0$.

Solve Algebraically

$$\cos 4x + \cos 2x = 0 \quad \text{Original equation}$$

$$2 \cos \left(\frac{4x + 2x}{2}\right) \cos \left(\frac{4x - 2x}{2}\right) = 0 \quad \text{Cosine Sum-to-Product Identity}$$

$$(2 \cos 3x)(\cos x) = 0 \quad \text{Simplify.}$$

Set each factor equal to zero and find solutions on the interval $[0, 2\pi]$.

$2 \cos 3x = 0$ — First factor set equal to 0

$\cos 3x = 0$ — Divide each side by 2.

$3x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ — Multiple angle solutions in $[0, 2\pi]$

$x = \frac{\pi}{6}$ or $\frac{\pi}{2}$ — Divide each solution by 3.

$\cos x = 0$ — Second factor set equal to 0

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ — Solutions in $[0, 2\pi]$

WatchOut

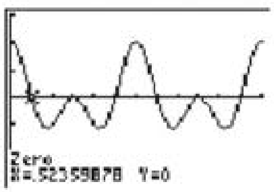
Periods for Multiple Angle Trigonometric Functions Recall from Lesson 4-4 that the periods of $y = \sin kx$ and $y = \cos kx$ are $\frac{2\pi}{k}$, not 2π.

The period of $y = \cos 3x$ is $\frac{2\pi}{3}$, so the solutions are

$$x = \frac{\pi}{6} + \frac{2\pi}{3}n, \quad \frac{\pi}{2} + \frac{2\pi}{3}n, \quad \frac{\pi}{2} + 2\pi n, \text{ or } \quad \frac{3\pi}{2} + 2\pi n, \ n \in \mathbb{Z}.$$

Support Graphically

The graph of $y = \cos 4x + \cos 2x$ has zeros at $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$ on the interval $[0, 2\pi]$. ✔



$[0, 2\pi]$ scl: $\frac{\pi}{6}$ by $[-3, 3]$ scl: 1

GuidedPractice

Solve each equation.

9A. $\sin x + \sin 5x = 0$

9B. $\cos 3x - \cos 5x = 0$

Exercises

= Step-by-Step Solutions begin on page R29.

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval. (Example 1)

1. $\cos\theta = \frac{3}{5}$, (270°, 360°)

2. $\tan\theta = \frac{8}{15}$, (180°, 270°)

3. $\cos\theta = -\frac{9}{41}$, (90°, 180°)

4. $\sin\theta = -\frac{7}{12}$, $\left(\frac{3\pi}{2}, 2\pi\right)$

5. $\tan\theta = -\frac{1}{2}$, $\left(\frac{3\pi}{2}, 2\pi\right)$

6. $\tan\theta = \sqrt{3}$, $\left(0, \frac{\pi}{2}\right)$

7. $\sin\theta = \frac{4}{5}$, $\left(\frac{\pi}{2}, \pi\right)$

8. $\cos\theta = -\frac{5}{13}$, $\left(\pi, \frac{3\pi}{2}\right)$

Solve each equation on the interval [0, 2π]. (Example 2)

9. $\sin 2\theta = \cos\theta$

10. $\cos 2\theta = \cos\theta$

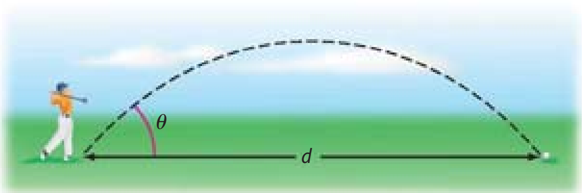
11. $\cos 2\theta - \sin\theta = 0$

12. $\tan 2\theta - \tan 2\theta \tan^2\theta = 2$

13. $\sin 2\theta \csc\theta = 1$

14. $\cos 2\theta + 4\cos\theta = -3$

15. **GOLF** A golf ball is hit with an initial velocity of 88 feet per second. The distance the ball travels is found by $d = \frac{v_0{}^2 \sin 2\theta}{32}$, where v_0 is the initial velocity, θ is the angle that the path of the ball makes with the ground, and 32 is in feet per second squared. (Example 2)



a. If the ball travels 242 feet, what is θ to the nearest degree?

b. Use a double-angle identity to rewrite the equation for d.

Rewrite each expression in terms with no power greater than 1. (Example 3)

16. $\cos^3\theta$

17. $\tan^3\theta$

18. $\sec^4\theta$

19. $\cot^3\theta$

20. $\cos^4\theta - \sin^4\theta$

21. $\sin^2\theta\cos^3\theta$

22. $\sin^2\theta - \cos^2\theta$

23. $\frac{\sin^4\theta}{\cos^2\theta}$

Solve each equation. (Example 4)

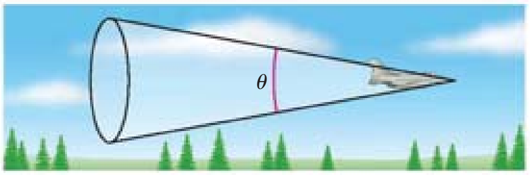
24. $1 - \sin^2\theta - \cos 2\theta = \frac{1}{2}$

25. $\cos^2\theta - \frac{3}{2}\cos 2\theta = 0$

26. $\sin^2\theta - 1 = \cos^2\theta$

27. $\cos^2\theta - \sin\theta = 1$

28. **MACH NUMBER** The angle θ at the vertex of the cone-shaped shock wave produced by a plane breaking the sound barrier is related to the mach number M describing the plane's speed by the half-angle equation $\sin\frac{\theta}{2} = \frac{1}{M}$. (Example 5)



a. Express the mach number of the plane in terms of cosine.

b. Use the expression found in part **a** to find the mach number of a plane if $\cos\theta = \frac{4}{5}$.

Find the exact value of each expression.

29. sin 67.5°

30. $\cos\frac{\pi}{12}$

31. tan 157.5°

32. $\sin\frac{11\pi}{12}$

Solve each equation on the interval [0, 2π]. (Example 6)

33. $\sin\frac{\theta}{2} + \cos\theta = 1$

34. $\tan\frac{\theta}{2} = \sin\frac{\theta}{2}$

35. $2\sin\frac{\theta}{2} = \sin\theta$

36. $1 - \sin^2\frac{\theta}{2} - \cos\frac{\theta}{2} = \frac{3}{4}$

Rewrite each product as a sum or difference. (Example 7)

37. $\cos 3\theta\cos\theta$

38. $\cos 12x \sin 5x$

39. $\sin 3x\cos 2x$

40. $\sin 8\theta\sin\theta$

Find the exact value of each expression. (Example 8)

41. 2 sin 135° sin 75°

42. $\cos\frac{7\pi}{12} + \cos\frac{\pi}{12}$

43. $\frac{2}{3}$ sin 172.5° sin 127.5°

44. sin 142.5° cos 352.5°

45. sin 75° + sin 195°

46. 2 cos 105° + 2 cos 195°

47. $3\sin\frac{17\pi}{12} - 3\sin\frac{\pi}{12}$

48. $\cos\frac{13\pi}{12} + \cos\frac{5\pi}{12}$

Solve each equation. (Example 9)

49. $\cos\theta - \cos 3\theta = 0$

50. $2\cos 4\theta + 2\cos 2\theta = 0$

51. $\sin 3\theta + \sin 5\theta = 0$

52. $\sin 2\theta - \sin\theta = 0$

53. $3\cos 6\theta - 3\cos 4\theta = 0$

54. $4\sin\theta + 4\sin 3\theta = 0$

Simplify each expression.

55. $\sqrt{\frac{1+\cos 6x}{2}}$

56. $\sqrt{\frac{1-\cos 16\theta}{2}}$

Write each expression as a sum or difference.

57. $\cos(a+b)\cos(a-b)$

58. $\sin(\theta-\pi)\sin(\theta+\pi)$

59. $\sin(b+\theta)\cos(b+\pi)$

60. $\cos(a-b)\sin(b-a)$

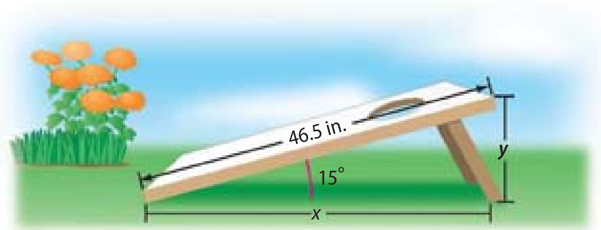
61. MAPS A Mercator projection is a flat projection of the globe in which the distance between the lines of latitude increases with their distance from the equator.

The calculation of a point on a Mercator projection contains the expression $\tan\left(45° + \frac{\ell}{2}\right)$, where ℓ is the latitude of the point.

a. Write the expression in terms of $\sin \ell$ and $\cos \ell$.

b. Find the value of this expression if $\ell = 60°$.

62. BEAN BAG TOSS Ivan constructed a bean bag tossing game as shown in the figure below.



a. Exactly how far will the back edge of the board be from the ground?

b. Exactly how long is the entire setup?

PROOF **Prove each identity.**

63. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

64. $\cos 2\theta = 2\cos^2\theta - 1$

65. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

66. $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

67. $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

68. $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

Verify each identity by using the power-reducing identities and then again by using the product-to-sum identities.

69. $2\cos^2 5\theta - 1 = \cos 10\theta$

70. $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

Rewrite each expression in terms of cosines of multiple angles with no power greater than 1.

71. $\sin^6\theta$

72. $\sin^8\theta$

73. $\cos^7\theta$

74. $\sin^4\theta\cos^4\theta$

75. MULTIPLE REPRESENTATIONS In this problem, you will investigate how graphs of functions can be used to find identities.

a. GRAPHICAL Use a graphing calculator to graph $f(x) = 4\left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)$ on the interval $[-2\pi, 2\pi]$.

b. ANALYTICAL Write a sine function $h(x)$ that models the graph of $f(x)$. Then verify that $f(x) = h(x)$ algebraically.

c. GRAPHICAL Use a graphing calculator to graph $g(x) = \cos^2\left(\theta - \frac{\pi}{3}\right) - \sin^2\left(\theta - \frac{\pi}{3}\right)$ on the interval $[-2\pi, 2\pi]$.

d. ANALYTICAL Write a cosine function $k(x)$ that models the graph of $g(x)$. Then verify that $g(x) = k(x)$ algebraically.

H.O.T. Problems Use Higher-Order Thinking Skills

76. CHALLENGE Verify the following identity.

$$\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = \sin\theta$$

REASONING **Consider an angle in the unit circle. Determine what quadrant a double angle and half angle would lie in if the terminal side of the angle is in each quadrant.**

77. I

78. II

79. III

CHALLENGE **Verify each identity.**

80. $\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$

81 $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

PROOF **Prove each identity.**

82. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

83. $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

84. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

85. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

86. $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

87. $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

88. $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

89. $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

90. WRITING IN MATH Describe the steps that you would use to find the exact value of $\cos 8\theta$ if $\cos\theta = \frac{\sqrt{2}}{5}$.

Spiral Review

Find the exact value of each trigonometric expression. (Lesson 5-4)

91. $\cos \frac{\pi}{12}$

92. $\cos \frac{19\pi}{12}$

93. $\sin \frac{5\pi}{6}$

94. $\sin \frac{13\pi}{12}$

95. $\cos \left(-\frac{7\pi}{6}\right)$

96. $\sin \left(-\frac{7\pi}{12}\right)$

97. GARDENING Eliza is waiting for the first day of spring in which there will be 14 hours of daylight to start a flower garden. The number of hours of daylight H in her town can be modeled by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year, $d = 1$ represents January 1, $d = 2$ represents January 2, and so on. On what day will Eliza begin gardening? (Lesson 5-3)

Find the exact value of each expression. If undefined, write *undefined*. (Lesson 4-3)

98. $\csc \left(-\frac{\pi}{3}\right)$

99. $\tan 210°$

100. $\sin \frac{19\pi}{4}$

101. $\cos (-3780°)$

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Lesson 2-1)

102. $f(x) = -\frac{1}{5}x^{\frac{2}{3}}$

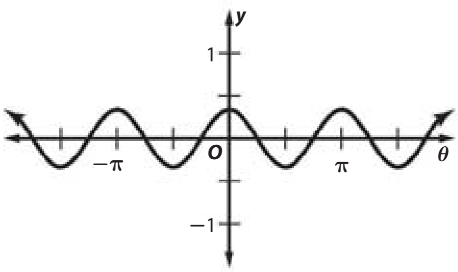
103. $f(x) = 4x^{\frac{5}{4}}$

104. $f(x) = -3x^6$

105. $f(x) = 4x^5$

Skills Review for Standardized Tests

106. REVIEW Identify the equation for the graph.



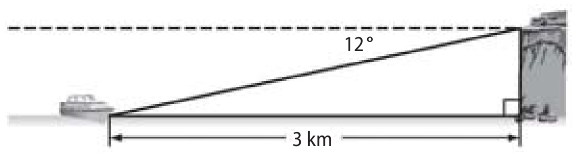
A $y = 3 \cos 2\theta$

B $y = \frac{1}{3} \cos 2\theta$

C $y = 3 \cos \frac{1}{2}\theta$

D $y = \frac{1}{3} \cos \frac{1}{2}\theta$

107. REVIEW From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point?



F $\frac{3}{\sin 12°}$

G $\frac{3}{\tan 12°}$

H $\frac{3}{\cos 12°}$

J $3 \tan 12°$

108. FREE RESPONSE Use the graph to answer each of the following.

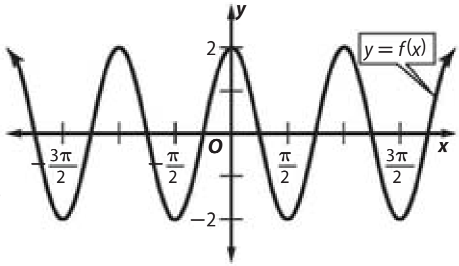
a. Write a function of the form $f(x) = a \cos (bx + c) + d$ that corresponds to the graph.

b. Rewrite $f(x)$ as a sine function.

c. Rewrite $f(x)$ as a cosine function of a single angle.

d. Find all solutions of $f(x) = 0$.

e. How do the solutions that you found in part **d** relate to the graph of $f(x)$?



CHAPTER 5

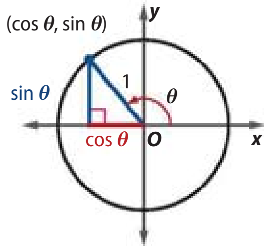
Study Guide and Review

Chapter Summary

KeyConcepts

Trigonometric Identities (Lesson 5-1)

- Trigonometric identities are identities that involve trigonometric functions and can be used to find trigonometric values.
- Trigonometric expressions can be simplified by writing the expression in terms of one trigonometric function or in terms of sine and cosine only.
- The most common trigonometric identity is the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.



Verifying Trigonometric Identities (Lesson 5-2)

- Start with the more complicated side of the identity and work to transform it into the simpler side.
- Use reciprocal, quotient, Pythagorean, and other basic trigonometric identities.
- Use algebraic operations such as combining fractions, rewriting fractions as sums or differences, multiplying expressions, or factoring expressions.
- Convert a denominator of the form $1 \pm u$ or $u \pm 1$ to a single term using its conjugate and a Pythagorean Identity.
- Work each side separately to reach a common expression.

Solving Trigonometric Equations (Lesson 5-3)

- Algebraic techniques that can be used to solve trigonometric equations include isolating the trigonometric expression, taking the square root of each side, and factoring.
- Trigonometric identities can be used to solve trigonometric equations by rewriting the equation using a single trigonometric function or by squaring each side to obtain an identity.

Sum and Difference Identities (Lesson 5-4)

- Sum and difference identities can be used to find exact values of trigonometric functions of uncommon angles.
- Sum and difference identities can also be used to rewrite a trigonometric expression as an algebraic expression.

Multiple-Angle and Product-Sum Identities (Lesson 5-5)

- Trigonometric identities can be used to find the values of expressions that otherwise could not be evaluated.

KeyVocabulary

cofunction (p. 314)
difference identity (p. 337)
double-angle identity (p. 346)
half-angle identity (p. 348)
identity (p. 312)
odd-even identity (p. 314)
power-reducing identity (p. 347)
product-to-sum identity (p. 350)
Pythagorean identity (p. 313)
quotient identity (p. 312)
reciprocal identity (p. 312)
reduction identity (p. 340)
sum identity (p. 337)
trigonometric identity (p. 312)
verify an identity (p. 320)

VocabularyCheck

Complete each identity by filling in the blank. Then name the identity.

1. $\sec \theta =$ ________
2. ________ $= \dfrac{\sin \theta}{\cos \theta}$
3. ________ $+ 1 = \sec^2 \theta$
4. $\cos(90° - \theta) =$ ________
5. $\tan(-\theta) =$ ________
6. $\sin(\alpha + \beta) = \sin \alpha$ ________ $+ \cos \alpha$ ________
7. ________ $= \cos^2 \alpha - \sin^2 \alpha$
8. ________ $= \pm\sqrt{\dfrac{1 + \cos \theta}{2}}$
9. $\dfrac{1 - \cos 2\theta}{2} =$ ________
10. ________ $= \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

Lesson-by-Lesson Review

5-1 Trigonometric Identities (pp. 312–319)

Find the value of each expression using the given information.

11. $\sec\theta$ and $\cos\theta$; $\tan\theta = 3$, $\cos\theta > 0$

12. $\cot\theta$ and $\sin\theta$; $\cos\theta = -\frac{1}{5}$, $\tan\theta < 0$

13. $\csc\theta$ and $\tan\theta$; $\cos\theta = \frac{3}{5}$, $\sin\theta < 0$

14. $\cot\theta$ and $\cos\theta$; $\tan\theta = \frac{2}{7}$, $\csc\theta > 0$

15. $\sec\theta$ and $\sin\theta$; $\cot\theta = -2$, $\csc\theta < 0$

16. $\cos\theta$ and $\sin\theta$; $\cot\theta = \frac{3}{8}$, $\sec\theta < 0$

Simplify each expression.

17. $\sin^2(-x) + \cos^2(-x)$

18. $\sin^2 x + \cos^2 x + \cot^2 x$

19. $\frac{\sec^2 x - \tan^2 x}{\cos(-x)}$

20. $\frac{\sec^2 x}{\tan^2 x + 1}$

21. $\frac{1}{1 - \sin x}$

22. $\frac{\cos x}{1 + \sec x}$

Example 1

If $\sec\theta = -3$ and $\sin\theta > 0$, find $\sin\theta$.

Since $\sin\theta > 0$ and $\sec\theta < 0$, θ must be in Quadrant II. To find $\sin\theta$, first find $\cos\theta$ using the Reciprocal Identity for $\sec\theta$ and $\cos\theta$.

$\cos\theta = \frac{1}{\sec\theta}$ — Reciprocal Identity

$= -\frac{1}{3}$ — $\sec\theta = -3$

Now you can use the Pythagorean identity that includes $\sin\theta$ and $\cos\theta$ to find $\sin\theta$.

$\sin^2\theta + \cos^2\theta = 1$ — Pythagorean Identity

$\sin^2\theta + \left(-\frac{1}{3}\right)^2 = 1$ — $\cos\theta = -\frac{1}{3}$

$\sin^2\theta + \frac{1}{9} = 1$ — Multiply.

$\sin^2\theta = \frac{8}{9}$ — Subtract.

$\sin\theta = \frac{\sqrt{8}}{3}$ or $\frac{2\sqrt{2}}{3}$ — Simplify.

5-2 Verifying Trigonometric Identities (pp. 320–326)

Verify each identity.

23. $\frac{\sin\theta}{1 - \cos\theta} + \frac{\sin\theta}{1 + \cos\theta} = 2\csc\theta$

24. $\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\csc\theta} = 1$

25. $\frac{\cot\theta}{1 + \csc\theta} + \frac{1 + \csc\theta}{\cot\theta} = 2\sec\theta$

26. $\frac{\cos\theta}{1 - \sin\theta} = \frac{1 + \sin\theta}{\cos\theta}$

27. $\frac{\cot^2\theta}{1 + \csc\theta} = \csc\theta - 1$

28. $\frac{\sec\theta}{\tan\theta} + \frac{\csc\theta}{\cot\theta} = \sec\theta + \csc\theta$

29. $\frac{\sec\theta + \csc\theta}{1 + \tan\theta} = \csc\theta$

30. $\cot\theta\csc\theta + \sec\theta = \csc^2\theta\sec\theta$

31. $\frac{\sin\theta}{\sin\theta + \cos\theta} = \frac{\tan\theta}{1 + \tan\theta}$

32. $\cos^4\theta - \sin^4\theta = \frac{1 - \tan^2\theta}{\sec^2\theta}$

Example 2

Verify that $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\csc\theta$.

The left-hand side of this identity is more complicated, so start with that expression.

$$\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1 + \cos\theta)^2}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{1 + 1 + 2\cos\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{2 + 2\cos\theta}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{2(1 + \cos\theta)}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{2}{\sin\theta}$$

$$= 2\csc\theta$$

Lesson-by-Lesson Review

5-3 Solving Trigonometric Equations (pp. 327–333)

Find all solutions of each equation on the interval $[0, 2\pi]$.

33. $2 \sin x = \sqrt{2}$

34. $4 \cos^2 x = 3$

35. $\tan^2 x - 3 = 0$

36. $9 + \cot^2 x = 12$

37. $2 \sin^2 x = \sin x$

38. $3 \cos x + 3 = \sin^2 x$

Solve each equation for all values of x.

39. $\sin^2 x - \sin x = 0$

40. $\tan^2 x = \tan x$

41. $3 \cos x = \cos x - 1$

42. $\sin^2 x = \sin x + 2$

43. $\sin^2 x = 1 - \cos x$

44. $\sin x = \cos x + 1$

Example 3

Solve the equation $\sin \theta = 1 - \cos \theta$ for all values of θ.

$\sin \theta = 1 - \cos \theta$	Original equation.
$\sin^2 \theta = (1 - \cos \theta)^2$	Square each side.
$\sin^2 \theta = 1 - 2 \cos \theta + \cos^2 \theta$	Expand.
$1 - \cos^2 \theta = 1 - 2 \cos \theta + \cos^2 \theta$	Pythagorean Identity
$0 = 2 \cos^2 \theta - 2 \cos \theta$	Subtract.
$0 = 2 \cos \theta(\cos \theta - 1)$	Factor.

Solve for x on $[0, 2\pi]$.

$\cos \theta = 0$ or $\cos \theta = 1$

$\theta = \cos^{-1} 0$ $\qquad$ $\theta = \cos^{-1} 1$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\qquad$ $\theta = 0$

A check shows that $\frac{3\pi}{2}$ is an extraneous solution. So the solutions are $\theta = \frac{\pi}{2} + 2n\pi$ or $\theta = 0 + 2n\pi$.

5-4 Sum and Difference Identities (pp. 336–343)

Find the exact value of each trigonometric expression.

45. $\cos 15°$

46. $\sin 345°$

47. $\tan \frac{13\pi}{12}$

48. $\sin \frac{7\pi}{12}$

49. $\cos -\frac{11\pi}{12}$

50. $\tan \frac{5\pi}{12}$

Simplify each expression.

51. $\dfrac{\tan \frac{\pi}{9} + \tan \frac{8\pi}{9}}{1 - \tan \frac{\pi}{9} \tan \frac{8\pi}{9}}$

52. $\cos 24° \cos 36° - \sin 24° \sin 36°$

53. $\sin 95° \cos 50° - \cos 95° \sin 50°$

54. $\cos \frac{2\pi}{9} \cos \frac{\pi}{18} + \sin \frac{2\pi}{9} \sin \frac{\pi}{18}$

Verify each identity.

55. $\cos (\theta + 30°) - \sin (\theta + 60°) = -\sin \theta$

56. $\cos \left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\cos \theta - \sin \theta)$

57. $\cos \left(\theta - \frac{\pi}{3}\right) + \cos \left(\theta + \frac{\pi}{3}\right) = \cos \theta$

58. $\tan \left(\theta + \frac{3\pi}{4}\right) = \frac{\tan \theta - 1}{\tan \theta + 1}$

Example 4

Find the exact value of $\tan \frac{23\pi}{12}$.

$\tan \frac{23\pi}{12} = \tan \left(\frac{5\pi}{4} + \frac{2\pi}{3}\right)$	$\frac{23\pi}{12} = \frac{5\pi}{4} + \frac{2\pi}{3}$
$= \dfrac{\tan \frac{5\pi}{4} + \tan \frac{2\pi}{3}}{1 - \tan \frac{5\pi}{4} \tan \frac{2\pi}{3}}$	Sum Identity
$= \dfrac{1 - \sqrt{3}}{1 - (-\sqrt{3})}$	Evaluate for tangent.
$= \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$	Simplify.
$= \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}} \cdot \dfrac{1 - \sqrt{3}}{1 - \sqrt{3}}$	Rationalize the denominator.
$= \dfrac{4 - 2\sqrt{3}}{1 - 3}$	Multiply.
$= \dfrac{4 - 2\sqrt{3}}{-2}$ or $-2 + \sqrt{3}$	Simplify.

Lesson-by-Lesson Review

5-5 Multiple-Angle and Product-Sum Identities (pp. 346–354)

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

59. $\cos\theta = \frac{1}{3}$, $(0°, 90°)$

60. $\tan\theta = 2$, $(180°, 270°)$

61. $\sin\theta = \frac{4}{5}$, $\left(\frac{\pi}{2}, \pi\right)$

62. $\sec\theta = \frac{13}{5}$, $\left(\frac{3\pi}{2}, 2\pi\right)$

Find the exact value of each expression.

63. $\sin 75°$

64. $\cos\frac{11\pi}{12}$

65. $\tan 67.5°$

66. $\cos\frac{3\pi}{8}$

67. $\sin\frac{15\pi}{8}$

68. $\tan\frac{13\pi}{12}$

Example 5

Find the values of sin 2θ, cos 2θ, and tan 2θ if θ is in the fourth quadrant and $\tan\theta = -\frac{24}{7}$.

θ is in the fourth quadrant, so $\cos\theta = \frac{7}{25}$ and $\sin\theta = -\frac{24}{25}$.

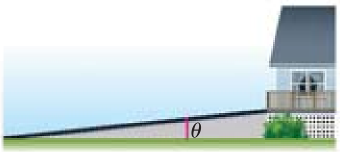
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{24}{25}\right)\frac{7}{25} \text{ or } -\frac{336}{625}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{7}{25}\right)^2 - 1 \text{ or } -\frac{527}{625}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{-\frac{527}{49}} \text{ or } \frac{336}{527}$$

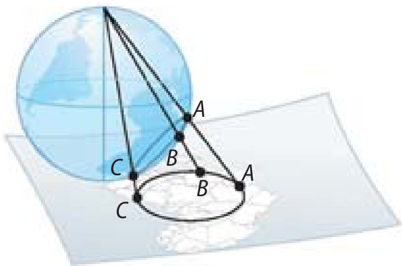
Applications and Problem Solving

69. CONSTRUCTION Find the tangent of the angle that the ramp makes with the building if $\sin\theta = \frac{\sqrt{145}}{145}$ and $\cos\theta = \frac{12\sqrt{145}}{145}$. (Lesson 5-1)



70. LIGHT The intensity of light that emerges from a system of two polarizing lenses can be calculated by $I = I_0 - \frac{I_0}{\csc^2\theta}$, where I_0 is the intensity of light entering the system and θ is the angle of the axis of the second lens with the first lens. Write the equation for the light intensity using only tan θ. (Lesson 5-1)

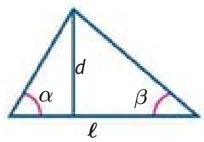
71. MAP PROJECTIONS Stereographic projection is used to project the contours of a three-dimensional sphere onto a two-dimensional map. Points on the sphere are related to points on the map using $r = \frac{\sin\alpha}{1-\cos\alpha}$. Verify that $r = \frac{1+\cos\alpha}{\sin\alpha}$. (Lesson 5-2)



72. PROJECTILE MOTION A ball thrown with an initial speed v_0 at an angle θ that travels a horizontal distance d will remain in the air t seconds, where $t = \frac{d}{v_0\cos\theta}$. Suppose a ball is thrown with an initial speed of 50 feet per second, travels 100 feet, and is in the air for 4 seconds. Find the angle at which the ball was thrown. (Lesson 5-3)

73. BROADCASTING Interference occurs when two waves pass through the same space at the same time. It is destructive if the amplitude of the sum of the waves is less than the amplitudes of the individual waves. Determine whether the interference is destructive when signals modeled by $y = 20\sin(3t + 45°)$ and $y = 20\sin(3t + 225°)$ are combined. (Lesson 5-4)

74. TRIANGULATION *Triangulation* is the process of measuring a distance d using the angles α and β and the distance ℓ using $\ell = \frac{d}{\tan\alpha} + \frac{d}{\tan\beta}$. (Lesson 5-5)



a. Solve the formula for d.

b. Verify that $d = \frac{\ell\sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$.

c. Verify that $d = \frac{\ell\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

d. Show that if $\alpha = \beta$, then $d = 0.5\ell\tan\alpha$.

CHAPTER 5 Practice Test

Find the value of each expression using the given information.

1. $\sin\theta$ and $\cos\theta$, $\csc\theta = -4$, $\cos\theta < 0$
2. $\csc\theta$ and $\sec\theta$, $\tan\theta = \frac{2}{5}$, $\csc\theta < 0$

Simplify each expression.

3. $\frac{\sin(90° - x)}{\tan(90° - x)}$
4. $\frac{\sec^2 x - 1}{\tan^2 x + 1}$
5. $\sin\theta(1 + \cot^2\theta)$

Verify each identity.

6. $\frac{\csc^2\theta - 1}{\csc^2\theta} + \frac{\sec^2\theta - 1}{\sec^2\theta} = 1$
7. $\frac{\cos\theta}{1 + \sin\theta} + \frac{1 - \sin\theta}{\cos\theta} = \frac{2\cos\theta}{1 + \sin\theta}$
8. $\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = 2\csc^2\theta$
9. $-\sec^2\theta\sin^2\theta = \frac{\cos^2\theta - 1}{\cos^2\theta}$
10. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

11. **MULTIPLE CHOICE** Which expression is *not* true?
    - **A** $\tan(-\theta) = -\tan\theta$
    - **B** $\tan(-\theta) = \frac{1}{\cot(-\theta)}$
    - **C** $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$
    - **D** $\tan(-\theta) + 1 = \sec(-\theta)$

Find all solutions of each equation on the interval $[0, 2\pi]$.

12. $\sqrt{2}\sin\theta + 1 = 0$
13. $\sec^2\theta = \frac{4}{3}$

Solve each equation for all values of θ.

14. $\tan^2\theta - \tan\theta = 0$
15. $\frac{1 - \sin\theta}{\cos\theta} = \cos\theta$
16. $\frac{1}{\sec\theta - 1} - \frac{1}{\sec\theta + 1} = 2$
17. $\sec\theta - 2\tan\theta = 0$

18. **CURRENT** The current produced by an alternator is given by $I = 40\sin 135\pi t$, where I is the current in amperes and t is the time in seconds. At what time t does the current first reach 20 amperes? Round to the nearest ten-thousandths.

Find the exact value of each trigonometric expression.

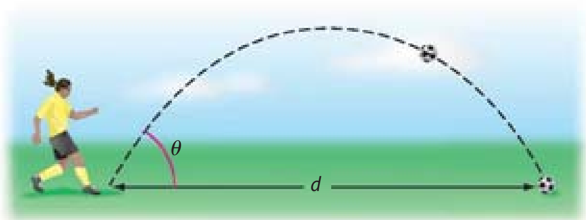
19. $\tan 165°$
20. $\cos -\frac{\pi}{12}$
21. $\sin 75°$
22. $\cos 465° - \cos 15°$
23. $6\sin 675° - 6\sin 45°$

24. **MULTIPLE CHOICE** Which identity is true?
    - **F** $\cos(\theta + \pi) = -\sin\pi$
    - **G** $\cos(\pi - \theta) = \cos\theta$
    - **H** $\sin\left(\theta - \frac{3\pi}{2}\right) = \cos\theta$
    - **J** $\sin(\pi + \theta) = \sin\theta$

Simplify each expression.

25. $\cos\frac{\pi}{8}\cos\frac{3\pi}{8} - \sin\frac{\pi}{8}\sin\frac{3\pi}{8}$
26. $\frac{\tan 135° - \tan 15°}{1 + \tan 135°\tan 15°}$

27. **PHYSICS** A soccer ball is kicked from ground level with an initial speed of v at an angle of elevation θ.



    a. The horizontal distance d the ball will travel can be determined using $d = \frac{v^2\sin 2\theta}{g}$, where g is the acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan\theta - \tan\theta\sin^2\theta)$.

    b. The maximum height h the object will reach can be determined using $h = \frac{v^2\sin^2\theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled.

Find the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for the given value and interval.

28. $\tan\theta = -3$, $\left(\frac{3\pi}{2}, 2\pi\right)$
29. $\cos\theta = \frac{1}{5}$, $(0°, 90°)$
30. $\cos\theta = \frac{5}{9}$, $\left(0, \frac{\pi}{2}\right)$

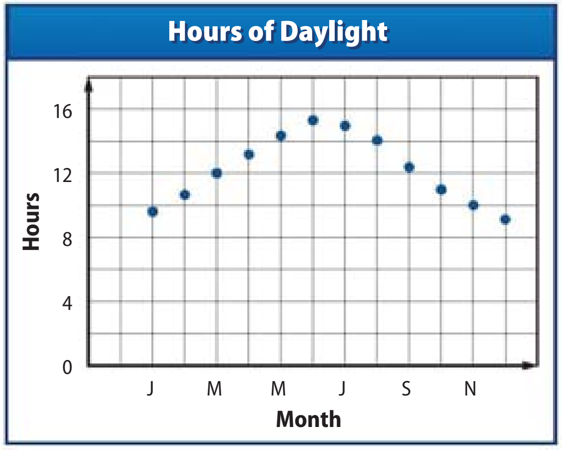
CHAPTER 5

Connect to AP Calculus
Rates of Change for Sine and Cosine

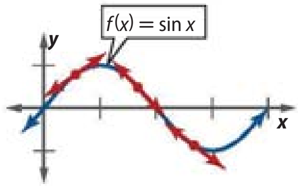
Objective

- Approximate rates of change for sine and cosine functions using the difference quotient.

In Chapter 4, you learned that many real-world situations exhibit periodic behavior over time and thus, can be modeled by sinusoidal functions. Using transformations of the parent functions sin *x* and cos *x*, trigonometric models can be used to represent data, analyze trends, and predict future values.



While you are able to model real-world situations using graphs of sine and cosine, differential calculus can be used to determine the rate that the model is changing at any point in time. Your knowledge of the difference quotient, the sum identities for sine and cosine, and the evaluation of limits now makes it possible to discover the rates of change for these functions at any point in time.



Activity 1 Approximate Rate of Change

Approximate the rate of change of $f(x) = \sin x$ at several points.

Step 1 Substitute $f(x) = \sin x$ into the difference quotient.

$$m = \frac{f(x+h) - f(x)}{h} \longrightarrow m = \frac{\sin(x+h) - \sin x}{h}$$

Step 2 Approximate the rate of change of $f(x)$ at $x = \frac{\pi}{2}$. Let $h = 0.1, 0.01, 0.001$, and 0.0001.

Step 3 Repeat Steps 1 and 2 for $x = 0$ and for $x = \pi$.

Analyze the Results

1. Use tangent lines and the graph of $f(x) = \sin x$ to interpret the values found in Steps 2 and 3.
2. What will happen to the rate of change of $f(x)$ as x increases?

Unlike the natural base exponential function $g(x) = e^x$ and the natural logarithmic function $h(x) = \ln x$, an expression to represent the rate of change of $f(x) = \sin x$ at any point is not as apparent. However, we can substitute $f(x)$ into the difference quotient and then simplify the expression.

$$m = \frac{f(x+h) - f(x)}{h}$$ Difference quotient

$$= \frac{\sin(x+h) - \sin x}{h}$$ $f(x) = \sin x$

$$= \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$ Sine Sum Identity

$$= \frac{(\sin x \cos h - \sin x) + \cos x \sin h}{h}$$ Group terms with sin *x*.

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$ Factor sin *x* and cos *x*.

We now have two expressions that involve h, $\sin x\left(\frac{\cos h - 1}{h}\right)$ and $\cos x\left(\frac{\sin h}{h}\right)$. To obtain an accurate approximation of the rate of change of $f(x)$ at a point, we want h to be as close to 0 as possible. Recall that in Chapter 1, we were able to substitute $h = 0$ into an expression to find the exact slope of a function at a point. However, both of the fractional expressions are undefined at $h = 0$.

$\sin x\left(\frac{\cos h - 1}{h}\right)$	Original expressions	$\cos x\left(\frac{\sin h}{h}\right)$
$= \sin x\left(\frac{\cos 0 - 1}{0}\right)$	$h = 0$	$= \cos x\left(\frac{\sin 0}{0}\right)$
undefined		undefined

We can approximate values for the two expressions by finding the limit of each as h approaches 0 using techniques discussed in Lesson 1-3.

Activity 2 Calculate Rate of Change

Find an expression for the rate of change of $f(x) = \sin x$.

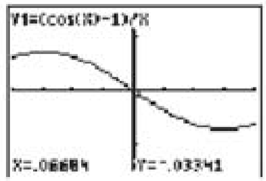
Step 1 Use a graphing calculator to estimate $\lim_{h \to 0} \frac{\cos h - 1}{h}$.

Step 2 Verify the value found in Step 1 by using the TABLE function of your calculator.

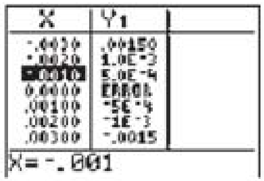
Step 3 Repeat Steps 1 and 2 to estimate $\lim_{h \to 0} \frac{\sin h}{h}$.

Step 4 Substitute the values found in Step 2 and Step 3 into the slope equation
$m = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$.

Step 5 Simplify the expression in Step 4.



$[-\pi, \pi]$ scl: $\frac{\pi}{4}$ by $[-1.5, 1.5]$ scl: 1

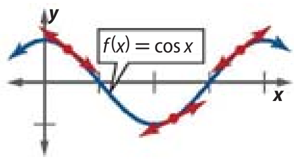


Analyze the Results

3. Find the rate of change of $f(x) = \sin x$ at $x = \frac{3\pi}{2}$, 2, and $\frac{5\pi}{2}$.
4. Make a conjecture as to why the rates of change for all trigonometric functions must be modeled by other trigonometric functions.

Model and Apply

5. In this problem, you will find an expression for the rate of change of $f(x) = \cos x$ at any point x.



   a. Substitute $f(x) = \cos x$ into the difference quotient.
   b. Simplify the expression from part **a.**
   c. Use a graphing calculator to find the limit of the two fractional expressions as h approaches 0.
   d. Substitute the values found in part **c** into the slope equation found in part **b.**
   e. Simplify the slope equation in part **d.**
   f. Find the rate of change of $f(x) = \cos x$ at $x = 0$, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

CHAPTER 6

Systems of Equations and Matrices

Then

In **Chapter 0**, you solved systems of equations and performed matrix operations.

Now

In **Chapter 6**, you will:

- Multiply matrices, and find determinants and inverses of matrices.
- Solve systems of linear equations.
- Write partial fraction decompositions of rational expressions.
- Use linear programming to solve applications.

Why?

BUSINESS Linear programming has become a standard tool for many businesses, like farming. Farmers must take into account many constraints in order to maximize profits from the sale of crops or livestock, including the cost of labor, land, and feed.

PREREAD Discuss what you already know about solving equations with a classmate. Then scan the lesson titles and write two or three predictions about what you will learn in this chapter.

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