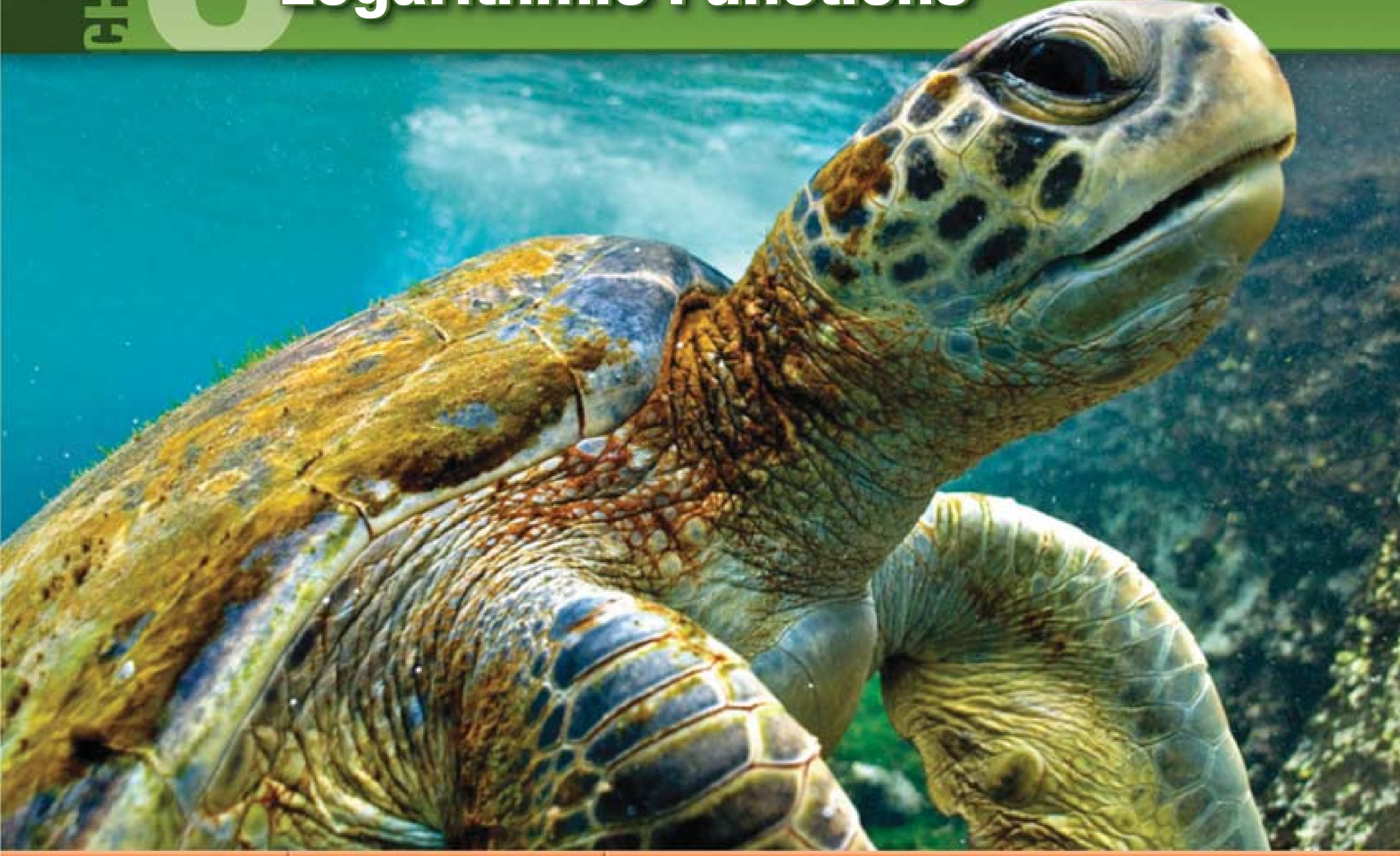
CHAPTER 3

Exponential and Logarithmic Functions



Then

- In **Chapter 2**, you graphed and analyzed power, polynomial, and rational functions.

Now

- In **Chapter 3,** you will:
  - Evaluate, analyze, and graph exponential and logarithmic functions.
  - Apply properties of logarithms.
  - Solve exponential and logarithmic equations.
  - Model data using exponential, logarithmic, and logistic functions.

Why? ▲

- **ENDANGERED SPECIES** Exponential functions are often used to model the growth and decline of populations of endangered species. For example, an exponential function can be used to model the population of the Galapagos Green Turtle since it became an endangered species.

  PREREAD Use the Concept Summary Boxes in the chapter to predict the organization of Chapter 3.

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Get Ready for the Chapter

Diagnose Readiness You have two options for checking Prerequisite Skills.

1 Textbook Option Take the Quick Check below.

QuickCheck

Simplify. (Lesson 0-4)

1. $(3x^2)^4 \cdot 2x^3$

2. $(3b^3)(2b^4)$

3. $\frac{y^7}{y^4}$

4. $\left(\frac{1}{2a}\right)^3$

5. $\frac{c^4d}{cd}$

6. $\frac{(2n^2)^4}{4n}$

7. **CARPET** The length of a bedroom carpet can be represented by $2a^2$ feet and the width by $5a^3$ feet. Determine the area of the carpet.

Use a graphing calculator to graph each function. Determine whether the inverse of the function is a function. (Lesson 1-7)

8. $f(x) = \sqrt{4 - x^2}$

9. $f(x) = \sqrt{x + 2}$

10. $f(x) = \frac{8 - x}{x}$

11. $g(x) = \frac{x - 3}{x}$

12. $f(x) = \frac{2}{\sqrt{1 - x}}$

13. $g(x) = \frac{5}{\sqrt{x + 7}}$

14. **STAMPS** The function $v(t) = 200(1.6)^t$ can be used to predict the value v of a rare stamp after t years. Graph the function, and determine whether the inverse of the function is a function.

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Lesson 2-1)

15. $f(x) = 2x^2$

16. $g(x) = 4x^3$

17. $h(x) = -3x^3$

18. $f(x) = -x^5$

2 Online Option Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

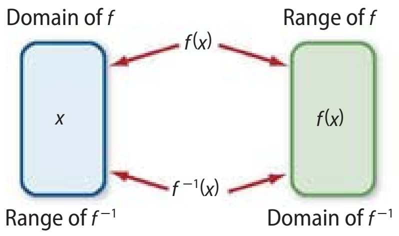
NewVocabulary

English		Español
algebraic functions	p. 158	funciones algebraicas
transcendental functions	p. 158	funciones transcendentales
exponential functions	p. 158	funciones exponenciales
natural base	p. 160	base natural
continuous compound interest	p. 163	interés compuesto continuo
logarithmic function with base *b*	p. 172	función logarítmica con base *b*
logarithm	p. 172	logaritmo
common logarithm	p. 173	logaritmo común
natural logarithm	p. 174	logaritmo natural
logistic growth function	p. 202	función de crecimiento logística
linearize	p. 204	linearize

ReviewVocabulary

one-to-one p. 66 de uno a uno a function that passes the horizontal line test, and no y-value is matched with more than one x-value

inverse functions p. 65 funciones inversas Two functions, f and f^{-1}, are inverse functions if and only if $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$.



end behavior p. 159 comportamiento de final describes the behavior of $f(x)$ as x increases or decreases without bound—becoming greater and greater or more and more negative

continuous function p. 159 función continua a function with a graph that has no breaks, holes or gaps

LESSON 3-1 Exponential Functions

Then

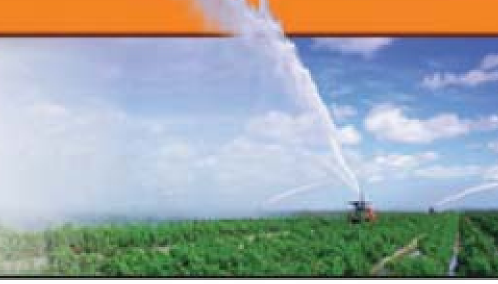
- You identified, graphed, and described several parent functions. (Lesson 1-5)

Now

- 1 Evaluate, analyze, and graph exponential functions.
- 2 Solve problems involving exponential growth and decay.

Why?

- Worldwide water consumption has increased rapidly over the last several decades. Most of the world's water is used for agriculture, and increasing population has resulted in an increasing agricultural demand. The increase in water consumption can be modeled using an exponential function.



NewVocabulary

algebraic function
transcendental function
exponential function
natural base
continuous compound interest

1 Exponential Functions

Exponential Functions In Chapter 2, you studied power, radical, polynomial, and rational functions. These are examples of **algebraic functions**—functions with values that are obtained by adding, subtracting, multiplying, or dividing constants and the independent variable or raising the independent variable to a rational power. In this chapter, we will explore exponential and logarithmic functions. These are considered to be **transcendental functions** because they cannot be expressed in terms of algebraic operations. In effect, they *transcend* algebra.

Consider functions $f(x) = x^3$ and $g(x) = 3^x$. Both involve a base raised to a power; however, in $f(x)$, a power function, the base is a variable and the exponent is a constant. In $g(x)$, the base is a constant and the exponent is a variable. Functions of a form similar to $g(x)$ are called **exponential functions**.

KeyConcept Exponential Function

An exponential function with base b has the form $f(x) = ab^x$, where x is any real number and a and b are real number constants such that $a \neq 0$, b is positive, and $b \neq 1$.

Examples

$f(x) = 4^x,\ f(x) = \left(\frac{1}{3}\right)^x,\ f(x) = 7^{-x}$

Nonexamples

$f(x) = 2x^{-3},\ f(x) = 5^{\pi},\ f(x) = 1^x$

When the inputs are rational numbers, exponential functions can be evaluated using the properties of exponents. For example, if $f(x) = 4^x$, then

$$f(2) = 4^2 \qquad f\left(\frac{1}{3}\right) = 4^{\frac{1}{3}} \qquad f(-3) = 4^{-3}$$

$$= 16 \qquad = \sqrt[3]{4} \qquad = \frac{1}{4^3}$$

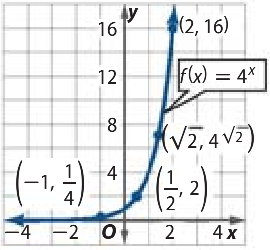
$$= \frac{1}{64}$$

Since exponential functions are defined for all real numbers, you must also be able to evaluate an exponential function for *irrational* values of x, such as $\sqrt{2}$. But what does the expression $4^{\sqrt{2}}$ mean?

The value of this expression can be approximated using successively closer rational approximations of $\sqrt{2}$ as shown below.

x	1	1.4	1.41	1.414	1.4142	1.41421	…
$f(x) = 4^x$	4	7.0	7.06	7.101	7.1029	7.10296	…

From this table, we can conclude that $4^{\sqrt{2}}$ is a real number approximately equal to 7.10. Since $f(x) = 4^x$ has real number values for every x-value in its domain, this function is continuous and can be graphed as a smooth curve as shown.



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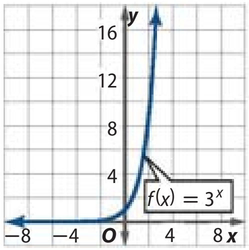
Example 1 Sketch and Analyze Graphs of Exponential Functions

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.

a. $f(x) = 3^x$

Evaluate the function for several x-values in its domain. Then use a smooth curve to connect each of these ordered pairs.

x	−4	−2	−1	0	2	4	6
$f(x)$	0.01	0.11	0.33	1	9	81	729



Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

y-Intercept: 1 Asymptote: x-axis

End behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

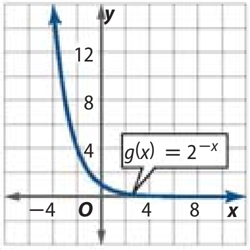
Increasing: $(-\infty, \infty)$

StudyTip

Negative Exponents Notice that $f(x) = \left(\frac{1}{b}\right)^x$ and $g(x) = b^{-x}$ are equivalent because $\frac{1}{b^x} = (b^{-1})^x$ or b^{-x}.

b. $g(x) = 2^{-x}$

x	−6	−4	−2	0	2	4	6
$f(x)$	64	16	4	1	0.25	0.06	0.02



Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

y-Intercept: 1 Asymptote: x-axis

End behavior: $\lim_{x \to -\infty} g(x) = \infty$ and $\lim_{x \to \infty} g(x) = 0$

Decreasing: $(-\infty, \infty)$

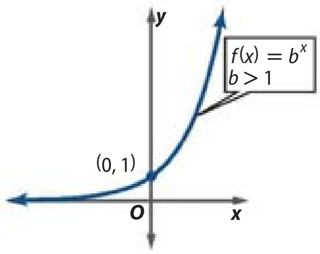
GuidedPractice

1A. $f(x) = 6^{-x}$ **1B.** $g(x) = 5^x$ **1C.** $h(x) = \left(\frac{1}{4}\right)^x + 1$

The increasing and decreasing graphs in Example 1 are typical of the two basic types of exponential functions: *exponential growth* and *exponential decay*.

KeyConcept Properties of Exponential Functions

Exponential Growth



Domain: $(-\infty, \infty)$ **Range:** $(0, \infty)$

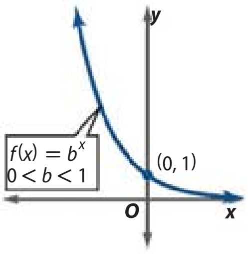
y-Intercept: 1 **x-Intercept:** none

Extrema: none **Asymptote:** x-axis

End Behavior: $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $(-\infty, \infty)$

Exponential Decay



Domain: $(-\infty, \infty)$ **Range:** $(0, \infty)$

y-Intercept: 1 **x-Intercept:** none

Extrema: none **Asymptote:** x-axis

End Behavior: $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 0$

Continuity: continuous on $(-\infty, \infty)$

StudyTip

Coefficient a An exponential function of the form $f(x) = ab^x$ has a y-intercept at $(0, a)$.

The same techniques that you used to transform graphs of algebraic functions can be applied to graphs of exponential functions.

Example 2 Graph Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to describe the transformation that results in each function. Then sketch the graphs of the functions.

a. $g(x) = 2^{x+1}$

This function is of the form $g(x) = f(x + 1)$. Therefore, the graph of $g(x)$ is the graph of $f(x) = 2^x$ translated 1 unit to the left (Figure 3.1.1).

b. $h(x) = 2^{-x}$

This function is of the form $h(x) = f(-x)$. Therefore, the graph of $h(x)$ is the graph of $f(x) = 2^x$ reflected in the y-axis (Figure 3.1.2).

c. $j(x) = -3(2^x)$

This function is of the form $j(x) = -3f(x)$. Therefore, the graph of $j(x)$ is the graph of $f(x) = 2^x$ reflected in the x-axis and expanded vertically by a factor of 3 (Figure 3.1.3).

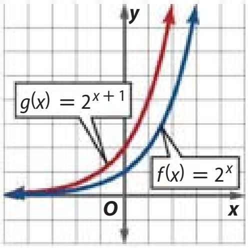


Figure 3.1.1

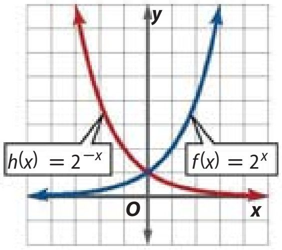


Figure 3.1.2

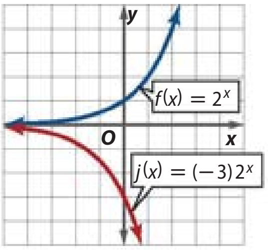


Figure 3.1.3

StudyTip

Analyzing Graphs Notice that the transformations of $f(x)$ given by $g(x)$, $h(x)$, and $j(x)$ do not affect the location of the horizontal asymptote, the x-axis. However, the transformations given by $h(x)$ and $g(x)$ do affect the y-intercept of the graph.

GuidedPractice

Use the graph of $f(x) = 4^x$ to describe the transformation that results in each function. Then sketch the graphs of the functions.

2A. $k(x) = 4^x - 2$

2B. $m(x) = -4^{x+2}$

2C. $p(x) = 2(4^{-x})$

It may surprise you to learn that for most real-world applications involving exponential functions, the most commonly used base is not 2 or 10 but an irrational number e called the **natural base**, where

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x.$$

By calculating the value of $\left(1 + \frac{1}{x}\right)^x$ for greater and greater values of x, we can estimate that the value of this expression approaches a number close to 2.7183. In fact, using calculus, it can be shown that this value approaches the irrational number we call e, named after the Swiss mathematician Leonhard Euler who computed e to 23 decimal places.

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10,000	2.71814...
100,000	2.71827...
1,000,000	2.71828...

$$e = 2.718281828\ldots$$

The number e can also be defined as $\lim_{x \to 0}(1 + x)^{\frac{1}{x}}$, since for fractional values of x closer and closer to 0, $(1 + x)^{\frac{1}{x}} = 2.718281828\ldots$ or e.

The function given by $f(x) = e^x$, is called the *natural base exponential function* (Figure 3.1.4) and has the same properties as those of other exponential functions.

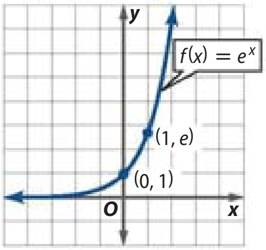


Figure 3.1.4

ReadingMath

Base *e* Expressions with base *e* are read similarly to exponential expressions with any other base. For example, the expression e^{4x} is read as *e to the four x*.

Example 3 Graph Natural Base Exponential Functions

Use the graph of $f(x) = e^x$ to describe the transformation that results in the graph of each function. Then sketch the graphs of the functions.

a. $a(x) = e^{4x}$

This function is of the form $a(x) = f(4x)$. Therefore, the graph of $a(x)$ is the graph of $f(x) = e^x$ compressed horizontally by a factor of 4 (Figure 3.1.5).

b. $b(x) = e^{-x} + 3$

This function is of the form $b(x) = f(-x) + 3$. Therefore, the graph of $b(x)$ is the graph of $f(x) = e^x$ reflected in the y-axis and translated 3 units up (Figure 3.1.6).

c. $c(x) = \frac{1}{2}e^x$

This function is of the form $c(x) = \frac{1}{2}f(x)$. Therefore, the graph of $c(x)$ is the graph of $f(x) = e^x$ compressed vertically by a factor of $\frac{1}{2}$ (Figure 3.1.7).

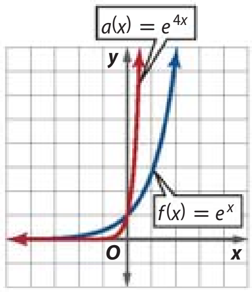


Figure 3.1.5

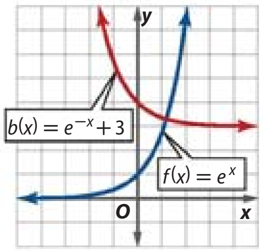


Figure 3.1.6

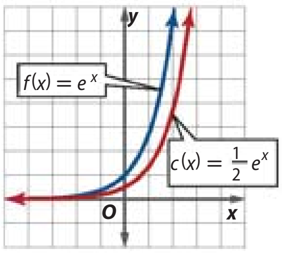


Figure 3.1.7

GuidedPractice

3A. $q(x) = e^{-x}$ **3B.** $r(x) = e^x - 5$ **3C.** $t(x) = 3e^x$

2 Exponential Growth and Decay

A common application of exponential growth is compound interest. Suppose an initial principal P is invested into an account with an annual interest rate r, and the interest is *compounded* or reinvested annually. At the end of each year, the interest earned is added to the account balance. This sum becomes the new principal for the next year.

Year	Account Balance After Each Compounding	
0	$A_0 = P$	P = original investment or principal
1	$A_1 = A_0 + A_0r$	Interest from year 0, A_0r, is added.
	$= \mathbf{A_0}(1 + r)$	Distributive Property
	$= \mathbf{P}(1 + r)$	$A_0 = P$
2	$A_2 = \mathbf{A_1}(1 + r)$	Interest from year 1 is added.
	$= \mathbf{P(1 + r)}(1 + r)$	$A_1 = P(1 + r)$
	$= P(1 + r)^2$	Simplify.
3	$A_3 = \mathbf{A_2}(1 + r)$	Interest from year 2 is added.
	$= \mathbf{P(1 + r)^2}(1 + r)$	$A_2 = P(1 + r)^2$
	$= P(1 + r)^3$	Simplify.
4	$A_4 = \mathbf{A_3}(1 + r)$	Interest from year 3 is added.
	$= \mathbf{P(1 + r)^3}(1 + r)$	$A_3 = P(1 + r)^3$
	$= P(1 + r)^3$	Simplify.

The pattern that develops leads to the following exponential function with base $(1 + r)$.

$A(t) = P(1 + r)^t$ Account balance after t years

To allow for quarterly, monthly, or even daily compoundings, let n be the number of times the interest is compounded each year. Then

- the rate per compounding $\frac{r}{n}$ is a fraction of the annual rate r, and
- the number of compoundings after t years is nt.

Replacing r with $\frac{r}{n}$ and t with nt in the formula $A(t) = P(1 + r)^t$, we obtain a general formula for compound interest.

KeyConcept Compound Interest Formula

If a principal P is invested at an annual interest rate r (in decimal form) compounded n times a year, then the balance A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example 4 Use Compound Interest

FINANCIAL LITERACY **Krysti invests \$300 in an account with a 6% interest rate, making no other deposits or withdrawals. What will Krysti's account balance be after 20 years if the interest is compounded:**

a. **semiannually?**

For semiannually compounding, $n = 2$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Compound Interest Formula}$$

$$= 300\left(1 + \frac{0.06}{2}\right)^{2(20)} \quad P = 300,\ r = 0.06,\ n = 2, \text{ and } t = 20$$

$$\approx 978.61 \quad \text{Simplify.}$$

With semiannual compounding, Krysti's account balance after 20 years will be \$978.61.

b. **monthly?**

For monthly compounding, $n = 12$, since there are 12 months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Compound Interest Formula}$$

$$= 300\left(1 + \frac{0.06}{12}\right)^{12(20)} \quad P = 300,\ r = 0.06,\ n = 12, \text{ and } t = 20$$

$$\approx 993.06 \quad \text{Simplify.}$$

With monthly compounding, Krysti's account balance after 20 years will be \$993.06.

StudyTip

Daily Compounding In this text, for problems involving interest compounded daily, we will assume a 365-day year.

c. **daily?**

For daily compounding, $n = 365$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Compound Interest Formula}$$

$$= 300\left(1 + \frac{0.06}{365}\right)^{365(20)} \quad P = 300,\ r = 0.06,\ t = 20, \text{ and } n = 365$$

$$\approx 995.94 \quad \text{Simplify.}$$

With daily compounding, Krysti's account balance after 20 years will be \$995.94.

GuidedPractice

4. **FINANCIAL LITERACY** If \$1000 is invested in an online savings account earning 8% per year, how much will be in the account at the end of 10 years if there are no other deposits or withdrawals and interest is compounded:

A. semiannually?
B. quarterly?
C. daily?

Notice that as the number of compoundings increases in Example 4, the account balance also increases. However, the increase is relatively small, only \$995.94 − \$993.06 or \$2.88.

The table below shows the amount A computed for several values of n. Notice that while the account balance is increasing, the amount of increase slows down as n increases. In fact, it appears that the amount tends towards a value close to $996.03.

Compounding	n	$A = 300\left(1+\frac{0.06}{n}\right)^{20n}$
annually	1	$962.14
semiannually	2	$978.61
quarterly	4	$987.20
monthly	12	$993.06
daily	365	$995.94
hourly	8760	$996.03

Suppose the interest were compounded *continuously* so that there was no waiting period between interest payments. We can derive a formula for **continuous compound interest** by first using algebra to manipulate the regular compound interest formula.

$$A = P\left(1+\frac{1}{\frac{n}{r}}\right)^{nt}$$

Compound interest formula with $\frac{r}{n}$ written as $\frac{1}{\frac{n}{r}}$.

$$= P\left(1+\frac{1}{x}\right)^{xrt}$$

Let $x = \frac{n}{r}$ and $n = xr$.

$$= P\left[\left(1+\frac{1}{x}\right)^{x}\right]^{rt}$$

Power Property of Exponents

The expression in brackets should look familiar. Recall from page 160 that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$. Since r is a fixed value and $x = \frac{n}{r}$, $x \to \infty$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = \lim_{x\to\infty} P\left[\left(1+\frac{1}{x}\right)^{x}\right]^{rt} = Pe^{rt}.$$

This leads us to the formula for calculating continuous compounded interest shown below.

KeyConcept Continuous Compound Interest Formula

If a principal P is invested at an annual interest rate r (in decimal form) compounded continuously, then the balance A in the account after t years is given by

$$A = Pe^{rt}.$$

Ariel Skelley/Blend Images/Getty Images

Real-WorldLink

The prime rate is the interest rate that banks charge their most credit-worthy borrowers. Changes in this rate can influence other rates, including mortgage interest rates.

Source: Federal Reserve System

Example 5 Use Continuous Compound Interest

FINANCIAL LITERACY Suppose Krysti finds an account that will allow her to invest her $300 at a 6% interest rate compounded continuously. If there are no other deposits or withdrawals, what will Krysti's account balance be after 20 years?

$A = Pe^{rt}$ Continuous Compound Interest Formula

$= 300e^{(0.06)(20)}$ $P = 300$, $r = 0.06$, and $t = 20$

≈ 996.04 Simplify.

With continuous compounding, Krysti's account balance after 20 years will be $996.04.

GuidedPractice

5. **ONLINE BANKING** If $1000 is invested in an online savings account earning 8% per year compounded continuously, how much will be in the account at the end of 10 years if there are no other deposits or withdrawals?

In addition to investments, populations of people, animals, bacteria, and amounts of radioactive material can also change at an exponential rate. Exponential growth and decay models apply to any situation where growth is proportional to the initial size of the quantity being considered.

KeyConcept Exponential Growth or Decay Formulas

If an initial quantity N_0 grows or decays at an exponential rate r or k (as a decimal), then the final amount N after a time t is given by the following formulas.

Exponential Growth or Decay

$$N = N_0(1 + r)^t$$

If r is a *growth rate*, then $r > 0$.

If r is a *decay rate*, then $r < 0$.

Continuous Exponential Growth or Decay

$$N = N_oe^{kt}$$

If k is a *continuous growth rate*, then $k > 0$.

If k is a *continuous decay rate*, then $k < 0$.

Continuous growth or decay is similar to continuous compound interest. The growth or decay is compounded continuously rather than just yearly, monthly, hourly, or at some other time interval. Population growth can be modeled exponentially, continuously, and by other models.

Real-World Link

In 2008, the population of Mexico was estimated to be growing at a rate of about 1.42% annually.

Source: *CIA–The World Fact Book*

Real-World Example 6 Model Using Exponential Growth or Decay

POPULATION **Mexico has a population of approximately 110 million. If Mexico's population continues to grow at the described rate, predict the population of Mexico in 10 and 20 years.**

a. **1.42% annually**

Use the exponential growth formula to write an equation that models this situation.

$N = N_0 (1 + r)^t$ Exponential Growth Formula

$= 110{,}000{,}000(1 + 0.0142)^t$ $N_0 = 110{,}000{,}000$ and $r = 0.0142$

$= 110{,}000{,}000(1.0142)^t$ Simplify.

Use this equation to find N when $t = 10$ and $t = 20$.

$N = 110{,}000{,}000(1.0142)^t$ Modeling equation $N = 110{,}000{,}000(1.0142)^t$

$= 110{,}000{,}000(1.0142)^{10}$ $t = 10$ or $t = 20$ $= 110{,}000{,}000(1.0142)^{20}$

$\approx 126{,}656{,}869$ Simplify. $= 145{,}836{,}022$

If the population of Mexico continues to grow at an annual rate of 1.42%, its population in 10 years will be about 126,656,869; and in 20 years, it will be about 145,836,022.

b. **1.42% continuously**

Use the continuous exponential growth formula to write a modeling equation.

$N = N_0e^{kt}$ Continuous Exponential Growth Formula

$= 110{,}000{,}000e^{0.0142t}$ $N_0 = 110{,}000{,}000$ and $k = 0.0142$

Use this equation to find N when $t = 10$ and $t = 20$.

$N = 110{,}000{,}000e^{0.0142t}$ Modeling equation $N = 110{,}000{,}000e^{1.0142t}$

$= 110{,}000{,}000e^{0.0142(10)}$ $t = 10$ and $t = 20$ $= 110{,}000{,}000e^{0.0142(20)}$

$\approx 126{,}783{,}431$ Simplify. $\approx 146{,}127{,}622$

If the population of Mexico continues to grow at a continuous rate of 1.42%, its population in 10 years will be about 126,783,431; in 20 years, it will be about 146,127,622.

GuidedPractice

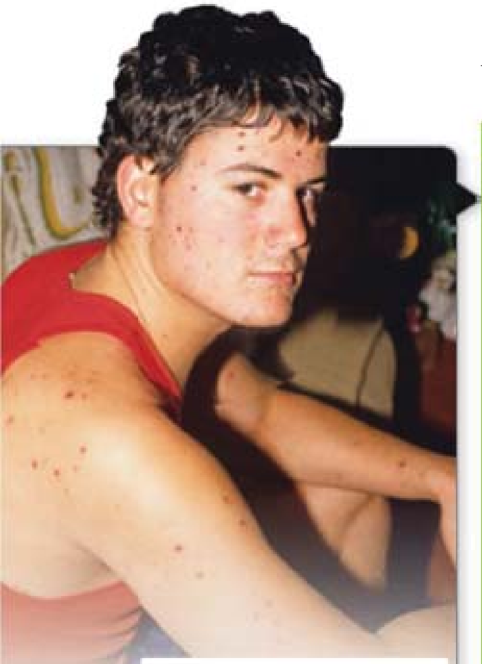
6. **POPULATION** The population of a town is declining at a rate of 6%. If the current population is 12,426 people, predict the population in 5 and 10 years using each model.

A. annually **B.** continuously

WatchOut!

Using Rates of Decay Remember to write rates of decay as negative values.

ArtPix/Alamy



Real-WorldLink

A chicken pox vaccine was first licensed for use in the United States in 1995.

Source: Centers for Disease Control

After finding a model for a situation, you can use the graph of the model to solve problems.

Real-World Example 7 Use the Graph of an Exponential Model

DISEASE **The table shows the number of reported cases of chicken pox in the United States in 1980 and 2005.**

U.S. Reported Cases of Chicken Pox

Year	Cases (thousands)
1980	190.9
2005	32.2

Source: U.S. Centers for Disease Control and Prevention

a. If the number of reported cases of chicken pox is decreasing at an exponential rate, identify the rate of decline and write an exponential equation to model this situation.

If we let $N(t)$ represent the number of cases t years after 1980 and assume exponential decay, then the initial number of cases $N_0 = 190.9$ and at time $t = 2005 - 1980$ or 25, the number of reported cases $N(25) = 32.2$. Use this information to find the rate of decay r.

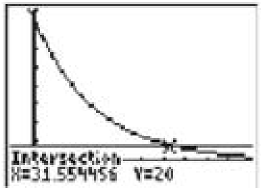
$N(t) = N_0(1 + r)^t$	Exponential Decay Formula
$32.2 = 190.9(1 + r)^{25}$	$N(25) = 32.2$, $N_0 = 190.9$, and $t = 25$
$\frac{32.2}{190.9} = (1 + r)^{25}$	Divide each side by 190.9.
$\sqrt[25]{\frac{32.2}{190.9}} = 1 + r$	Take the positive 25th root of each side.
$\sqrt[25]{\frac{32.2}{190.9}} - 1 = r$	Subtract 1 from each side.
$-0.069 \approx r$	Simplify.

The number of reported cases is decreasing at a rate of approximately 6.9% per year. Therefore, an equation modeling this situation is $N(t) = 190.9[1 + (-0.069)]^t$ or $N(t) = 190.9(0.931)^t$.

b. Use your model to predict when the number of cases will drop below 20,000.

To find when the number of cases will drop below 20,000, find the intersection of the graph of $N(t) = 190.9(0.931)^t$ and the line $N(t) = 20$. A graphing calculator shows that value of t for which $190.9(0.931)^t = 20$ is about 32.

Since t is the number of years after 1980, this model suggests that after the year 1980 + 32 or 2012, the number of cases will drop below 20,000 if this rate of decline continues.



```
Intersection
X=31.554456  Y=20
```

$[-5, 50]$ scl: 5 by $[-25, 200]$ scl: 25

GuidedPractice

7. **POPULATION** Use the data in the table and assume that the population of Miami-Dade County is growing exponentially.

Estimated Population of Miami-Dade County, Florida

Year	Population (million)
1990	1.94
2000	2.25

Source: U.S. Census Bureau

A. Identify the rate of growth and write an exponential equation to model this growth.

B. Use your model to predict in which year the population of Miami-Dade County will surpass 2.7 million.

David R. Frazier Photolibrary, Inc./Alamy

Exercises

● = Step-by-Step Solutions begin on page R29.

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing. (Example 1)

1. $f(x) = 2^{-x}$

2. $r(x) = 5^x$

3. $h(x) = 0.2^{x+2}$

4. $k(x) = 6^x$

5. $m(x) = -(0.25)^x$

6. $p(x) = 0.1^{-x}$

7. $q(x) = \left(\frac{1}{6}\right)^x$

8. $g(x) = \left(\frac{1}{3}\right)^x$

9. $c(x) = 2^x - 3$

10. $d(x) = 5^{-x} + 2$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$. (Examples 2 and 3)

11. $f(x) = 4^x$; $g(x) = 4^x - 3$

12. $f(x) = \left(\frac{1}{2}\right)^x$; $g(x) = \left(\frac{1}{2}\right)^{x+4}$

13. $f(x) = 3^x$; $g(x) = -2(3^x)$

14. $f(x) = 2^x$; $g(x) = 2^{x-2} + 5$

15. $f(x) = 10^x$; $g(x) = 10^{-x+3}$

16. $f(x) = e^x$; $g(x) = e^{2x}$

17. $f(x) = e^x$; $g(x) = e^{x+2} - 1$

18. $f(x) = e^x$; $g(x) = e^{-x+1}$

19. $f(x) = e^x$; $g(x) = 3e^x$

20. $f(x) = e^x$; $g(x) = -(e^x) + 4$

FINANCIAL LITERACY **Copy and complete the table below to find the value of an investment A for the given principal P, rate r, and time t if the interest is compounded n times annually.** (Examples 4 and 5)

n	1	4	12	365	continuously
A					

21. $P = \$500$, $r = 3\%$, $t = 5$ years

22. $P = \$1000$, $r = 4.5\%$, $t = 10$ years

23. $P = \$1000$, $r = 5\%$, $t = 20$ years

24. $P = \$5000$, $r = 6\%$, $t = 30$ years

25 **FINANCIAL LITERACY** Brady acquired an inheritance of \$20,000 at age 8, but he will not have access to it until he turns 18. (Examples 4 and 5)

a. If his inheritance is placed in a savings account earning 4.6% interest compounded monthly, how much will Brady's inheritance be worth on his 18th birthday?

b. How much will Brady's inheritance be worth if it is placed in an account earning 4.2% interest compounded continuously?

26. **FINANCIAL LITERACY** Katrina invests \$1200 in a certificate of deposit (CD). The table shows the interest rates offered by the bank on 3- and 5-year CDs. (Examples 4 and 5)

CD Offers		
Years	3	5
Interest	3.45%	4.75%
Compounded	continuously	monthly

a. How much would her investment be worth with each option?

b. How much would her investment be worth if the 5-year CD was compounded continuously?

POPULATION **Copy and complete the table to find the population N of an endangered species after a time t given its initial population N_0 and annual rate r or continuous rate k of increase or decline.** (Example 6)

t	5	10	15	20	50
N					

27. $N_0 = 15{,}831$, $r = -4.2\%$

28. $N_0 = 23{,}112$, $r = 0.8\%$

29. $N_0 = 17{,}692$, $k = 2.02\%$

30. $N_0 = 9689$, $k = -3.7\%$

31. **WATER** Worldwide water usage in 1950 was about 294.2 million gallons. If water usage has grown at the described rate, estimate the amount of water used in 2000 and predict the amount in 2050. (Example 6)

a. 3% annually

b. 3.05% continuously

32. **WAGES** Jasmine receives a 3.5% raise at the end of each year from her employer to account for inflation. When she started working for the company in 1994, she was earning a salary of \$31,000. (Example 6)

a. What was Jasmine's salary in 2000 and 2004?

b. If Jasmine continues to receive a raise at the end of each year, how much money will she earn during her final year if she plans on retiring in 2024?

33. **PEST CONTROL** Consider the termite guarantee made by Exterm-inc in their ad below.

Exterm-inc
Locally Owned & Operated

Termite & Pest Control
SINCE 1995

FREE TERMITE INSPECTION

Termite Guarantee: **60%** of termite colony eliminated with each treatment. All but **1%** eliminated after **just 3 treatments!**

FINANCING AVAILABLE
555-3267

If the first statement in this claim is true, assess the validity of the second statement. Explain your reasoning. (Example 6)

34. **INFLATION** The Consumer Price Index (CPI) is an index number that measures the average price of consumer goods and services. A change in the CPI indicates the growth rate of inflation. In 1958 the CPI was 28.6, and in 2008 the CPI was 211.08. (Example 7)

a. Determine the growth rate of inflation between 1958 and 2008. Use this rate to write an exponential equation to model this situation.

b. What will the CPI be in 2015? At this rate, when will the CPI exceed 350?

㉟ **GASOLINE** Jordan wrote an exponential equation to model the cost of gasoline. He found the average cost per gallon of gasoline for two years and used these data for his model. (Example 7)

Average Cost per Gallon of Gasoline

Year	Cost($)
1990	1.19
2011	3.91

a. If the average cost of gasoline increased at an exponential rate, identify the rate of increase. Write an exponential equation to model this situation.

b. Use your model to predict the average cost of a gallon of gasoline in 2015 and 2017.

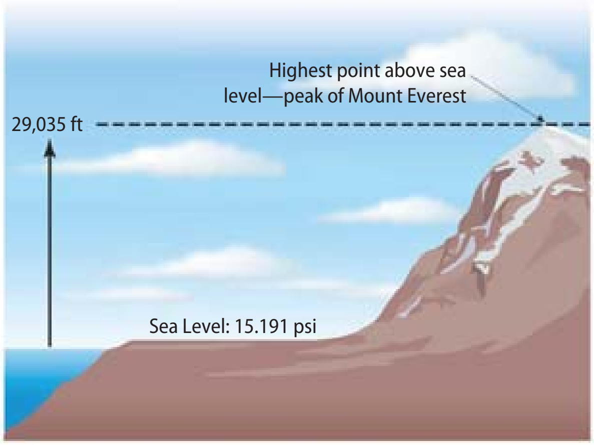
c. When will the average cost per gallon of gasoline exceed $7?

d. Why might an exponential model not be an accurate representation of average gasoline prices?

36. **PHYSICS** The pressure of the atmosphere at sea level is 15.191 pounds per square inch (psi). It decreases continuously at a rate of 0.004% as altitude increases by x feet.

a. Write a modeling function for the continuous exponential decay representing the atmospheric pressure $a(x)$.

b. Use the model to approximate the atmospheric pressure at the top of Mount Everest.



c. If a certain rescue helicopter can fly only in atmospheric pressures greater than 5.5 pounds per square inch, how high can it fly up Mount Everest?

37. **RADIOACTIVITY** The half-life of a radioactive substance is the amount of time it takes for half of the atoms of the substance to disintegrate. Uranium-235 is used to fuel a commercial power plant. It has a half-life of 704 million years.

a. How many grams of uranium-235 will remain after 1 million years if you start with 200 grams?

b. How many grams of uranium-235 will remain after 4540 million years if you start with 200 grams?

38. **BOTANY** Under the right growing conditions, a particular species of plant has a doubling time of 12 days. Suppose a pasture contains 46 plants of this species. How many plants will there be after 20, 65, and x days?

39. **RADIOACTIVITY** Radiocarbon dating uses carbon-14 to estimate the age of organic materials found commonly at archaeological sites. The half-life of carbon-14 is approximately 5.73 thousand years.

a. Write a modeling equation for the exponential decay.

b. How many grams of carbon-14 will remain after 12.82 thousand years if you start with 7 grams?

c. Use your model to estimate when only 1 gram of the original 7 grams of carbon-14 will remain.

40. **MICROBIOLOGY** A certain bacterium used to treat oil spills has a doubling time of 15 minutes. Suppose a colony begins with a population of one bacterium.

a. Write a modeling equation for this exponential growth.

b. About how many bacteria will be present after 55 minutes?

c. A population of 8192 bacteria is sufficient to clean a small oil spill. Use your model to predict how long it will take for the colony to grow to this size.

41. **ENCYCLOPEDIA** The number of articles making up an online open-content encyclopedia increased exponentially during its first few years. The number of articles, $A(t)$, t years after 2001 can be modeled by $A(t) = 16{,}198 \cdot 2.13^t$.

a. According to this model, how many articles made up the encyclopedia in 2001? At what rate is the number of articles increasing?

b. During which year did the encyclopedia reach 1 million articles?

c. Predict the number of articles there will be at the beginning of 2018.

42. **RISK** The chance of having an automobile accident increases exponentially if the driver has consumed alcohol. The relationship can be modeled by $A(c) = 6e^{12.8c}$, where A is the percent chance of an accident and c is the driver's blood alcohol concentration (BAC).

a. The legal BAC is 0.08. What is the percent chance of having a car accident at this concentration?

b. What BAC would correspond to a 50% chance of having a car accident?

43. **GRAPHING CALCULATOR** The table shows the number of blogs, in millions, in existence every six months.

Months	1	7	13	19	25	31
Blogs	0.7	2	4	8	16	31

- **a.** Using the calculator's exponential regression tool, find a function that models the data.
- **b.** After how many months did the number of blogs reach 20 million?
- **c.** Predict the number of blogs after 48 months.

44. **LANGUAGES** *Glottochronology* is an area of linguistics that studies the divergence of languages. The equation $c = e^{-Lt}$, where c is the proportion of words that remain unchanged, t is the time since two languages diverged, and L is the rate of replacement, models this divergence.

- **a.** If two languages diverged 5 years ago and the rate of replacement is 43.13%, what proportion of words remains unchanged?
- **b.** After how many years will only 1% of the words remain unchanged?

45. **FINANCIAL LITERACY** A couple just had a child and wants to immediately start a college fund. Use the information below to determine how much money they should invest.

GRAPHING CALCULATOR **Determine the value(s) of x that make(s) each equation or inequality below true. Round to the nearest hundredth, if necessary.**

46. $2^x < 4$

47. $e^{2x} = 3$

48. $-e^x > -2$

49. $2^{-4x} \leq 8$

Describe the domain, range, continuity, and increasing/decreasing behavior for an exponential function with the given intercept and end behavior. Then graph the function.

50. $f(0) = -1, \lim\limits_{x \to -\infty} f(x) = 0, \lim\limits_{x \to \infty} f(x) = -\infty$

51. $f(0) = 4, \lim\limits_{x \to -\infty} f(x) = \infty, \lim\limits_{x \to \infty} f(x) = 3$

52. $f(0) = 3, \lim\limits_{x \to -\infty} f(x) = 2, \lim\limits_{x \to \infty} f(x) = \infty$

Determine the equation of each function after the given transformation of the parent function.

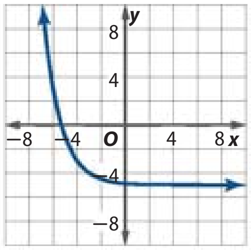
53. $f(x) = 5^x$ translated 3 units left and 4 units down

54. $f(x) = 0.25^x$ compressed vertically by a factor of 3 and translated 9 units left and 12 units up

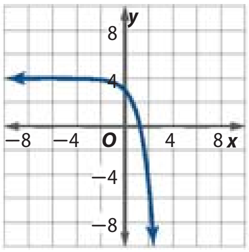
55. $f(x) = 4^x$ reflected in the x-axis and translated 1 unit left and 6 units up

Determine the transformations of the given parent function that produce each graph.

56. $f(x) = \left(\frac{1}{2}\right)^x$



57. $f(x) = 3^x$



58. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the average rate of change for exponential functions.

- **a.** **GRAPHICAL** Graph $f(x) = b^x$ for $b = 2, 3, 4,$ or 5.
- **b.** **ANALYTICAL** Find the average rate of change of each function on the interval $[0, 2]$.
- **c.** **VERBAL** What can you conclude about the average rate of change of $f(x) = b^x$ as b increases? How is this shown in the graphs in part **a**?
- **d.** **GRAPHICAL** Graph $f(x) = b^{-x}$ for $b = 2, 3, 4,$ or 5.
- **e.** **ANALYTICAL** Find the average rate of change of each function on the interval $[0, 2]$.
- **f.** **VERBAL** What can you conclude about the average rate of change of $f(x) = b^{-x}$ as b increases. How is this shown in the graphs in part **d**?

H.O.T. Problems Use Higher-Order Thinking Skills

(59) **ERROR ANALYSIS** Eric and Sonja are determining the worth of a $550 investment after 12 years in a savings account earning 3.5% interest compounded monthly. Eric thinks the investment is worth $837.08, while Sonja thinks it is worth $836.57. Is either of them correct? Explain.

REASONING **State whether each statement is *true* or *false*. Explain your reasoning.**

60. Exponential functions can never have restrictions on the domain.

61. Exponential functions always have restrictions on the range.

62. Graphs of exponential functions always have an asymptote.

63. **OPEN ENDED** Write an example of an increasing exponential function with a negative y-intercept.

64. **CHALLENGE** Trina invests $1275 in an account that compounds quarterly at 8%, but at the end of each year she takes 100 out. How much is the account worth at the end of the fifth year?

65. **REASONING** Two functions of the form $f(x) = b^x$ *sometimes*, *always*, or *never* have at least one ordered pair in common.

66. **WRITING IN MATH** Compare and contrast the domain, range, intercepts, symmetry, continuity, increasing/decreasing behavior, and end behavior of exponential and power parent functions.

Spiral Review

Solve each inequality. (Lesson 2-6)

67. $(x - 3)(x + 2) \leq 0$

68. $x^2 + 6x \leq -x - 4$

69. $3x^2 + 15 \geq x^2 + 15x$

Find the domain of each function and the equations of any vertical or horizontal asymptotes, noting any holes. (Lesson 2-5)

70. $f(x) = \frac{3}{x^2 - 4x + 4}$

71. $f(x) = \frac{x - 1}{x^2 + 4x - 5}$

72. $f(x) = \frac{x^2 - 8x + 16}{x - 4}$

73. **TEMPERATURE** A formula for converting degrees Celsius to Fahrenheit is $F(x) = \frac{9}{5}x + 32$. (Lesson 1-7)

a. Find the inverse $F^{-1}(x)$. Show that $F(x)$ and $F^{-1}(x)$ are inverses.

b. Explain what purpose $F^{-1}(x)$ serves.

74. **SHOPPING** Lily wants to buy a pair of inline skates that are on sale for 30% off the original price of \$149. The sales tax is 5.75%. (Lesson 1-6)

a. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax.

b. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

c. How much will Lily pay for the inline skates?

75. **EDUCATION** The table shows the number of freshmen who applied to and the number of freshmen attending selected universities in a certain year. (Lesson 1-1)

a. State the relation as a set of ordered pairs.

b. State the domain and range of the relation.

c. Determine whether the relation is a function. Explain.

d. Assuming the relation is a function, is it reasonable to determine a prediction equation for this situation? Explain.

University	Applied	Attending
Auburn University	13,264	4184
University of California-Davis	27,954	4412
University of Illinois-Urbana-Champaign	21,484	6366
Florida State University	13,423	4851
State University of New York-Stony Brook	16,849	2415
The Ohio State University	19,563	5982
Texas A&M University	17,284	6949

Source: *How to Get Into College*

Skills Review for Standardized Tests

76. **SAT/ACT** A set of n numbers has an average (arithmetic mean) of $3k$ and a sum of $12m$, where k and m are positive. What is the value of n in terms of k and m?

A $\frac{4m}{k}$

B $36km$

C $\frac{4k}{m}$

D $\frac{m}{4k}$

E $\frac{k}{4m}$

77. The number of bacteria in a colony were growing exponentially. Approximately how many bacteria were there at 7 P.M.?

F 15,700

G 159,540

H 1,011,929

J 6,372,392

Time	Number of Bacteria
2 P.M.	100
4 P.M.	4000

78. **REVIEW** If $4^{x+2} = 48$, then $4^x = ?$

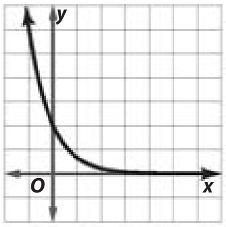
A 3.0

B 6.4

C 6.9

D 12.0

79. **REVIEW** What is the equation of the function?



F $y = 2(3)^x$

G $y = 2\left(\frac{1}{3}\right)^x$

H $y = 3\left(\frac{1}{2}\right)^x$

J $y = 3(2)^x$

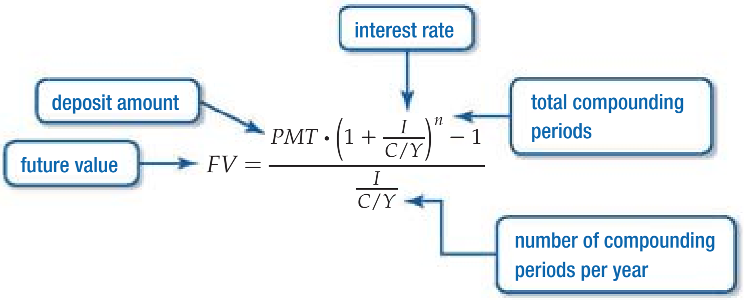
Graphing Technology Lab
Financial Literacy: Exponential Functions

Objective

- Calculate future values of annuities and monthly payments.

In Lesson 3-1, you used exponential functions to calculate compounded interest. In the compounding formula, you assume that an initial deposit is made and the investor never deposits nor withdrawals any money. Other types of investments do not follow this simple compounding rule.

When an investor takes out an *annuity*, he or she makes identical deposits into the account at regular intervals or periods. The compounding interest is calculated at the time of each deposit. We can determine the *future value* of an annuity, or its value at the end of a period, using the formula below.



$$FV = \frac{PMT \cdot \left(1 + \frac{I}{C/Y}\right)^n - 1}{\frac{I}{C/Y}}$$

StudyTip

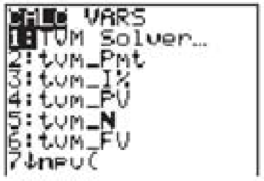
Future Value Formula The payments must be periodic and of equal value in order for the formula to be accurate.

Because solving this equation by hand can be tedious, you can use the finance application on a TI-84. The *time value of money* solver can be used to find any unknown value in this formula. The known variables are all entered and zeros are entered for the unknown variables.

Activity 1 Find a Future Value of an Annuity

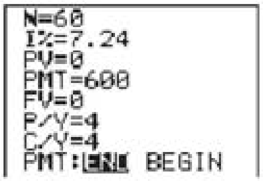
An investor pays $600 quarterly into an annuity. The annuity earns 7.24% annual interest. What will be the value of the annuity after 15 years?

Step 1 **Select** Finance **in the** APPS **Menu. Then select** CALC, TVM Solver.



Step 2 **Enter the data.**

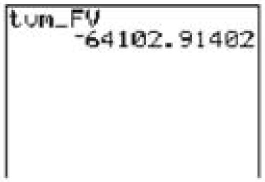
Payments are made quarterly over 15 years, so there are 4 • 15 or 60 payments. The present value, or amount at the beginning, is $0. The future value is unknown, 0 is used as a placeholder. Interest is compounded quarterly, so P/Y and C/Y are 4. (C/Y and P/Y are identical.) Payment is made at the end of each month, so select end.



```
N=60
I%=7.24
PV=0
PMT=600
FV=0
P/Y=4
C/Y=4
PMT:END BEGIN
```

Step 3 **Calculate.**

Quit the screen then go back into the Finance application. Select tvm_FV to calculate the future value. Then press ENTER. The result is the future value subtracted from the present value, so the negative sign is ignored.



```
tvm_FV
        -64102.91402
```

After 15 years, the value of the annuity will be about $64,103.

StudyTip

Down Payments When a consumer makes a down payment, that amount is subtracted from the present value of the loan before anything else is calculated.

When taking out a loan for a large purchase like a home or car, consumers are typically concerned with how much their monthly payment will be. While the exponential function below can be used to determine the monthly payment, it can also be calculated using the finance application in the TI-84.

$$PMT = \frac{\overset{\text{present value}}{PV} \cdot \frac{I}{C/Y}}{1-\left(1+\frac{I}{C/Y}\right)^{-n}}$$

Activity 2 Calculate Monthly Payment

You borrow $170,000 from the bank to purchase a home. The 30-year loan has an annual interest rate of 4.5%. Calculate your monthly payment and the amount paid after 30 years.

Step 1 **Select** Finance **in the** APPS **Menu. Then select** CALC, TVM Solver.

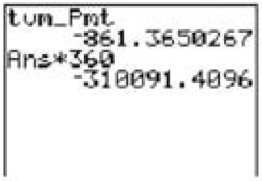
Step 2 **Enter the data.**

The number of payments is $N = 30 \cdot 12$ or 360.
The interest rate I is 4.5%.
The present value of the loan PV is $170,000.
The monthly payment and future value are unknown.
The number of payments per year P/Y and C/Y is 12.
Payment is made at the end of month, so select end.

```
N=360
I%=4.5
PV=170000
PMT=0
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

Step 3 **Calculate.**

Select tvm_Pmt to calculate the monthly payment.
Then press ENTER. Multiply the monthly payment by 360.



```
tvm_Pmt
        -861.3650267
Ans*360
        -310091.4096
```

Your monthly payment will be $861.37 and the total that will be repaid is $310,091.41.

Exercises

Calculate the future value of each annuity.

1. $800 semiannually, 12 years, 4%
2. $400 monthly, 6 years, 5.5%
3. $200 monthly, 3 years, 7%
4. $1000 annually, 14 years, 6.25%
5. $450, quarterly, 8 years, 5.5%
6. $300 bimonthly, 18 years, 4.35%

Calculate the monthly payment and the total amount to be repaid for each loan.

7. $220,000, 30 years, 5.5%
8. $140,000, 20 years, 6.75%
9. $20,000, 5 years, 8.5%
10. $5000, 5 years, 4.25%
11. $45,000, 10 years, 3.5%
12. $180,000, 30 years, 6.5%

13. **CHANGING VALUES** Changing a value of any of the variables may dramatically affect the loan payments. The monthly payment for a 30-year loan for $150,000 at 6% interest is $899.33, with a total payment amount of $323,757.28. Calculate the monthly payment and the total amount of the loan for each scenario.
    - **a.** Putting down $20,000 on the purchase.
    - **b.** Paying 4% interest instead of 6%.
    - **c.** Paying the loan off in 20 years instead of 30.
    - **d.** Making 13 payments per year.
    - **e.** Which saved the most money? Which had the lowest monthly payment?

LESSON 3-2 Logarithmic Functions

Then

- You graphed and analyzed exponential functions. (Lesson 3-1)

Now

- 1 Evaluate expressions involving logarithms.
- 2 Sketch and analyze graphs of logarithmic functions.

Why?

- The intensity level of sound is measured in decibels. A whisper measures 20 decibels, a normal conversation 60 decibels, and a vacuum cleaner at 80 decibels. The music playing in headphones maximizes at 100 decibels. The decibel scale is an example of a logarithmic scale.

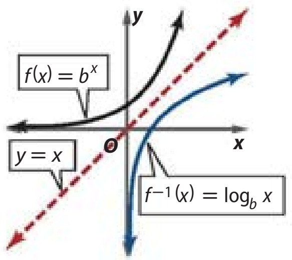
NewVocabulary

logarithmic function with base b
logarithm
common logarithm
natural logarithm

1 Logarithmic Functions and Expressions

Recall from Lesson 1-7 that graphs of functions that pass the horizontal line test are said to be *one-to-one* and have inverses that are also functions. Looking back at the graphs on page 159, you can see that exponential functions of the form $f(x) = b^x$ pass the horizontal line test and are therefore one-to-one with inverses that are functions.

The inverse of $f(x) = b^x$ is called **a logarithmic function with base b**, denoted $\log_b x$ and read *log base b of x*. This means that if $f(x) = b^x$, $b > 0$ and $b \neq 1$, then $f^{-1}(x) = \log_b x$, as shown in the graph of these two functions. Notice that the graphs are reflections of each other in the line $y = x$.



This inverse definition provides a useful connection between exponential and logarithmic equations.

KeyConcept Relating Logarithmic and Exponential Forms

If $b > 0$, $b \neq 1$, and $x > 0$, then

Logarithmic Form		Exponential Form
$\log_{\underset{\text{base}}{b}} x = \underset{\text{exponent}}{y}$	if and only if	$\underset{\text{base}}{b}^{\underset{\text{exponent}}{y}} = x.$

The statement above indicates that $\log_b x$ is the exponent to which b must be raised in order to obtain x. Therefore, when evaluating **logarithms**, remember that a logarithm is an exponent.

Example 1 Evaluate Logarithms

Evaluate each logarithm.

a. $\log_3 81$

$\log_3 81 = y$	Let $\log_3 81 = y$.
$3^y = 81$	Write in exponential form.
$3^y = 3^4$	$81 = 3^4$
$y = 4$	Prop. of Equality for Exponents

Therefore, $\log_3 81 = \mathbf{4}$, because $3^{\mathbf{4}} = 81$.

b. $\log_5 \sqrt{5}$

$\log_5 \sqrt{5} = y$	Let $\log_5 \sqrt{5} = y$.
$5^y = \sqrt{5}$	Write in exponential form.
$5^y = 5^{\frac{1}{2}}$	$5^{\frac{1}{2}} = \sqrt{5}$
$y = \frac{1}{2}$	Prop. of Equality for Exponents

Therefore, $\log_5 \sqrt{5} = \mathbf{\frac{1}{2}}$, because $5^{\mathbf{\frac{1}{2}}} = \sqrt{5}$.

Somos/Veer/Getty Images

c. $\log_7 \frac{1}{49}$

$\log_7 \frac{1}{49} = -2$, because $7^{-2} = \frac{1}{7^2}$ or $\frac{1}{49}$.

d. $\log_2 2$

$\log_2 2 = 1$, because $2^1 = 2$.

GuidedPractice

1A. $\log_8 512$

1B. $\log_4 4^{3.2}$

1C. $\log_2 \frac{1}{32}$

1D. $\log_{16} \sqrt{2}$

Example 1 and other examples suggest the following basic properties of logarithms.

KeyConcept Basic Properties of Logarithms

If $b > 0$, $b \neq 1$, and x is a real number, then the following statements are true.

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b b^x = x$
- $b^{\log_b x} = x,\ x > 0$

} Inverse Properties

StudyTip

Inverse Functions The inverse properties of logarithms also follow from the inverse relationship between logarithmic and exponential functions and the definition of inverse functions. If $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, then the following statements are true.

$f^{-1}[f(x)] = \log_b b^x = x$

$f[f^{-1}(x)] = b^{\log_b x} = x$

These properties follow directly from the statement relating the logarithmic and exponential forms of equations.

$\log_b 1 = 0$, because $b^0 = 1$.

$\log_b b^y = y$, because $b^y = b^y$.

$\log_b b = 1$, because $b^1 = b$.

$b^{\log_b x} = x$, because $\log_b x = \log_b x$.

You can use these basic properties to evaluate logarithmic and exponential expressions.

Example 2 Apply Properties of Logarithms

Evaluate each expression.

a. $\log_5 125$

$\log_5 125 = \log_5 5^3$ — $5^3 = 125$

$= 3$ — $\log_b b^x = x$

b. $12^{\log_{12} 4.7}$

$12^{\log_{12} 4.7} = 4.7$ — $b^{\log_b x} = x$

GuidedPractice

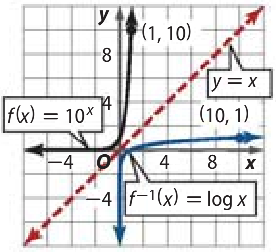
2A. $\log_9 81$

2B. $3^{\log_3 1}$

A logarithm with base 10 or $\log_{10}$ is called a **common logarithm** and is often written without the base. The common logarithm function $y = \log x$ is the inverse of the exponential function $y = 10^x$. Therefore,

$$y = \log x \qquad \text{if and only if} \qquad 10^y = x, \text{ for all } x > 0.$$

The properties for logarithms also hold true for common logarithms.



KeyConcept Basic Properties of Common Logarithms

If x is a real number, then the following statements are true.

- $\log 1 = 0$
- $\log 10 = 1$
- $\log 10^x = x$
- $10^{\log x} = x,\ x > 0$

} Inverse Properties

Common logarithms can be evaluated using the basic properties described above. Approximations of common logarithms of positive real numbers can be found by using LOG on a calculator.

Example 3 Common Logarithms

Evaluate each expression.

a. **log 0.001**

$\log 0.001 = \log 10^{-3}$ $\quad 0.001 = \frac{1}{10^3}$ or 10^{-3}

$= -3$ $\quad \log 10^x = x$

b. $\mathbf{10^{\log 5}}$

$10^{\log 5} = 5$ $\quad 10^{\log x} = x$

c. **log 26**

$\log 26 \approx 1.42$ Use a calculator.

CHECK Since 26 is between 10 and 100, log 26 is between log 10 and log 100. Since log 10 = 1 and log 100 = 2, log 26 has a value between 1 and 2. ✓

d. **log (−5)**

Since $f(x) = \log_b x$ is only defined when $x > 0$, $\log(-5)$ is undefined on the set of real numbers.

GuidedPractice

3A. log 10,000

3B. log 0.081

3C. log −0

3D. $10^{\log 3}$

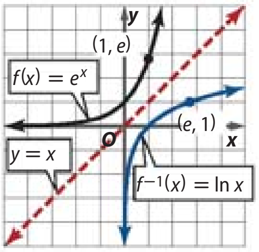
TechnologyTip

Error Message If you try to take the common logarithm of a negative number, your calculator will display either the error message ERR: NONREAL ANS or an imaginary number.

A logarithm with base e or $\log_e$ is called a **natural logarithm** and is denoted ln. The natural logarithmic function $y = \ln x$ is the inverse of the exponential function $y = e^x$. Therefore,

$$y = \ln x \qquad \text{if and only if} \qquad e^y = x, \text{ for all } x > 0.$$

The properties for logarithms also hold true for natural logarithms.



KeyConcept Basic Properties of Natural Logarithms

If x is a real number, then the following statements are true.

- $\ln 1 = 0$
- $\ln e = 1$
- $\ln e^x = x$
- $e^{\ln x} = x, x > 0$

} Inverse Properties

Natural logarithms can be evaluated using the basic properties described above. Approximations of natural logarithms of positive real numbers can be found by using LN on a calculator.

Example 4 Natural Logarithms

Evaluate each expression.

a. $\mathbf{\ln e^{0.73}}$

$\ln e^{0.73} = 0.73$ $\quad \ln e^x = x$

b. **ln (−5)**

ln (−5) is undefined.

c. $\mathbf{e^{\ln 6}}$

$e^{\ln 6} = 6$ $\quad e^{\ln x} = x$

d. **ln 4**

$\ln 4 \approx 1.39$ Use a calculator.

GuidedPractice

4A. ln 32

4B. $e^{\ln 4}$

4C. $\ln\left(\frac{1}{e^3}\right)$

4D. $-\ln 9$

2 Graphs of Logarithmic Functions

You can use the inverse relationship between exponential and logarithmic functions to graph functions of the form $y = \log_b x$.

Example 5 Graphs of Logarithmic Functions

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.

a. $f(x) = \log_3 x$

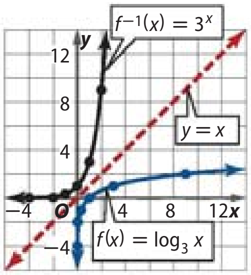
Construct a table of values and graph the inverse of this logarithmic function, the exponential function $f^{-1}(x) = 3^x$.

x	−4	−2	−1	0	1	2
$f^{-1}(x)$	0.01	0.11	0.33	1	3	9

Since $f(x) = \log_3 x$ and $f^{-1}(x) = 3^x$ are inverses, you can obtain the graph of $f(x)$ by plotting the points $(f^{-1}(x), x)$.

$f^{-1}(x)$	0.01	0.11	0.33	1	3	9
x	−4	−2	−1	0	1	2

The graph of $f(x) = \log_3 x$ has the following characteristics.



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x-intercept: 1

Asymptote: y-axis

End behavior: $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

Increasing: $(0, \infty)$

StudyTip

Graphs To graph a logarithmic function, first graph the inverse using your graphing calculator. Then, utilize the TABLE function to quickly obtain multiple coordinates of the inverse. Use these points to sketch the graph of the logarithmic function.

b. $g(x) = \log_{\frac{1}{2}} x$

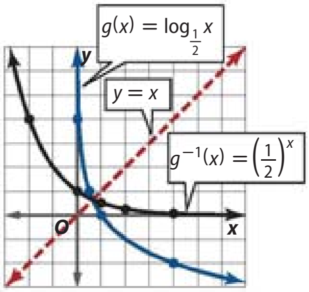
Construct a table of values and graph the inverse of this logarithmic function, the exponential function $g^{-1}(x) = \left(\frac{1}{2}\right)^x$.

x	−4	−2	0	1	2	4
$g^{-1}(x)$	16	4	1	0.5	0.25	0.06

Graph $g(x)$ by plotting the points $(g^{-1}(x), x)$.

$g^{-1}(x)$	16	4	1	0.5	0.25	0.06
x	−4	−2	0	1	2	4

The graph of $g(x) = \log_{\frac{1}{2}} x$ has the following characteristics.



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x-intercept: 1

Asymptote: y-axis

End behavior: $\lim_{x \to 0^+} g(x) = \infty$ and $\lim_{x \to \infty} g(x) = -\infty$

Decreasing: $(0, \infty)$

GuidedPractice

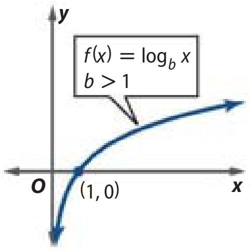
5A. $h(x) = \log_2 x$

5B. $j(x) = \log_{\frac{1}{3}} x$

The characteristics of typical *logarithmic growth*, or increasing logarithmic functions, and *logarithmic decay*, or decreasing logarithmic functions, are summarized below.

KeyConcept Properties of Logarithmic Functions

Logarithmic Growth



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

y-Intercept: none

x-Intercept: 1

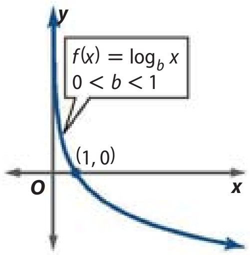
Extrema: none

Asymptote: y-axis

End Behavior: $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

Continuity: continuous on $(0, \infty)$

Logarithmic Decay



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

y-Intercept: none

x-Intercept: 1

Extrema: none

Asymptote: y-axis

End Behavior: $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$

Continuity: continuous on $(0, \infty)$

The same techniques used to transform the graphs of exponential functions can be applied to the graphs of logarithmic functions.

WatchOut!

Transformations Remember that horizontal translations are dependent on the constant *inside* the parentheses, and vertical translations are dependent on the constant *outside* of the parentheses.

Example 6 Graph Transformations of Logarithmic Functions

Use the graph of $f(x) = \log x$ to describe the transformation that results in each function. Then sketch the graphs of the functions.

a. $k(x) = \log (x + 4)$

This function is of the form $k(x) = f(x + 4)$. Therefore, the graph of $k(x)$ is the graph of $f(x)$ translated 4 units to the left (Figure 3.2.1).

b. $m(x) = -\log x - 5$

The function is of the form $m(x) = -f(x) - 5$. Therefore, the graph of $m(x)$ is the graph of $f(x)$ reflected in the x-axis and then translated 5 units down (Figure 3.2.2).

c. $p(x) = 3 \log (x + 2)$

The function is of the form $p(x) = 3f(x + 2)$. Therefore, the graph of $p(x)$ is the graph of $f(x)$ expanded vertically by a factor of 3 and then translated 2 units to the left. (Figure 3.2.3).

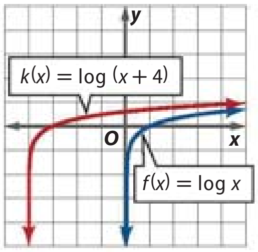


Figure 3.2.1

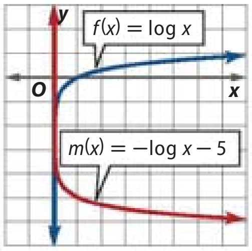


Figure 3.2.2

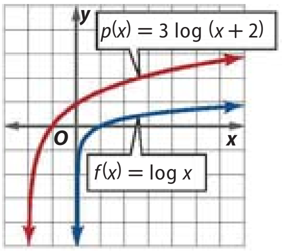


Figure 3.2.3

GuidedPractice

Use the graph of $f(x) = \ln x$ to describe the transformation that results in each function. Then sketch the graphs of the functions.

6A. $a(x) = \ln (x - 6)$

6B. $b(x) = 0.5 \ln x - 2$

6C. $c(x) = \ln (x + 4) + 3$

Real-WorldCareer

Sound Engineer Sound engineers operate and maintain sound recording equipment. They also regulate the signal strength, clarity, and range of sounds of recordings or broadcasts. To become a sound engineer, you should take high school courses in math, physics, and electronics.

Michael Newman/PhotoEdit

Logarithms can be used in scientific calculations, such as with pH acidity levels and the intensity level of sound.

Real-World Example 7 Use Logarithmic Functions

SOUND The intensity level of a sound, measured in decibels, can be modeled by $d(w) = 10 \log \frac{w}{w_0}$, where w is the intensity of the sound in watts per square meter and w_0 is the constant 1.0×10^{-12} watts per square meter.

a. If the intensity of the sound of a person talking loudly is 3.16×10^{-8} watts per square meter, what is the intensity level of the sound in decibels?

Evaluate $d(w)$ when $w = 3.16 \times 10^{-8}$.

$d(w) = 10 \log \frac{w}{w_0}$ Original function

$= 10 \log \frac{3.16 \times 10^{-8}}{1.0 \times 10^{-12}}$ $w = 3.16 \times 10^{-8}$ and $w_0 = 1.0 \times 10^{-12}$

≈ 45 Use a calculator.

The intensity level of the sound is 45 decibels.

b. If the threshold of hearing for a certain person with hearing loss is 5 decibels, will a sound with an intensity level of 2.1×10^{-12} watts per square meter be audible to that person?

Evaluate $d(w)$ when $w = 2.1 \times 10^{-12}$.

$d(w) = 10 \log \frac{w}{w_0}$ Original function

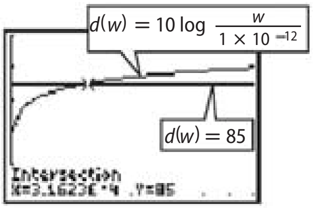
$= 10 \log \frac{2.1 \times 10^{-12}}{1.0 \times 10^{-12}}$ $w = 2.1 \times 10^{-12}$ and $w_0 = 1.0 \times 10^{-12}$

≈ 3.22 Use a calculator.

Because the person can only hear sounds that are 5 decibels or higher, he or she would not be able to hear a sound with an intensity level of 3.22 decibels.

c. Sounds in excess of 85 decibels can cause hearing damage. Determine the intensity of a sound with an intensity level of 85 decibels.

Use a graphing calculator to graph $d(w) = 10 \log \frac{w}{1 \times 10^{-12}}$ and $d(w) = 85$ on the same screen and find the point of intersection.



```
Intersection
X=3.1623E-4 .Y=85
```

[0, 0.001] scl: 0.0001 by [50, 100] scl: 10

When the intensity level of the sound is 85 decibels, the intensity of the sound is 3.1623×10^{-4} watts per square meter.

GuidedPractice

7. **TECHNOLOGY** The number of machines infected by a specific computer virus can be modeled by $c(d) = 6.8 + 20.1 \ln d$, where d is the number of days since the first machine was infected.
   - **A.** About how many machines were infected on day 12?
   - **B.** How many more machines were infected on day 30 than on day 12?
   - **C.** On about what day will the number of infected machines reach 75?

Exercises

● = Step-by-Step Solutions begin on page R29.

Evaluate each expression. (Examples 1–4)

1. $\log_2 8$

2. $\log_{10} 10$

3. $\log_6 \frac{1}{36}$

4. $4^{\log_4 1}$

5. $\log_{11} 121$

6. $\log_2 2^3$

7. $\log_{\sqrt{9}} 81$

8. $\log 0.01$

9. $\log 42$

10. $\log_x x^2$

11. $\log 5275$

12. $\ln e^{-14}$

13. $3 \ln e^4$

14. $\ln (5 - \sqrt{6})$

15. $\log_{36} \sqrt[5]{6}$

16. $4 \ln (7 - \sqrt{2})$

17. $\log 635$

18. $\frac{\ln 2}{\ln 7}$

19. $\ln (-6)$

20. $\ln \left(\frac{1}{e^{12}}\right)$

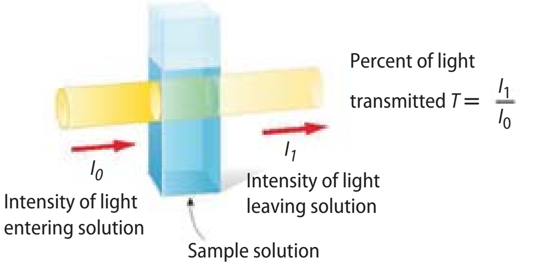
21. $\ln 8$

22. $\log_{\sqrt[7]{4}} 64$

23. $\frac{7}{\ln e}$

24. $\log 1000$

25. **LIGHT** The amount of light A absorbed by a sample solution is given by $A = 2 - \log 100T$, where T is the fraction of the light transmitted through the solution as shown in the diagram below. (Example 3)



In an experiment, a student shines light through two sample solutions containing different concentrations of a certain dye.

a. If the percent of light transmitted through the first sample solution is 72%, how much light does the sample solution absorb to the nearest hundredth?

b. If the absorption of the second sample solution is 0.174, what percent of the light entering the solution is transmitted?

26. **SOUND** While testing the speakers for a concert, an audio engineer notices that the sound level reached a relative intensity of 2.1×10^8 watts per square meter. The equation $D = \log I$ represents the loudness in decibels D given the relative intensity I. What is the level of the loudness of this sound in decibels? Round to the nearest thousandth if necessary. (Example 3)

27. **MEMORY** The students in Mrs. Ross' class were tested on exponential functions at the end of the chapter and then were retested monthly to determine the amount of information they retained. The average exam scores can be modeled by $f(x) = 85.9 - 9 \ln x$, where x is the number of months since the initial exam. What was the average exam score after 3 months? (Example 4)

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing. (Example 5)

28. $f(x) = \log_4 x$

29. $g(x) = \log_5 x$

30. $h(x) = \log_8 x$

31. $j(x) = \log_{\frac{1}{4}} x$

32. $m(x) = \log_{\frac{1}{5}} x$

33. $n(x) = \log_{\frac{1}{8}} x$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$. (Example 6)

34. $f(x) = \log_2 x; g(x) = \log_2 (x + 4)$

35. $f(x) = \log_3 x; g(x) = \log_3 (x - 1)$

36. $f(x) = \log x; g(x) = \log 2x$

37. $f(x) = \ln x; g(x) = 0.5 \ln x$

38. $f(x) = \log x; g(x) = -\log (x - 2)$

39. $f(x) = \ln x; g(x) = 3 \ln (x) + 1$

40. $f(x) = \log x; g(x) = -2 \log x + 5$

41. $f(x) = \ln x; g(x) = \ln (-x)$

42. **INVESTING** The annual growth rate for an investment can be found using $r = \frac{1}{t} \ln \frac{P}{P_0}$, where r is the annual growth rate, t is time in years, P is the present value, and P_0 is the original investment. An investment of \$10,000 was made in 2002 and had a value of \$15,000 in 2009. What was the average annual growth rate of the investment? (Example 7)

Determine the domain, range, x-intercept, and vertical asymptote of each function.

43. $y = \log (x + 7)$

44. $y = \log x - 1$

45. $y = \ln (x - 3)$

46. $y = \ln \left(x + \frac{1}{4}\right) - 3$

Find the inverse of each equation.

47. $y = e^{3x}$

48. $y = \log 2x$

㊾ $y = 4e^{2x}$

50. $y = 6 \log 0.5x$

51. $y = 20^x$

52. $y = 4(2^x)$

Determine the domain and range of the inverse of each function.

53. $y = \log x - 6$

54. $y = 0.25e^{x+2}$

55. **COMPUTERS** Gordon Moore, the cofounder of Intel, made a prediction in 1975 that is now known as Moore's Law. He predicted that the number of transistors on a computer processor at a given price point would double every two years.

a. Write Moore's Law for the predicted number of transistors P in terms of time in years t and the initial number of transistors.

b. In October 1985, a specific processor had 275,000 transistors. About how many years later would you expect a processor at the same price to have about 4.4 million transistors?

Describe the domain, range, symmetry, continuity, and increasing/decreasing behavior for each logarithmic function with the given intercept and end behavior. Then sketch a graph of the function.

56. $f(1) = 0;\ \lim_{x \to 0} f(x) = -\infty;\ \lim_{x \to \infty} f(x) = \infty$

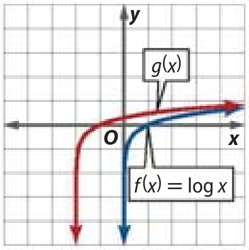
57. $g(-2) = 0;\ \lim_{x \to -3} g(x) = -\infty;\ \lim_{x \to \infty} g(x) = \infty$

58. $h(-1) = 0;\ \lim_{x \to -\infty} h(x) = \infty;\ \lim_{x \to 0} h(x) = -\infty$

59. $j(1) = 0;\ \lim_{x \to 0} j(x) = \infty;\ \lim_{x \to \infty} j(x) = -\infty$

Use the parent graph of $f(x) = \log x$ to find the equation of each function.

60.



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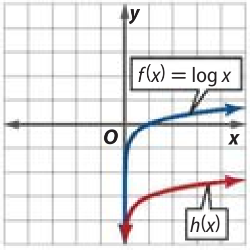

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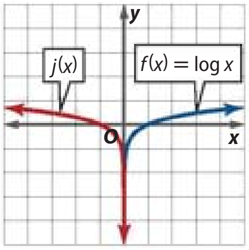

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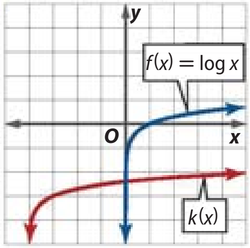


GRAPHING CALCULATOR **Create a scatter plot of the values shown in the table. Then use the graph to determine whether each statement is *true* or *false*.**

x	1	3	9	27
y	0	1	2	3

64. y is an exponential function of x.

65. x is an exponential function of y.

66. y is a logarithmic function of x.

67. y is inversely proportional to x.

68. **BACTERIA** The function $t = \frac{\ln B - \ln A}{2}$ models the amount of time t in hours for a specific bacteria to reach amount B from the initial amount A.

a. If the initial number of bacteria present is 750, how many hours would it take for the number of bacteria to reach 300,000? Round to the nearest hour.

b. Determine the average rate of change in bacteria per hour for the amounts in part **a**.

69. **MULTIPLE REPRESENTATIONS** In this problem, you will compare the average rates of change for an exponential, a power, and a radical function.

a. **GRAPHICAL** Graph $f(x) = 2^x$ and $g(x) = x^2$ for $0 \le x \le 8$.

b. **ANALYTICAL** Find the average rate of change of each function from part **a** on the interval [4, 6].

c. **VERBAL** Compare the growth rates of the functions from part **a** as x increases.

d. **GRAPHICAL** Graph $f(x) = \ln x$ and $g(x) = \sqrt{x}$.

e. **ANALYTICAL** Find the average rate of change of each function from part **d** on the interval [4, 6].

f. **VERBAL** Compare the growth rates of the functions from part **d** as x increases.

H.O.T. Problems Use Higher-Order Thinking Skills

70. **WRITING IN MATH** Compare and contrast the domain, range, intercepts, symmetry, continuity, increasing/decreasing behavior and end behavior of logarithmic functions with $a(x) = x^n$, $b(x) = x^{-1}$, $c(x) = a^x$, and $d(x) = e^x$.

71. **REASONING** Explain why b cannot be negative in $f(x) = \log_b x$.

72. **CHALLENGE** For $f(x) = \log_{10} (x - k)$, where k is a constant, what are the coordinates of the x-intercept?

73. **WRITING IN MATH** Compare the large-scale behavior of exponential and logarithmic functions with base b for $b = 2$, 6, and 10.

REASONING **Determine whether each statement is *true* or *false*.**

74. Logarithmic functions will always have a restriction on the domain.

75. Logarithmic functions will never have a restriction on the range.

76. Graphs of logarithmic functions always have an asymptote.

77. **WRITING IN MATH** Use words, graphs, tables, and equations to compare logarithmic and exponential functions.

Spiral Review

78. AVIATION When kerosene is purified to make jet fuel, pollutants are removed by passing the kerosene through a special clay filter. Suppose a filter is fitted in a pipe so that 15% of the impurities are removed for every foot that the kerosene travels. (Lesson 3-1)

a. Write an exponential function to model the percent of impurity left after the kerosene travels x feet.

b. Graph the function.

c. About what percent of the impurity remains after the kerosene travels 12 feet?

d. Will the impurities ever be completely removed? Explain.

Solve each inequality. (Lesson 2-6)

79. $x^2 - 3x - 2 > 8$

80. $4 \geq -(x - 2)^3 + 3$

81. $\frac{2}{x} + 3 > \frac{29}{x}$

82. $\frac{(x - 3)(x - 4)}{(x - 5)(x - 6)^2} \leq 0$

83. $\sqrt{2x + 3} - 4 \leq 5$

84. $\sqrt{x - 5} + \sqrt{x + 7} \leq 4$

Solve each equation. (Lesson 2-5)

85. $\frac{2a - 5}{a - 9} + \frac{a}{a + 9} = \frac{-6}{a^2 - 81}$

86. $\frac{2q}{2q + 3} - \frac{2q}{2q - 3} = 1$

87. $\frac{4}{z - 2} - \frac{z + 6}{z + 1} = 1$

Graph and analyze each function. Describe its domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Lesson 2-1)

88. $f(x) = -\frac{1}{2}x^7$

89. $g(x) = 3x^{-6}$

90. $h(x) = 2x^{-\frac{3}{4}}$

91. MICROBIOLOGY One model for the population P of bacteria in a sample after t days is given by $P(t) = 1000 - 19.75t + 20t^2 - \frac{1}{3}t^3$. (Lesson 1-2)

a. What type of function is $P(t)$?

b. When is the bacteria population increasing?

c. When is it decreasing?

Skills Review for Standardized Tests

92. SAT/ACT The table below shows the per unit revenue and cost of three products at a sports equipment factory.

Product	Revenue per Unit ($)	Cost per Unit ($)
football	f	4
baseball	b	3
soccer ball	6	y

If profit equals revenue minus cost, how much profit do they make if they produce and sell two of each item?

A $2f + 2b - 2y - 2$

B $2y - 2b - 2f - 2$

C $f + b - y - 1$

D $b + 2f + y - 7$

E $2f + 2b - 2y - 26$

93. What is the value of n if $\log_3 3^{4n - 1} = 11$?

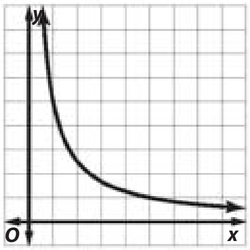
F 3

G 4

H 6

J 12

94. REVIEW The curve represents a portion of the graph of which function?



A $y = 50 - x$

B $y = \log x$

C $y = e^{-x}$

D $y = \frac{5}{x}$

95. REVIEW A radioactive element decays over time according to

$$y = x\left(\frac{1}{4}\right)^{\frac{t}{200}},$$

where x = the number of grams present initially and t = time in years. If 500 grams were present initially, how many grams will remain after 400 years?

F 12.5 grams

G 31.25 grams

H 62.5 grams

J 125 grams

LESSON 3-3 Properties of Logarithms

Then

- You evaluated logarithmic expressions with different bases. (Lesson 3-2)

Now

1. Apply properties of logarithms.
2. Apply the Change of Base Formula.

Why?

- Plants take in carbon-14 through photosynthesis, and animals and humans take in carbon-14 by ingesting plant material. When an organism dies, it stops taking in new carbon, and the carbon-14 already in its system starts to decay. Scientists can calculate the age of organic materials using a logarithmic function that estimates the decay of carbon-14. Properties of logarithms can be used to analyze this function.

1 Properties of Logarithms

Recall that the following properties of exponents, where b, x, and y are positive real numbers.

Product Property	Quotient Property	Power Property
$b^x \cdot b^y = b^{x+y}$	$\frac{b^x}{b^y} = b^{x-y}$	$(b^x)^y = b^{xy}$

Since logarithms and exponents have an inverse relationship, these properties of exponents imply these corresponding properties of logarithms.

KeyConcept Properties of Logarithms

If b, x, and y are positive real numbers, $b \neq 1$, and p is a real number, then the following statements are true.

Product Property	$\log_b xy = \log_b x + \log_b y$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$
Power Property	$\log_b x^p = p \log_b x$

You will prove the Quotient and Power Properties in Exercises 113 and 114.

To show that the Product Property of Logarithms is true, let $m = \log_b x$ and $n = \log_b y$. Then, using the definition of logarithm, $b^m = x$ and $b^n = y$.

$\log_b xy = \log_b b^m b^n$	$x = b^m$ and $y = b^n$
$= \log_b b^{m+n}$	Product Property of Exponents
$= m + n$	Inverse Property of Logarithms
$= \log_b x + \log_b y$	$m = \log_b x$ and $n = \log_b y$

These properties can be used to express logarithms in terms of other logarithms.

Example 1 Use the Properties of Logarithms

Express each logarithm in terms of ln 2 and ln 3.

a. $\ln 54$

$\ln 54 = \ln (2 \cdot 3^3)$	$54 = 2 \cdot 3^3$
$= \ln 2 + \ln 3^3$	Product Property
$= \ln 2 + 3 \ln 3$	Power Property

b. $\ln \frac{9}{8}$

$\ln \frac{9}{8} = \ln 9 - \ln 8$	Quotient Property
$= \ln 3^2 - \ln 2^3$	$3^2 = 9$ and $2^3 = 8$
$= 2 \ln 3 - 3 \ln 2$	Power Property

GuidedPractice

Express each logarithm in terms of log 5 and log 3.

1A. $\log 75$

1B. $\log 5.4$

Michel Gunther/Photolibrary

Math History Link

Joost Burgi

(1550–1617)

A Swiss mathematician, Burgi was a renowned clockmaker who also created and designed astronomical instruments. His greatest works in mathematics came when he discovered logarithms independently from John Napier.

The Product, Quotient, and Power Properties can also be used to simplify logarithms.

Example 2 Simplify Logarithms

Evaluate each logarithm.

a. $\log_4 \sqrt[5]{64}$

Since the base of the logarithm is 4, express $\sqrt[5]{64}$ as a power of 4.

$\log_4 \sqrt[5]{64} = \log_4 64^{\frac{1}{5}}$ Rewrite using rational exponents.

$= \log_4 (4^3)^{\frac{1}{5}}$ $4^3 = 64$

$= \log_4 4^{\frac{3}{5}}$ Power Property of Exponents

$= \frac{3}{5} \log_4 4$ Power Property of Logarithms

$= \frac{3}{5}(1) \text{ or } \frac{3}{5}$ $\log_x x = 1$

b. $5 \ln e^2 - \ln e^3$

$5 \ln e^2 - \ln e^3 = 5(2 \ln e) - 3 \ln e$ Power Property of Logarithms

$= 10 \ln e - 3 \ln e$ Multiply.

$= 10(1) - 3(1) \text{ or } 7$ $\ln e = 1$

Guided Practice

2A. $\log_6 \sqrt[3]{36}$

2B. $\ln e^9 + 4 \ln e^3$

The properties of logarithms provide a way of expressing logarithmic expressions in forms that use simpler operations, converting multiplication into addition, division into subtraction, and powers and roots into multiplication.

Example 3 Expand Logarithmic Expressions

Expand each expression.

a. $\log 12x^5y^{-2}$

The expression is the logarithm of the product of 12, x^5, and y^2.

$\log 12x^5y^{-2} = \log 12 + \log x^5 + \log y^{-2}$ Product Property

$= \log 12 + 5 \log x - 2 \log y$ Power Property

b. $\ln \frac{x^2}{\sqrt{4x+1}}$

The expression is the logarithm of the quotient of x^2 and $\sqrt{4x+1}$.

$\ln \frac{x^2}{\sqrt{4x+1}} = \ln x^2 - \ln \sqrt{4x+1}$ Quotient Property

$= \ln x^2 - \ln (4x+1)^{\frac{1}{2}}$ $\sqrt{4x+1} = (4x+1)^{\frac{1}{2}}$

$= 2 \ln x - \frac{1}{2} \ln (4x+1)$ Power Property

Guided Practice

3A. $\log_{13} 6a^3bc^4$

3B. $\ln \frac{3y+2}{4\sqrt[3]{y}}$

The same methods used to expand logarithmic expressions can be used to condense them.

Example 4 Condense Logarithmic Expressions

Condense each expression.

a. $\mathbf{4 \log_3 x - \frac{1}{3} \log_3 (x + 6)}$

$$\begin{aligned} 4 \log_3 x - \frac{1}{3} \log_3 (x + 6) &= \log_3 x^4 - \log_3 (x + 6)^{\frac{1}{3}} && \text{Power Property} \\ &= \log_3 x^4 - \log_3 \sqrt[3]{x + 6} && (x + 6)^{\frac{1}{3}} = \sqrt[3]{x + 6} \\ &= \log_3 \frac{x^4}{\sqrt[3]{x + 6}} && \text{Quotient Property} \\ &= \log_3 \frac{x^4 \sqrt[3]{(x + 6)^2}}{x + 6} && \text{Rationalize the denominator.} \end{aligned}$$

b. $\mathbf{6 \ln (x - 4) + 3 \ln x}$

$$\begin{aligned} 6 \ln (x - 4) + 3 \ln x &= \ln (x - 4)^6 + \ln x^3 && \text{Power Property} \\ &= \ln x^3(x - 4)^6 && \text{Product Property} \end{aligned}$$

WatchOut!

Logarithm of a Sum The logarithm of a sum or difference does not equal the sum or difference of logarithms. For example, $\ln (x \pm 4) \neq \ln x \pm \ln 4$.

GuidedPractice

4A. $-5 \log_2 (x + 1) + 3 \log_2 (6x)$

4B. $\ln (3x + 5) - 4 \ln x - \ln (x - 1)$

2 Change of Base Formula

Sometimes you may need to work with a logarithm that has an inconvenient base. For example, evaluating $\log_3 5$ presents a challenge because calculators have no key for evaluating base 3 logarithms. The Change of Base Formula provides a way of changing such an expression into a quotient of logarithms with a different base.

KeyConcept Change of Base Formula

For any positive real numbers *a*, *b*, and *x*, $a \neq 1$, $b \neq 1$,

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

You will prove the Change of Base Formula in Exercise 115.

Most calculators have only two keys for logarithms, LOG for base 10 logarithms and LN for base *e* logarithms. Therefore, you will often use the Change of Base Formula in one of the following two forms. Either method will provide the correct answer.

$$\log_b x = \frac{\log x}{\log b} \qquad \log_b x = \frac{\ln x}{\ln b}$$

Example 5 Use the Change of Base Formula

Evaluate each logarithm.

a. $\mathbf{\log_3 5}$

$$\begin{aligned} \log_3 5 &= \frac{\ln 5}{\ln 3} && \text{Change of Base Formula} \\ &\approx 1.47 && \text{Use a calculator.} \end{aligned}$$

b. $\mathbf{\log_{\frac{1}{2}} 6}$

$$\begin{aligned} \log_{\frac{1}{2}} 6 &= \frac{\log 6}{\log \frac{1}{2}} && \text{Change of Base Formula} \\ &\approx -2.58 && \text{Use a calculator.} \end{aligned}$$

StudyTip

Check for Reasonableness You can check your answer in Example 5a by evaluating $3^{1.47}$. Because $3^{1.47} \approx 5$, the answer is reasonable.

GuidedPractice

5A. $\log_{78} 4212$

5B. $\log_{15} 33$

5C. $\log_{\frac{1}{3}} 10$

You can use properties of logarithms to solve real-world problems. For example, the ratio of the frequencies of a note in one octave and the same note in the next octave is 2 : 1. Therefore, further octaves will occur at 2^n times the frequency of that note, where n is an integer. This relationship can be used to find the difference in pitch between any two notes.

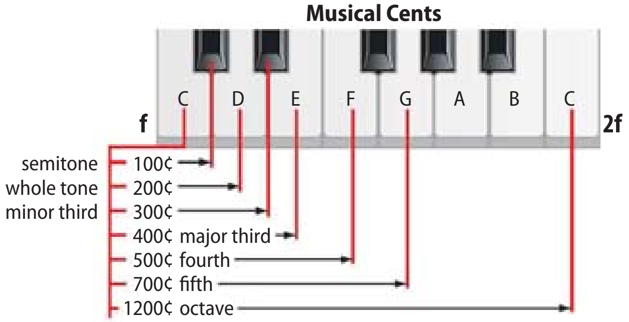
Real-WorldLink

Standard pitch, also called concert pitch, is the pitch used by orchestra members to tune their instruments. The frequency of standard pitch is 440 hertz, which is equivalent to the note A in the fourth octave.

Source: *Encyclopaedia Britannica*

Example 6 Use the Change of Base Formula

MUSIC **The musical *cent* (¢) is a unit of relative pitch. One octave consists of 1200 cents.**



The formula to determine the difference in cents between two notes with beginning frequency a and ending frequency b is $n = 1200\left(\log_2 \frac{a}{b}\right)$. Find the difference in pitch between each of the following pairs of notes.

a. **493.9 Hz, 293.7 Hz**

Let $a = 493.9$ and $b = 293.7$. Substitute for the values of a and b and solve.

$$n = 1200\left(\log_2 \frac{a}{b}\right) \quad \text{Original equation}$$

$$= 1200\left(\log_2 \frac{493.9}{293.7}\right) \quad a = 493.9 \text{ and } b = 293.7$$

$$= 1200\left(\frac{\log \frac{493.9}{293.7}}{\log 2}\right) \quad \text{Change of Base Formula}$$

$$\approx 899.85 \quad \text{Simplify.}$$

The difference in pitch between the notes is approximately 899.85 cents.

b. **3135.9 Hz, 2637 Hz**

Let $a = 3135.9$ and $b = 2637$. Substitute for the values of a and b and solve.

$$n = 1200\left(\log_2 \frac{a}{b}\right) \quad \text{Original equation}$$

$$= 1200\left(\log_2 \frac{3135.9}{2637}\right) \quad a = 3135.9 \text{ and } b = 2637$$

$$= 1200\left(\frac{\log \frac{3135.9}{2637}}{\log 2}\right) \quad \text{Change of Base Formula}$$

$$\approx 299.98 \quad \text{Simplify.}$$

The difference in pitch between the notes is approximately 299.98 cents.

GuidedPractice

6. **PHOTOGRAPHY** In photography, exposure is the amount of light allowed to strike the film. Exposure can be adjusted by the number of stops used to take a photograph. The change in the number of stops n needed is related to the change in exposure c by $n = \log_2 c$.

A. How many stops would a photographer use to triple the exposure?

B. How many stops would a photographer use to get $\frac{1}{5}$ the exposure?

Stockbyte/Getty Images

Exercises

● = Step-by-Step Solutions begin on page R29.

Express each logarithm in terms of ln 2 and ln 5. (Example 1)

1. $\ln \frac{4}{5}$

2. $\ln 200$

3. $\ln 80$

4. $\ln 12.5$

5. $\ln \frac{0.8}{2}$

6. $\ln \frac{2}{5}$

7. $\ln 2000$

8. $\ln 1.6$

Express each logarithm in terms of ln 3 and ln 7. (Example 1)

9. $\ln 63$

10. $\ln \frac{49}{81}$

11. $\ln \frac{7}{9}$

12. $\ln 147$

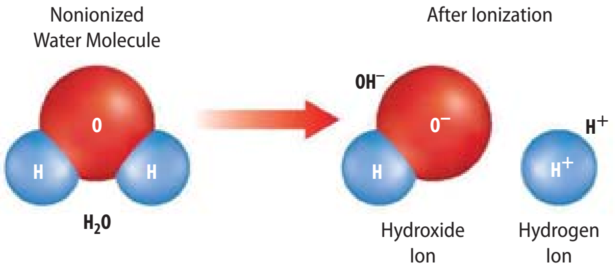
13. $\ln 1323$

14. $\ln \frac{343}{729}$

15. $\ln \frac{2401}{81}$

16. $\ln 1701$

17. **CHEMISTRY** The ionization constant of water K_w is the product of the concentrations of hydrogen (H^+) and hydroxide (OH^-) ions.



The formula for the ionization constant of water is $K_w = [H^+][OH^-]$, where the brackets denote concentration in moles per liter. (Example 1)

a. Express $\log K_w$ in terms of $\log [H^+]$ and $\log [OH^-]$.

b. The value of the constant K_w is 1×10^{-14}. Simplify your equation from part **a** to reflect the numerical value of K_w.

c. If the concentration of hydrogen ions in a sample of water is 1×10^{-9} moles per liter, what is the concentration of hydroxide ions?

18. **TORNADOES** The distance d in miles that a tornado travels is $d = 10^{\frac{w-65}{93}}$, where w is the wind speed in miles per hour of the tornado. (Example 1)

a. Express w in terms of $\log d$.

b. If a tornado travels 100 miles, estimate the wind speed.

Evaluate each logarithm. (Example 2)

19. $\log_5 \sqrt[4]{25}$

20. $8 \ln e^2 - \ln e^{12}$

21. $9 \ln e^3 + 4 \ln e^5$

22. $\log_2 \sqrt[5]{32}$

23. $2 \log_3 \sqrt{27}$

24. $3 \log_7 \sqrt[6]{49}$

25. $4 \log_2 \sqrt{8}$

26. $50 \log_5 \sqrt{125}$

27. $\log_3 \sqrt[6]{243}$

28. $36 \ln e^{0.5} - 4 \ln e^5$

Expand each expression. (Example 3)

29. $\log_9 6x^3y^5z$

30. $\ln \frac{x^7}{\sqrt[3]{x+2}}$

31. $\log_3 \frac{p^2q}{\sqrt[5]{3q-1}}$

32. $\ln \frac{4df^5}{\sqrt[8]{1-3d}}$

33. $\log_{11} ab^{-4}c^{12}d^7$

34. $\log_7 h^2j^{11}k^{-5}$

35. $\log_4 10t^2uv^{-3}$

36. $\log_5 a^6b^{-3}c^4$

37. $\ln \frac{3a^4b^7c}{\sqrt[4]{b-9}}$

38. $\log_2 \frac{3x+2}{\sqrt[7]{1-5x}}$

Condense each expression. (Example 4)

39. $3 \log_5 x - \frac{1}{2} \log_5 (6-x)$

40. $5 \log_7 (2x) - \frac{1}{3} \log_7 (5x+1)$

41. $7 \log_3 a + \log_3 b - 2 \log_3 (8c)$

42. $4 \ln (x+3) - \frac{1}{5} \ln (4x+7)$

43. $2 \log_8 (9x) - \log_8 (2x-5)$

44. $\ln 13 + 7 \ln a - 11 \ln b + \ln c$

45. $2 \log_6 (5a) + \log_6 b + 7 \log_6 c$

46. $\log_2 x - \log_2 y - 3 \log_2 z$

47. $\frac{1}{4} \ln (2a-b) - \frac{1}{5} \ln (3b+c)$

48. $\log_3 4 - \frac{1}{2} \log_3 (6x-5)$

Evaluate each logarithm. (Example 5)

49. $\log_6 14$

50. $\log_3 10$

51. $\log_7 5$

52. $\log_{128} 2$

53. $\log_{12} 145$

54. $\log_{22} 400$

55. $\log_{100} 101$

56. $\log_{\frac{1}{2}} \frac{1}{3}$

57. $\log_{-2} 8$

58. $\log_{13,000} 13$

59 **COMPUTERS** Computer programs are written in sets of instructions called *algorithms*. To execute a task in a computer program, the algorithm coding in the program must be analyzed. The running time in seconds R that it takes to analyze an algorithm of n steps can be modeled by $R = \log_2 n$. (Example 6)

a. Determine the running time to analyze an algorithm of 240 steps.

b. To the nearest step, how many steps are in an algorithm with a running time of 8.45 seconds?

60. **TRUCKING** Bill's Trucking Service purchased a new delivery truck for \$56,000. Suppose $t = \log_{(1-r)} \frac{V}{P}$ represents the time t in years that has passed since the purchase given its initial price P, present value V, and annual rate of depreciation r. (Example 6)

a. If the truck's present value is \$40,000 and it has depreciated at a rate of 15% per year, how much time has passed since its purchase to the nearest year?

b. If the truck's present value is \$34,000 and it has depreciated at a rate of 10% per year, how much time has passed since its purchase to the nearest year?

Estimate each logarithm to the nearest whole number.

61. $\log_4 5$

62. $\log_2 13$

63. $\log_3 10$

64. $\log_7 400$

65. $\log_5 \frac{1}{124}$

66. $\log_{12} 177$

67. $\log_{\frac{1}{5}} \frac{1}{6}$

68. $\log_4 \frac{1}{165}$

Expand each expression.

69. $\ln \sqrt[5]{x^3(x+3)}$

70. $\log_5 \frac{x^2y^5}{\sqrt[3]{4x-y}}$

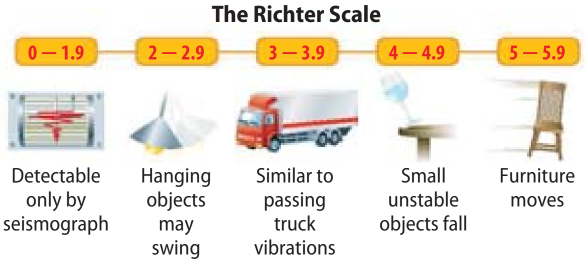
(71) $\log_{14} \frac{11}{\sqrt[4]{x^5(8x-1)}}$

72. $\ln \frac{9x^2yz^3}{(y-5)^4}$

73. $\log_8 \sqrt[7]{x^3y^2(z-1)}$

74. $\log_{12} \frac{5x}{\sqrt[6]{x^7(x+13)}}$

75. **EARTHQUAKES** The Richter scale measures the intensity of an earthquake. The magnitude M of the seismic energy in joules E released by an earthquake can be calculated by $M = \frac{2}{3} \log \frac{E}{10^{4.4}}$.


The Richter Scale

a. Use the properties of logarithms to expand the equation.

b. What magnitude would an earthquake releasing 7.94×10^{11} joules have?

c. The 2007 Alum Rock earthquake in California released 4.47×10^{12} joules of energy. The 1964 Anchorage earthquake in Alaska released 1.58×10^{18} joules of energy. How many times as great was the magnitude of the Anchorage earthquake as the magnitude of the Alum Rock earthquake?

d. Generally, earthquakes cannot be felt until they reach a magnitude of 3 on the Richter scale. How many joules of energy does an earthquake of this magnitude release?

Condense each expression.

76. $\frac{3}{4} \ln x + \frac{7}{4} \ln y + \frac{5}{4} \ln z$

77. $\log_2 15 + 6 \log_2 x - \frac{4}{3} \log_2 x - \frac{1}{3} \log_2 (x+3)$

78. $\ln 14 - \frac{2}{3} \ln 3x - \frac{4}{3} \ln (4-3x)$

79. $3 \log_6 2x + 9 \log_6 y - \frac{4}{5} \log_6 x - \frac{8}{5} \log_6 y - \frac{1}{5} \log_6 z$

80. $\log_4 25 - \frac{5}{2} \log_4 x - \frac{7}{2} \log_4 y - \frac{3}{2} \log_4 (z+9)$

81. $\frac{5}{2} \ln x + \frac{1}{2} \ln (y+8) - 3 \ln y - \ln (10-x)$

Use the properties of logarithms to rewrite each logarithm below in the form $a \ln 2 + b \ln 3$, where a and b are constants. Then approximate the value of each logarithm given that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$.

82. $\ln 4$

83. $\ln 48$

84. $\ln 162$

85. $\ln 216$

86. $\ln \frac{3}{2}$

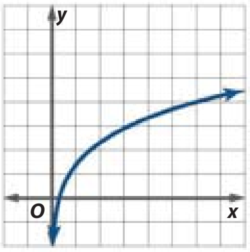
87. $\ln \frac{4}{9}$

88. $\ln \frac{4}{27}$

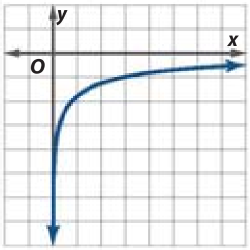
89. $\ln \frac{32}{9}$

Determine the graph that corresponds to each equation.

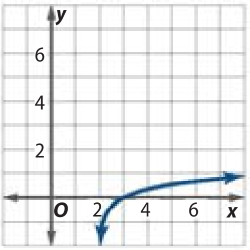
a.



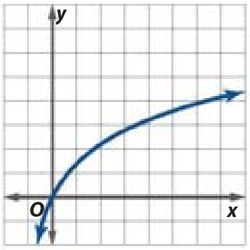
b.



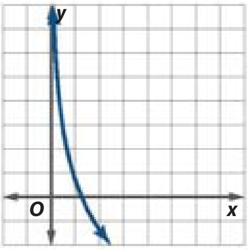
c.



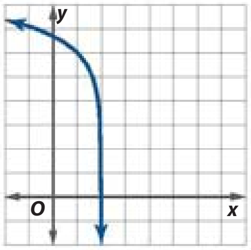
d.



e.



f.



90. $f(x) = \ln x + \ln (x+3)$

91. $f(x) = \ln x - \ln (x+5)$

92. $f(x) = 2 \ln (x+1)$

93. $f(x) = 0.5 \ln (x-2)$

94. $f(x) = \ln (2-x) + 6$

95. $f(x) = \ln 2x - 4 \ln x$

Write each set of logarithmic expressions in increasing order.

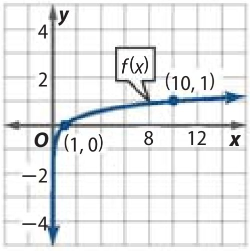
96. $\log_3 \frac{12}{4}$, $\log_3 \frac{36}{3} + \log_3 4$, $\log_3 12 - 2 \log_3 4$

97. $\log_5 55$, $\log_5 \sqrt{100}$, $3 \log_5 \sqrt[3]{75}$

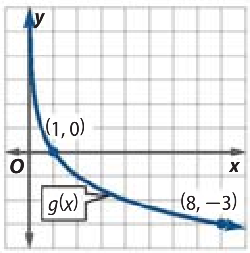
98. BIOLOGY The generation time for bacteria is the time that it takes for the population to double. The generation time G can be found using $G = \frac{t}{3.3 \log_b f}$, where t is the time period, b is the number of bacteria at the beginning of the experiment, and f is the number of bacteria at the end of the experiment. The generation time for mycobacterium tuberculosis is 16 hours. How long will it take 4 of these bacteria to multiply into 1024 bacteria?

Write an equation for each graph.

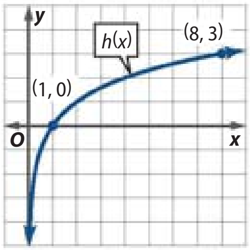
99.



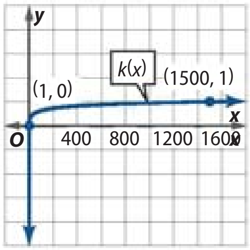
100.



101.



102.



103. CHEMISTRY pK_a is the logarithmic acid dissociation constant for the acid HF, which is composed of ions H^+ and F^-. The pK_a can be calculated by $pK_a = -\log \frac{[H^+][F^-]}{[HF]}$, where $[H^+]$ is the concentration of H^+ ions, $[F^-]$ is the concentration of F^- ions, and [HF] is the concentration of the acid solution. All of the concentrations are measured in moles per liter.

a. Use the properties of logs to expand the equation for pK_a.

b. What is the pK_a of a reaction in which $[H^+] = 0.01$ moles per liter, $[F^-] = 0.01$ moles per liter, and $[HF] = 2$ moles per liter?

c. The acid dissociation constant K_a of a substance can be calculated by $K_a = \frac{[H^+][F^-]}{[HF]}$. If a substance has a $pK_a = 25$, what is its K_a?

d. Aldehydes are a common functional group in organic molecules. Aldehydes have a pK_a around 17. To what K_a does this correspond?

Evaluate each expression.

104. $\ln\left[\ln\left(e^{e^6}\right)\right]$

105. $10^{\log e^{\ln 4}}$

106. $4 \log_{17} 17^{\log_{10} 100}$

107. $e^{\log_4 4^{\ln 2}}$

Simplify each expression.

108. $(\log_3 6)(\log_6 13)$

109 $(\log_2 7)(\log_5 2)$

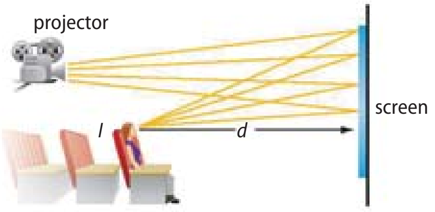
110. $(\log_4 9) \div (\log_4 2)$

111. $(\log_5 12) \div (\log_8 12)$

112. MOVIES Traditional movies are a sequence of still pictures which, if shown fast enough, give the viewer the impression of motion. If the frequency of the stills shown is too small, the moviegoer notices a flicker between each picture. Suppose the minimum frequency f at which the flicker first disappears is given by $f = K \log I$, where I is the intensity of the light from the screen that reaches the viewer and K is the constant of proportionality.

a. The intensity of the light perceived by a moviegoer who sits at a distance d from the screen is given by $I = \frac{k}{d^2}$, where k is a constant of proportionality.

Show that $f = K(\log k - 2 \log d)$.



b. Suppose you notice the flicker from a movie projection and move to double your distance from the screen. In terms of K, how does this move affect the value of f? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

PROOF Investigate graphically and then prove each of the following properties of logarithms.

113. Quotient Property

114. Power Property

115. PROOF Prove that $\log_b x = \frac{\log_a x}{\log_a b}$.

116. REASONING How can the graph of $g(x) = \log_4 x$ be obtained using a transformation of the graph of $f(x) = \ln x$?

117. CHALLENGE If $x \in \mathbb{N}$, for what values of x can $\ln x$ not be simplified?

118. ERROR ANALYSIS Omar and Nate expanded $\log_2 \left(\frac{xy}{z}\right)^4$ using the properties of logarithms. Is either of them correct? Explain.

Omar: $4 \log_2 x + 4 \log_2 y - 4 \log_2 z$

Nate: $2 \log_4 x + 2 \log_4 y - 2 \log_4 z$

119 **PROOF** Use logarithmic properties to prove $\frac{\log_5 (nt)^2}{\log_4 \frac{t}{r}} = \frac{2 \log n \log 4 + 2 \log t \log 4}{\log 5 \log t - \log 5 \log r}$.

120. WRITING IN MATH The graph of $g(x) = \log_b x$ is actually a transformation of $f(x) = \log x$. Use the Change of Base Formula to find the transformation that relates these two graphs. Then explain the effect that different values of b have on the common logarithm graph.

Spiral Review

Sketch and analyze each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing. (Lesson 3-2)

121. $f(x) = \log_6 x$

122. $g(x) = \log_{\frac{1}{3}} x$

123. $h(x) = \log_5 x - 2$

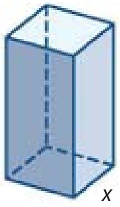
Use the graph of $f(x)$ to describe the transformation that yields the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$. (Lesson 3-1)

124. $f(x) = 2^x$; $g(x) = -2^x$

125. $f(x) = 5^x$; $g(x) = 5^{x+3}$

126. $f(x) = \left(\frac{1}{4}\right)^x$; $g(x) = \left(\frac{1}{4}\right)^x - 2$

127. GEOMETRY The volume of a rectangular prism with a square base is fixed at 120 cubic feet. (Lesson 2-5)



- **a.** Write the surface area of the prism as a function $A(x)$ of the length of the side of the square x.
- **b.** Graph the surface area function.
- **c.** What happens to the surface area of the prism as the length of the side of the square approaches 0?

Divide using synthetic division. (Lesson 2-3)

128. $(x^2 - x + 4) \div (x - 2)$

129. $(x^3 + x^2 - 17x + 15) \div (x + 5)$

130. $(x^3 - x^2 + 2) \div (x + 1)$

Show that f and g are inverse functions. Then graph each function on the same graphing calculator screen. (Lesson 1-7)

131. $f(x) = -\frac{2}{3}x + \frac{1}{6}$

$g(x) = -\frac{3}{2}x + \frac{1}{4}$

132. $f(x) = \frac{1}{x+2}$

$g(x) = \frac{1}{x} - 2$

133. $f(x) = (x - 3)^3 + 4$

$g(x) = \sqrt[3]{x - 4} + 3$

134. SCIENCE Specific heat is the amount of energy per unit of mass required to raise the temperature of a substance by one degree Celsius. The table lists the specific heat in joules per gram for certain substances. The amount of energy transferred is given by $Q = cmT$, where c is the specific heat for a substance, m is its mass, and T is the change in temperature. (Lesson 1-5)

- **a.** Find the function for the change in temperature.
- **b.** What is the parent graph of this function?
- **c.** What is the relevant domain of this function?

Substance	Specific Heat (j/g)
aluminum	0.902
gold	0.129
mercury	0.140
iron	0.45
ice	2.03
water	4.179
air	1.01

Skills Review for Standardized Tests

135. SAT/ACT If $b \neq 0$, let $a \triangle b = \frac{a^2}{b^2}$. If $x \triangle y = 1$, then which statement must be true?

A $x = y$

B $x = -y$

C $x^2 - y^2 = 0$

D $x > 0$ and $y > 0$

E $x = |y|$

136. REVIEW Find the value of x for $\log_2 (9x + 5) = 2 + \log_2 (x^2 - 1)$.

F -0.4

G 0

H 1

J 3

137. To what is $2 \log_5 12 - \log_5 8 - 2 \log_5 3$ equal?

A $\log_5 2$

B $\log_5 3$

C $\log_5 0.5$

D 1

138. REVIEW The weight of a bar of soap decreases by 2.5% each time it is used. If the bar of soap weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses?

F 58 g

G 59 g

H 65 g

J 93 g

CHAPTER 3

Mid-Chapter Quiz

Lessons 3-1 through 3-3

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing. (Lesson 3-1)

1. $f(x) = 5^{-x}$

2. $f(x) = \left(\frac{2}{3}\right)^x + 3$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$. (Lesson 3-1)

3. $f(x) = \left(\frac{3}{2}\right)^x$; $g(x) = \left(\frac{3}{2}\right)^{-x}$

4. $f(x) = 3^x$; $g(x) = 2 \cdot 3^{x-2}$

5. $f(x) = e^x$; $g(x) = -e^x - 6$

6. $f(x) = 10^x$; $g(x) = 10^{2x}$

7. **MULTIPLE CHOICE** In the formula for compound interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$, which variable has NO effect on the amount of time it takes an investment to double? (Lesson 3-1)

A P

B r

C n

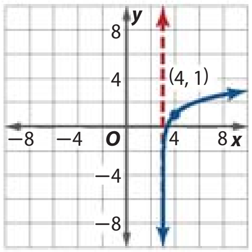
D t

8. **FINANCIAL LITERACY** Clarissa has saved $1200 from working summer jobs and would like to invest it so that she has some extra money when she graduates from college in 5 years. (Lesson 3-1)

a. How much money will Clarissa have if she invests at an annual rate of 7.2% compounded monthly?

b. How much money will Clarissa have if she invests at an annual rate of 7.2% compounded continuously?

9. **MULTIPLE CHOICE** The parent function for the graph shown is $f(x) = \log_2 x$.



The graph contains the given point and has the vertical asymptote shown. Which of the following is the function for the graph? (Lesson 3-2)

F $f(x) = \log_2 (x + 3) + 1$

G $f(x) = \log_2 (x - 4) + 1$

H $f(x) = -\log_2 (x - 3) + 1$

J $f(x) = \log_2 (x - 3) + 1$

Evaluate each expression. (Lesson 3-2)

10. $\log_2 64$

11. $\log_5 \frac{1}{125}$

12. $\ln e^{23}$

13. $\log 0.001$

14. **DISEASE** The number of children infected by a virus can be modeled by $c(d) = 4.9 + 11.2 \ln d$, where d is the number of days since the first child was infected. About how many children are infected on day 8? (Lesson 3-2)

Evaluate each function for the given value. (Lesson 3-2)

15. $T(x) = 2 \ln (x + 3)$; $x = 18$

16. $H(a) = 4 \log \frac{2a}{5} - 8$; $a = 25$

Express each logarithm in terms of ln 3 and ln 4. (Lesson 3-3)

17. $\ln 48$

18. $\ln 2.25$

19. $\ln \frac{64}{27}$

20. $\ln \frac{9}{16}$

21. **CHEMISTRY** The half-life of a radioactive isotope is 7 years. (Lesson 3-3)

a. If there is initially 75 grams of the substance, how much of the substance will remain after 14 years?

b. After how many years will there be $\frac{1}{16}$ of the original amount remaining?

c. The time it takes for a substance to decay from N_0 to N can be modeled by $t = 7 \log_{0.5} \frac{N}{N_0}$. Approximately how many years will it take for any amount of the radioactive substance to decay to $\frac{1}{3}$ its original amount?

Expand each expression. (Lesson 3-3)

22. $\log_3 \sqrt[4]{x^2y^3z^5}$

23. $\log_9 \frac{3x^3}{y}$

Condense each expression. (Lesson 3-3)

24. $5 \log_4 a + 6 \log_4 b - \frac{1}{3} \log_4 7c$

25. $2 \log (x + 1) - \log (x^2 - 1)$

LESSON 3-4

Exponential and Logarithmic Equations

Then

- You applied the inverse properties of exponents and logarithms to simplify expressions. (Lesson 3-2)

Now

- **1** Apply the One-to-One Property of Exponential Functions to solve equations.

  2 Apply the One-to-One Property of Logarithmic Functions to solve equations.

Why?

- The intensity of an earthquake can be calculated using $R = \log \frac{a}{T} + B$, where R is the Richter scale number, a is the amplitude of the vertical ground motion, T is the period of the seismic wave in seconds, and B is a factor that accounts for the weakening of the seismic waves.

1 One-to-One Property of Exponential Functions

In Lesson 3-2, exponential functions were shown to be one-to-one. Recall from Lesson 1-7 that if a function f is one-to-one, no y-value is matched with more than one x-value. That is, $f(a) = f(b)$ if and only if $a = b$. This leads us to the following One-to-One Property of Exponential Functions.

KeyConcept One-to-One Property of Exponential Functions

Words	For $b > 0$ and $b \neq 1$, $b^x = b^y$ if and only if $x = y$.	
Examples	If $3^x = 3^5$, then $x = 5$.	If $\log x = 3$, then $10^{\log x} = 10^3$.

This is also known as the Property of Equality for exponential functions.

The *if and only if* wording in this property implies two separate statements. One of them, $b^x = b^y$ if $x = y$, can be used to solve some simple exponential equations by first expressing both sides of the equation in terms of a common base.

Example 1 Solve Exponential Equations Using One-to-One Property

Solve each equation.

a. $36^{x+1} = 6^{x+6}$

$36^{x+1} = 6^{x+6}$	Original equation
$(6^2)^{x+1} = 6^{x+6}$	$6^2 = 36$
$6^{2x+2} = 6^{x+6}$	Power of a Power
$2x + 2 = x + 6$	One-to-One Property
$x + 2 = 6$	Subtract x from each side.
$x = 4$	Subtract 2 from each side. Check this solution in the original equation.

b. $\left(\frac{1}{2}\right)^c = 64^{\frac{1}{2}}$

$\left(\frac{1}{2}\right)^c = 64^{\frac{1}{2}}$	Original equation
$2^{-c} = (2^6)^{\frac{1}{2}}$	$2^{-1} = \frac{1}{2}$, $2^6 = 64$
$2^{-c} = 2^3$	Power of a Power
$-c = 3$	One-to-One Property
$c = -3$	Solve for c. Check this solution in the original equation.

GuidedPractice

1A. $16^{x+3} = 4^{4x+7}$

1B. $\left(\frac{2}{3}\right)^{x-5} = \left(\frac{9}{4}\right)^{\frac{3x}{4}}$

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Another statement that follows from the One-to-One Property of Exponential Functions, if $x = y$, then $b^x = b^y$, can be used to solve *logarithmic equations* such as $\log_2 x = 3$.

$$\begin{aligned} \log_2 x &= 3 && \text{Original equation} \\ 2^{\log_2 x} &= 2^3 && \text{One-to-One Property} \\ x &= 2^3 && \text{Inverse Property} \end{aligned}$$

This application of the One-to-One Property is called *exponentiating* each side of an equation. Notice that the effect of exponentiating each side of $\log_2 x = 3$ is to convert the equation from logarithmic to exponential form.

Example 2 Solve Logarithmic Equations Using One-to-One Property

Solve each logarithmic equation. Round to the nearest hundredth if necessary.

a. $\ln x = 6$

Method 1 Use exponentiation.

$$\begin{aligned} \ln x &= 6 && \text{Original equation} \\ e^{\ln x} &= e^6 && \text{Exponentiate each side.} \\ x &= e^6 && \text{Inverse Property} \\ x &\approx 403.43 && \text{Use a calculator.} \end{aligned}$$

Method 2 Write in exponential form.

$$\begin{aligned} \ln x &= 6 && \text{Original equation} \\ x &= e^6 && \text{Write in exponential form.} \\ x &\approx 403.43 && \text{Use a calculator.} \end{aligned}$$

CHECK $\ln 403.43 \approx 6$ ✓

StudyTip

Solutions to Logarithmic Equations While it is always a good idea to check your solutions to equations, this is especially true of logarithmic equations, since logarithmic functions are only defined on the set of positive real numbers.

b. $6 + 2 \log 5x = 18$

$$\begin{aligned} 6 + 2 \log 5x &= 18 && \text{Original equation} \\ 2 \log 5x &= 12 && \text{Subtract 6 from each side.} \\ \log 5x &= 6 && \text{Divide each side by 2.} \\ 5x &= 10^6 && \text{Write in exponential form.} \\ x &= \frac{10^6}{5} && \text{Divide each side by 5.} \\ x &= 200{,}000 && \text{Simplify. Check this solution in the original equation.} \end{aligned}$$

c. $\log_8 x^3 = 12$

$$\begin{aligned} \log_8 x^3 &= 12 && \text{Original equation} \\ 3 \log_8 x &= 12 && \text{Power Property} \\ \log_8 x &= 4 && \text{Divide each side by 3.} \\ x &= 8^4 \text{ or } 4096 && \text{Write in exponential form and simplify. Check this solution.} \end{aligned}$$

GuidedPractice

2A. $-3 \ln x = -24$

2B. $4 - 3 \log (5x) = 16$

2C. $\log_3 (x^2 - 1) = 4$

2 One-to-One Property of Logarithmic Functions

Logarithmic functions are also one-to-one. Therefore, we can state the following One-to-One Property of Logarithmic Functions.

StudyTip

Property of Equality The One-to-One Property of Logarithmic Functions is also known as the Property of Equality for Logarithmic Functions.

KeyConcept One-to-One Property of Logarithmic Functions

Words	For $b > 0$ and $b \neq 1$, $\log_b x = \log_b y$ if and only if $x = y$.	
Examples	If $\log_2 x = \log_2 6$, then $x = 6$.	If $e^y = 2$, then $\ln e^y = \ln 2$.

One statement implied by this property is that $\log_b x = \log_b y$ if $x = y$. You can use this statement to solve some simple logarithmic equations by first condensing each side of an equation into logarithms with the same base.

Example 3 Solve Exponential Equations Using One-to-One Property

Solve each equation.

a. $\log_4 x = \log_4 3 + \log_4 (x - 2)$

$\log_4 x = \log_4 3 + \log_4 (x - 2)$	Original equation
$\log_4 x = \log_4 3(x - 2)$	Product Property
$\log_4 x = \log_4 (3x - 6)$	Distributive Property
$x = 3x - 6$	One-to-One Property
$-2x = -6$	Subtract $3x$ from each side.
$x = 3$	Divide each side by -2. Check this solution.

b. $\log_3 (x^2 + 3) = \log_3 52$

$\log_3 (x^2 + 3) = \log_3 52$	Original equation
$x^2 + 3 = 52$	One-to-One Property
$x^2 = 49$	Subtract 3 from each side.
$x = \pm 7$	Take the square root of each side. Check this solution.

GuidedPractice

3A. $\log_6 2x = \log_6 (x^2 - x + 2)$

3B. $\log_{12} (x + 3) = \log_{12} x + \log_{12} 4$

Another statement that follows from the One-to-One Property of Logarithmic Functions, if $x = y$, then $\log_b x = \log_b y$, can be used to solve exponential equations such as $e^x = 3$.

$e^x = 3$	Original equation
$\ln e^x = \ln 3$	One-to-One Property
$x = \ln 3$	Inverse Property

This application of the One-to-One Property is called *taking the logarithm of each side* of an equation. While natural logarithms are more convenient to use when the base of the exponential expression is e, you can use logarithms to any base to help solve exponential equations.

StudyTip

Alternate Solution The problem in Example 4a could also have been solved by taking the $\log_4$ of each side. The result would be $x = \log_4 13$. Notice that when the Change of Base Formula is applied, this is equivalent to the solution $x = \frac{\log 13}{\log 4}$.

Example 4 Solve Exponential Equations

Solve each equation. Round to the nearest hundredth.

a. $4^x = 13$

$4^x = 13$	Original equation
$\log 4^x = \log 13$	Take the common logarithm of each side.
$x \log 4 = \log 13$	Power Property
$x = \frac{\log 13}{\log 4}$ or about 1.85	Divide each side by log 4 and use a calculator.

b. $e^{4 - 3x} = 6$

$e^{4 - 3x} = 6$	Original equation
$\ln e^{4 - 3x} = \ln 6$	Take the natural logarithm of each side.
$4 - 3x = \ln 6$	Inverse Property
$x = \frac{\ln 6 - 4}{-3}$ or about 0.74	Solve for x and use a calculator.

GuidedPractice

4A. $8^y = 0.165$

4B. $1.43^a + 3.1 = 8.48$

4C. $e^{2 + 5w} = 12$

Example 5 Solve in Logarithmic Terms

Solve $4^{3x-1} = 3^{2-x}$. Round to the nearest hundredth.

Solve Algebraically

$4^{3x-1} = 3^{2-x}$ Original equation

$\ln 4^{3x-1} = \ln 3^{2-x}$ Take the natural logarithm of each side.

$(3x - 1)\ln 4 = (2 - x)\ln 3$ Power Property

$3x \ln 4 - \ln 4 = 2 \ln 3 - x \ln 3$ Distributive Property

$3x \ln 4 + x \ln 3 = 2 \ln 3 + \ln 4$ Isolate the variables on the left side of the equation.

$x(3 \ln 4 + \ln 3) = 2 \ln 3 + \ln 4$ Distributive Property

$x(\ln 4^3 + \ln 3) = \ln 3^2 + \ln 4$ Power Property

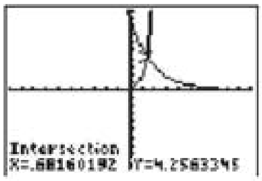
$x \ln [3(4^3)] = \ln 36$ Product Property

$x \ln 192 = \ln 36$ $3(4^3) = 192$

$x = \frac{\ln 36}{\ln 192}$ or about 0.68 Divide each side by ln 192.

Confirm Graphically

Graph $y = 4^{3x-1}$ and $y = 3^{2-x}$. The point of intersection of these two graphs given by the calculator is approximately 0.68, which is consistent with our algebraic solution.



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GuidedPractice

Solve each equation. Round to the nearest hundredth.

5A. $6^{2x+4} = 5^{-x+1}$

5B. $4^{3x+2} = 6^{2x-1}$

WatchOut!

Simplifying Notice that the Quotient Property cannot be used to further simplify $\frac{\ln 36}{\ln 192}$.

Equations involving multiple exponential expressions can be solved by applying quadratic techniques, such as factoring or the Quadratic Formula. Be sure to check for extraneous solutions.

Example 6 Solve Exponential Equations in Quadratic Form

Solve $e^{2x} + 6e^x - 16 = 0$.

$e^{2x} + 6e^x - 16 = 0$ Original equation

$u^2 + 6u - 16 = 0$ Write in quadratic form by letting $u = e^x$.

$(u + 8)(u - 2) = 0$ Factor.

$u = -8$ or $u = 2$ Zero Product Property

$e^x = -8$ $e^x = 2$ Replace u with e^x.

$\ln e^x = \ln (-8)$ $\ln e^x = \ln 2$ Take the natural logarithm of each side.

$x = \ln (-8)$ $x = \ln 2$ or about 0.69 Inverse Property

The only solution is $x = \ln 2$ because $\ln (-8)$ is extraneous. Check this solution.

CHECK $e^{2x} + 6e^x - 16 = 0$ Original equation

$e^{2(\ln 2)} + 6e^{\ln 2} - 16 \stackrel{?}{=} 0$ Replace x with ln 2.

$e^{\ln 2^2} + 6e^{\ln 2} - 16 \stackrel{?}{=} 0$ Power Property

$2^2 + 6(2) - 16 = 0$ ✓ Inverse Property

GuidedPractice

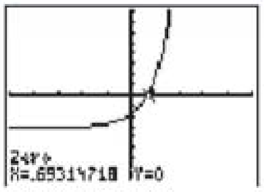
Solve each equation.

6A. $e^{2x} + 2e^x = 8$

6B. $4e^{4x} + 8e^{2x} = 5$

TechnologyTip

Finding Zeros You can confirm the solution of $e^{2x} + 6e^x - 16 = 0$ graphically by using a graphing calculator to locate the zero of $y = e^{2x} + 6e^x - 16$. The graphical solution of about 0.69 is consistent with the algebraic solution of $\ln 2 \approx 0.69$.



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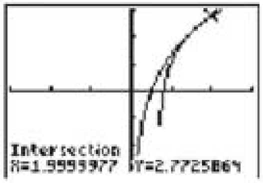
Equations having multiple logarithmic expressions may be solved by first condensing expressions using the Power, Product, and Quotient Properties, and then applying the One-to-One Property.

Example 7 Solve Logarithmic Equations

Solve $\ln (x + 2) + \ln (3x - 2) = 2 \ln 2x$.

$\ln (x + 2) + \ln (3x - 2) = 2 \ln 2x$	Original equation
$\ln (x + 2)(3x - 2) = \ln (2x)^2$	Product and Power Property
$\ln (3x^2 + 4x - 4) = \ln 4x^2$	Simplify.
$3x^2 + 4x - 4 = 4x^2$	One-to-One Property
$0 = x^2 - 4x + 4$	Simplify.
$0 = (x - 2)(x - 2)$	Factor.
$x = 2$	Zero Product Property

CHECK You can check this solution in the original equation, or confirm graphically by locating the intersection of the graphs of $y = \ln (x + 2) + \ln (3x - 2)$ and $y = 2 \ln 2x$.



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CheckYour Progress

Solve each equation.

7A. $\ln (7x + 3) - \ln (x + 1) = \ln (2x)$

7B. $\ln (2x + 1) + \ln (2x - 3) = 2 \ln (2x - 2)$

It may not be obvious that a solution of a logarithmic equation is extraneous until you check it in the original equation.

Example 8 Check for Extraneous Solutions

Solve $\log_{12} 12x + \log_{12} (x - 1) = 2$.

$\log_{12} 12x + \log_{12} (x - 1) = 2$	Original equation
$\log_{12} 12x(x - 1) = 2$	Product Property
$\log_{12} (12x^2 - 12x) = 2$	Distributive Property
$\log_{12} (12x^2 - 12x) = \log_{12} 12^2$	Inverse Property
$\log_{12} (12x^2 - 12x) = \log_{12} 144$	$12^2 = 144$
$12x^2 - 12x = 144$	One-to-One Property
$12x^2 - 12x - 144 = 0$	Subtract 144 from each side.
$12(x - 4)(x + 3) = 0$	Factor.
$x = 4$ or $x = -3$	Zero Product Property

CHECK

$$\log_{12} 12x + \log_{12} (x - 1) = 2$$
$$\log_{12} 12(4) + \log_{12} (4 - 1) \stackrel{?}{=} 2$$
$$\log_{12} 48 + \log_{12} 3 \stackrel{?}{=} 2$$
$$\log_{12} 48 \cdot 3 \stackrel{?}{=} 2$$
$$\log_{12} 144 = 2 \checkmark$$

$$\log_{12} 12x + \log_{12} (x - 1) = 2$$
$$\log_{12} 12(-3) + \log_{12} (-3 - 1) \stackrel{?}{=} 2$$
$$\log_{12} (-36) + \log_{12} (-4) \stackrel{?}{=} 2$$

Since neither $\log_{12} (-36)$ nor $\log_{12} (-4)$ is defined, $x = -3$ is an extraneous solution.

GuidedPractice

Solve each equation.

8A. $\ln (6y + 2) - \ln (y + 1) = \ln (2y - 1)$

8B. $\log (x - 12) = 2 + \log (x - 2)$

StudyTip

Identify the Domain of an Equation Another way to check for extraneous solutions is to identify the domain of the equation. In Example 8, the domain of $\log_{12} 12x$ is $x > 0$ while the domain of $\log_{12} (x - 1)$ is $x > 1$. Therefore, the domain of the equation is $x > 1$. Since $-3 \not> 1$, -3 cannot be a solution of the equation.

You can use information about growth or decay to write the equation of an exponential function.

Real-World Example 9 Model Exponential Growth

INTERNET **The table shows the number of hits a new Web site received by the end of January and the end of April of the same year.**

Web Site Traffic	
Month	**Number of Hits**
January	125
April	2000

a. **If the number of hits is increasing at an exponential rate, identify the continuous rate of growth. Then write an exponential equation to model this situation.**

Let $N(t)$ represent the number of hits at the end of t months and assume continuous exponential growth. Then the initial number N_0 is 125 hits and the number of hits N after a time of 3 months, the number of months from January to April, is 2000. Use this information to find the continuous growth rate k.

$N(t) = N_0e^{kt}$ Exponential Growth Formula

$2000 = 125e^{k(3)}$ $N(3) = 2000$, $N_0 = 125$, and $t = 3$

$16 = e^{3k}$ Divide each side by 125.

$\ln 16 = \ln e^{3k}$ Take the natural logarithm of each side.

$\ln 16 = 3k$ Inverse Property

$\frac{\ln 16}{3} = k$ Divide each side by 3.

$0.924 \approx k$ Use a calculator.

The number of hits is increasing at a continuous rate of approximately 92.4% per month. Therefore, an equation modeling this situation is $N(t) = 125e^{0.924t}$.

b. **Use your model to predict the number of months it will take for the Web site to receive 2 million hits.**

$N(t) = 125e^{0.924t}$ Exponential growth model

$2{,}000{,}000 = 125e^{0.924t}$ $N(t) = 2{,}000{,}000$

$16{,}000 = e^{0.924t}$ Divide each side by 125.

$\ln 16{,}000 = \ln e^{0.924t}$ Take the natural logarithm of each side.

$\ln 16{,}000 = 0.924t$ Inverse Property

$\frac{\ln 16{,}000}{0.924} = t$ Divide each side by 0.924.

$10.48 \approx t$ Use a calculator.

According to this model, the Web site will receive 2,000,000 hits in about 10.48 months.

GuidedPractice

9. **MEMORABILIA** The table shows revenue from sales of T-shirts and other memorabilia sold by two different vendors during and one week after the World Series.

World Series Memorabilia Sales		
Days after Series	**Vendor A Sales ($)**	**Vendor B Sales ($)**
0	300,000	200,000
7	37,000	49,000

A. If the sales are decreasing at an exponential rate, identify the continuous rate of decline for each vendor's sales. Then write an exponential equation to model each situation.

B. Use your models to predict the World Series memorabilia sales by each vendor 4 weeks after the series ended.

C. Will the two vendors' sales ever be the same? If so, at what point in time?

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Real-WorldLink

Championship hats and shirts are printed for both teams before a major athletic contest like the Bowl Championship Series. The losing team's merchandise is often donated to nonprofit organizations that distribute it to families in need in other countries. In 2007, an estimated $2.5 million of unusable sports clothing was donated.

Source: World Vision

Exercises

● = Step-by-Step Solutions begin on page R29.

Solve each equation. (Example 1)

1. $4^{x+7} = 8^{x+3}$

2. $8^{x+4} = 32^{3x}$

3. $49^{x+4} = 7^{18-x}$

4. $32^{x-1} = 4^{x+5}$

5. $\left(\frac{9}{16}\right)^{3x-2} = \left(\frac{3}{4}\right)^{5x+4}$

6. $12^{3x+11} = 144^{2x+7}$

7. $25^{\frac{x}{3}} = 5^{x-4}$

8. $\left(\frac{5}{6}\right)^{4x} = \left(\frac{36}{25}\right)^{9-x}$

9. **INTERNET** The number of people P in millions using two different search engines to surf the Internet t weeks after the creation of the search engine can be modeled by $P_1(t) = 1.5^{t+4}$ and $P_2(t) = 2.25^{t-3.5}$, respectively. During which week did the same number of people use each search engine? (Example 1)

10. **FINANCIAL LITERACY** Brandy is planning on investing \$5000 and is considering two savings accounts. The first account is continuously compounded and offers a 3% interest rate. The second account is annually compounded and also offers a 3% interest rate, but the bank will match 4% of the initial investment. (Example 1)

- **a.** Write an equation for the balance of each savings account at time t years.
- **b.** How many years will it take for the continuously compounded account to catch up with the annually compounded savings account?
- **c.** If Brandy plans on leaving the money in the account for 30 years, which account should she choose?

Solve each logarithmic equation. (Example 2)

11. $\ln a = 4$

12. $-8 \log b = -64$

13. $\ln(-2) = c$

14. $2 + 3 \log 3d = 5$

15. $14 + 20 \ln 7x = 54$

16. $100 + 500 \log_4 g = 1100$

17. $7000 \ln h = -21{,}000$

18. $-18 \log_0 j = -126$

19. $12{,}000 \log_2 k = 192{,}000$

20. $\log_2 m^4 = 32$

㉑ **CARS** If all other factors are equal, the higher the displacement D in liters of the air/fuel mixture of an engine, the more horsepower H it will produce. The horsepower of a naturally aspirated engine can be modeled by $H = \log_{1.003} \frac{D}{1.394}$. Find the displacement when horsepower is 200. (Example 2)

Solve each equation. (Example 3)

22. $\log_6 (x^2 + 5) = \log_6 41$

23. $\log_8 (x^2 + 11) = \log_8 92$

24. $\log_9 (x^4 - 3) = \log_9 13$

25. $\log_7 6x = \log_7 9 + \log_7 (x - 4)$

26. $\log_5 x = \log_5 (x + 6) - \log_5 4$

27. $\log_{11} 3x = \log_{11} (x + 5) - \log_{11} 2$

Solve each equation. Round to the nearest hundredth. (Example 4)

28. $6^x = 28$

29. $1.8^x = 9.6$

30. $3e^{4x} = 45$

31. $e^{3x+1} = 51$

32. $8^x - 1 = 3.4$

33. $2e^{7x} = 84$

34. $8.3e^{9x} = 24.9$

35. $e^{2x} + 5 = 16$

36. $2.5e^{x+4} = 14$

37. $0.75e^{3.4x} - 0.3 = 80.1$

38. **GENETICS** PCR (Polymerase Chain Reaction) is a technique commonly used in forensic labs to amplify DNA. PCR uses an enzyme to cut a designated nucleotide sequence from the DNA and then replicates the sequence. The number of identical nucleotide sequences N after t minutes can be modeled by $N(t) = 100 \cdot 1.17^t$. (Example 4)

- **a.** At what time will there be 1×10^4 sequences?
- **b.** At what time will the DNA have been amplified to 1 million sequences?

Solve each equation. Round to the nearest hundredth. (Example 5)

39. $7^{2x+1} = 3^{x+3}$

40. $11^{x+1} = 7^{x-1}$

41. $9^{x+2} = 2^{5x-4}$

42. $4^{x-3} = 6^{2x-1}$

43. $3^{4x+3} = 8^{-x+2}$

44. $5^{3x-1} = 4^{x+1}$

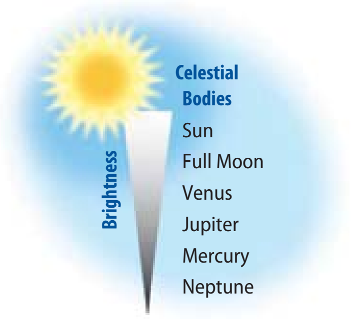
45. $6^{x-2} = 5^{2x+3}$

46. $8^{-2x-1} = 5^{-x+2}$

47. $2^{5x+6} = 4^{2x+1}$

48. $6^{-x-2} = 9^{-x-1}$

49. **ASTRONOMY** The brightness of two celestial bodies as seen from Earth can be compared by determining the variation in brightness between the two bodies. The variation in brightness V can be calculated by $V = 2.512^{m_f - m_b}$, where m_f is the magnitude of brightness of the fainter body and m_b is the magnitude of brightness of the brighter body. (Example 5)



- **a.** The Sun has $m = -26.73$, and the full Moon has $m = -12.6$. Determine the variation in brightness between the Sun and the full Moon.
- **b.** The variation in brightness between Mercury and Venus is 5.25. Venus has a magnitude of brightness of -3.7. Determine the magnitude of brightness of Mercury.
- **c.** Neptune has a magnitude of brightness of 7.7, and the variation in brightness of Neptune and Jupiter is 15,856. What is the magnitude of brightness of Jupiter?

Solve each equation. (Example 6)

50. $e^{2x} + 3e^x - 130 = 0$

51. $e^{2x} - 15e^x + 56 = 0$

52. $e^{2x} + 3e^x = -2$

53. $6e^{2x} - 5e^x = 6$

54. $9e^{2x} - 3e^x = 6$

55. $8e^{4x} - 15e^{2x} + 7 = 0$

56. $2e^{8x} + e^{4x} - 1 = 0$

57. $2e^{5x} - 7e^{2x} - 15e^{-x} = 0$

58. $10e^x - 15 - 45e^{-x} = 0$

59. $11e^x - 51 - 20e^{-x} = 0$

Solve each logarithmic equation. (Example 7)

60. $\ln x + \ln (x + 2) = \ln 63$

61. $\ln x + \ln (x + 7) = \ln 18$

62. $\ln (3x + 1) + \ln (2x - 3) = \ln 10$

63. $\ln (x - 3) + \ln (2x + 3) = \ln (-4x^2)$

64. $\log (5x^2 + 4) = 2 \log 3x^2 - \log (2x^2 - 1)$

65. $\log (x + 6) = \log (8x) - \log (3x + 2)$

66. $\ln (4x^2 - 3x) = \ln (16x - 12) - \ln x$

67. $\ln (3x^2 - 4) + \ln (x^2 + 1) = \ln (2 - x^2)$

68. **SOUND** Noise-induced hearing loss (NIHL) accounts for 25% of hearing loss in the United States. Exposure to sounds of 85 decibels or higher for an extended period can cause NIHL. Recall that the decibels (dB) produced by a sound of intensity I can be calculated by $dB = 10 \log \left(\frac{I}{1 \times 10^{-12}}\right)$. (Example 7)

Intensity (W/m^3)	Sound
316.227	fireworks
31.623	jet plane
3.162	ambulance
0.316	rock concert
0.032	headphones
0.003	hair dryer

Source: Dangerous Decibels

a. Which of the sounds listed in the table produce enough decibels to cause NIHL?

b. Determine the number of hair dryers that would produce the same number of decibels produced by a rock concert. Round to the nearest whole number.

c. How many jet planes would it take to produce the same number of decibels as a firework display? Round to the nearest whole number.

Solve each logarithmic equation. (Example 8)

69. $\log_2 (2x - 6) = 3 + \log_2 x$

70. $\log (3x + 2) = 1 + \log 2x$

(71) $\log x = 1 - \log (x - 3)$

72. $\log 50x = 2 + \log (2x - 3)$

73. $\log_9 9x - 2 = -\log_9 x$

74. $\log (x - 10) = 3 + \log (x - 3)$

Solve each logarithmic equation. (Example 8)

75. $\log (29{,}995x + 40{,}225) = 4 + \log (3x + 4)$

76. $\log_{\frac{1}{4}} \left(\frac{1}{4}x\right) = -\log_{\frac{1}{4}} (x + 8) - \frac{5}{2}$

77. $\log x = 3 - \log (100x + 900)$

78. $\log_5 \frac{x^2}{8} - 3 = \log_5 \frac{x}{40}$

79. $\log 2x + \log \left(4 - \frac{16}{x}\right) = 2 \log (x - 2)$

80. **BUSINESS** A chain of retail computer stores opened 2 stores in its first year of operation. After 8 years of operation, the chain consisted of 206 stores. (Example 9)

a. Write a continuous exponential equation to model the number of stores N as a function of year of operation t. Round k to the nearest hundredth.

b. Use the model you found in part **a** to predict the number of stores in the 12th year of operation.

81. **STOCK** The price per share of a coffee chain's stock was \$0.93 in a month during its first year of trading. During its fifth year of trading, the price per share of stock was \$3.52 during the same month. (Example 9)

a. Write a continuous exponential equation to model the price of stock P as a function of year of trading t. Round k to the nearest ten-thousandth.

b. Use the model you found in part **a** to predict the price of the stock during the ninth year of trading.

Solve each logarithmic equation.

82. $5 + 5 \log_{100} x = 20$

83. $6 + 2 \log_{e^2} x = 30$

84. $5 - 4 \log_{\frac{1}{2}} x = -19$

85. $36 + 3 \log_3 x = 60$

86. **ACIDITY** The acidity of a substance is determined by its concentration of H$^+$ ions. Because the H$^+$ concentration of substances can vary by several orders of magnitude, the logarithmic pH scale is used to indicate acidity. pH can be calculated by $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ is the concentration of H$^+$ ions in moles per liter.

Item	pH
ammonia	11.0
baking soda	8.3
human blood	7.4
water	7.0
milk	6.6
apples	3.0
lemon juice	2.0

a. Determine the H$^+$ concentration of baking soda.

b. How many times as acidic is milk than human blood?

c. By how many orders of magnitude is the $[\text{H}^+]$ of lemon juice greater than $[\text{H}^+]$ of ammonia?

d. How many moles of H$^+$ ions are in 1500 liters of human blood?

GRAPHING CALCULATOR **Solve each equation algebraically, if possible. If not possible, approximate the solution to the nearest hundredth using a graphing calculator.**

87. $x^3 = 2^x$

88. $\log_2 x = \log_8 x$

89. $3^x = x(5^x)$

90. $\log_x 5 = \log_5 x$

91. RADIOACTIVITY The isotopes phosphorous-32 and sulfur-35 both exhibit radioactive decay. The half-life of phosphorous-32 is 14.282 days. The half-life of sulfur-35 is 87.51 days.

a. Write equations to express the radioactive decay of phosphorous-32 and sulfur-35 in terms of time t in days and ratio R of remaining isotope using the general equation for radioactive decay, $A = t \cdot \frac{\ln R}{-0.693}$, where A is the number of days the isotope has decayed and t is the half-life in days.

b. At what value of R will sulfur-35 have been decaying 5 days longer than phosphorous-32?

Solve each exponential inequality.

92. $2 \le 2^x \le 32$

93. $9 < 3^y < 27$

94. $\frac{1}{4096} \le 8^p \le \frac{1}{64}$

95. $\frac{1}{2197} < 13^f \le \frac{1}{13}$

96. $10 < 10^d < 100{,}000$

97. $4000 > 5^q > 125$

98. $49 < 7^z < 1000$

99. $10{,}000 < 10^a < 275{,}000$

100. $\frac{1}{15} \ge 4^b \ge \frac{1}{64}$

101. $\frac{1}{2} \ge e^c \ge \frac{1}{100}$

102. FORENSICS Forensic pathologists perform autopsies to determine time and cause of death. The time t in hours since death can be calculated by $t = -10 \ln\left(\frac{T - R_t}{98.6 - R_t}\right)$, where T is the temperature of the body and R_t is the room temperature.

a. A forensic pathologist measures the body temperature to be 93°F in a room that is 72°F. What is the time of death?

b. A hospital patient passed away 4 hours ago. If the hospital has an average temperature of 75°F, what is the body temperature?

c. A patient's temperature was 89°F 3.5 hours after the patient passed away. Determine the room temperature.

103. MEDICINE Fifty people were treated for a virus on the same day. The virus is highly contagious, and the individuals must stay in the hospital until they have no symptoms. The number of people p who show symptoms after t days can be modeled by $p = \frac{52.76}{1 + 0.03e^{0.75t}}$.

a. How many show symptoms after 5 days?

b. Solve the equation for t.

c. How many days will it take until only one person shows symptoms?

Solve each equation.

104. $27 = \frac{12}{1 - \frac{1}{2}e^{-x}}$

105. $22 = \frac{L}{1 + \frac{L-3}{3}e^{-15}}$

106. $1000 = \frac{10{,}000}{1 + 19e^{-t}}$

107. $300 = \frac{400}{1 + 3e^{-2k}}$

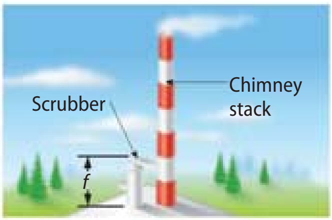
108. $16^x + 4^x - 6 = 0$

109. $\frac{e^x + e^{-x}}{e^x - e^{-x}} = 6$

110. $\frac{\ln(4x + 2)}{\ln(4x - 2)} = 3$

111. $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$

112. POLLUTION Some factories have added filtering systems called *scrubbers* to their smokestacks in order to reduce pollution emissions. The percent of pollution P removed after f feet of length of a particular scrubber can be modeled by $P = \frac{0.9}{1 + 70e^{-0.28f}}$.



a. Graph the percent of pollution removed as a function of scrubber length.

b. Determine the maximum percent of pollution that can be removed by the scrubber. Explain your reasoning.

c. Approximate the maximum length of scrubber that a factory should choose to use. Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

113. REASONING What is the maximum number of extraneous solutions that a logarithmic equation can have? Explain your reasoning.

114. OPEN ENDED Give an example of a logarithmic equation with infinite solutions.

115. CHALLENGE If an investment is made with an interest rate r compounded monthly, how long will it take for the investment to triple?

116. REASONING How can you solve an equation involving logarithmic expressions with three different bases?

117. CHALLENGE For what x values do the domains of $f(x) = \log(x^4 - x^2)$ and $g(x) = \log x + \log x + \log(x - 1) + \log(x + 1)$ differ?

118. WRITING IN MATH Explain how to algebraically solve for t in $P = \frac{L}{1 + \left(\frac{L - I}{I}\right)e^{-kt}}$.

Spiral Review

Evaluate each logarithm. (Lesson 3-3)

119. $\log_8 15$

120. $\log_2 8$

121. $\log_5 625$

122. SOUND An equation for loudness L, in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound. (Lesson 3-2)

a. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels.

b. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels.

c. How many times as intense is the fireworks display as the concert? In other words, find the ratio of their intensities.

Sound	Decibels
fireworks	130–190
car racing	100–130
parades	80–120
yard work	95–115
movies	90–110
concerts	75–110

For each function, (a) apply the leading term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function. (Lesson 2-2)

123. $f(x) = x^3 - 8x^2 + 7x$

124. $f(x) = x^3 + 6x^2 + 8x$

125. $f(x) = -x^4 + 6x^3 - 32x$

Solve each equation. (Lesson 2-1)

126. $\frac{1}{6}(12a)^{\frac{1}{3}} = 1$

127. $\sqrt[3]{x - 4} = 3$

128. $(3y)^{\frac{1}{3}} + 2 = 5$

Use logical reasoning to determine the end behavior or limit of the function as x approaches infinity. Explain your reasoning. (Lesson 1-3)

129. $f(x) = x^{10} - x^9 + 5x^8$

130. $g(x) = \dfrac{x^2 + 5}{7 - 2x^2}$

131. $h(x) = |(x - 3)^2 - 1|$

Find the variance and standard deviation of each population to the nearest tenth. (Lesson 0-8)

132. {48, 36, 40, 29, 45, 51, 38, 47, 39, 37}

133. {321, 322, 323, 324, 325, 326, 327, 328, 329, 330}

134. {43, 56, 78, 81, 47, 42, 34, 22, 78, 98, 38, 46, 54, 67, 58, 92, 55}

Skills Review for Standardized Tests

135. SAT/ACT In a movie theater, 2 boys and 3 girls are randomly seated together in a row. What is the probability that the 2 boys are seated next to each other?

A $\frac{1}{5}$

B $\frac{3}{5}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{2}{5}$

136. REVIEW Which equation is equivalent to $\log_4 \frac{1}{16} = x$?

F $\frac{1^4}{16} = x^4$

G $\left(\frac{1}{16}\right)^4 = x$

H $4^x = \frac{1}{16}$

J $4^{\frac{1}{16}} = x$

137. If $2^4 = 3^x$, then what is the approximate value of x?

A 0.63

B 2.34

C 2.52

D 2.84

138. REVIEW The pH of a person's blood is given by $\text{pH} = 6.1 + \log_{10} B - \log_{10} C$, where B is the concentration base of bicarbonate in the blood and C is the concentration of carbonic acid in the blood. Determine which substance has a pH closest to a person's blood if their ratio of bicarbonate to carbonic acid is 17.5:2.25.

F lemon juice

G baking soda

H milk

J ammonia

Substance	pH
lemon juice	2.3
milk	6.4
baking soda	8.4
ammonia	11.9

EXTEND 3-4

Graphing Technology Lab

Solving Exponential and Logarithmic Inequalities

Objective

- Solve exponential and logarithmic inequalities algebraically and graphically.

In Lesson 3-4, you solved exponential equations algebraically and confirmed solutions graphically. You can use similar techniques and the following properties to solve inequalities involving exponential functions.

KeyConcept Properties of Inequality for Exponential Functions

Words If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example If $5x < 5^4$, then $x < 4$.

This property also holds for ≤ and ≥.

Activity 1 Exponential Inequalities

Solve $5^{2x-6} > 0.04^{x-3}$.

Solve Algebraically

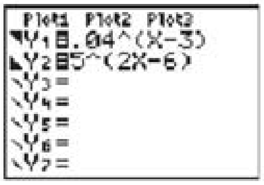
$5^{2x-6} > 0.04^{x-3}$	Original inequality
$5^{2x-6} > \left(\frac{1}{25}\right)^{x-3}$	Rewrite 0.04 as $\frac{1}{25}$.
$5^{2x-6} > (5^{-2})^{x-3}$	Rewrite $\frac{1}{25}$ as $\frac{1}{5^2}$ or 5^{-2} so each side has the same base.
$5^{2x-6} > 5^{-2x+6}$	Power of a Power
$2x - 6 > -2x + 6$	Property of Inequality for Exponential Functions
$4x > 12$	Addition Property of Inequalities
$x > 3$	Division Property of Inequalities

The solution set is $\{x \mid x > 3, x \in \mathbb{R}$ or $(3, \infty)$.

Confirm Graphically

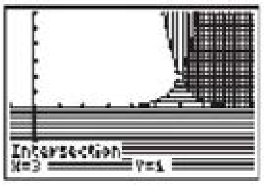
Step 1 Replacing each side of the inequality with y yields the system of inequalities $y > 0.04^{x-3}$ and $y < 5^{2x-6}$.

Enter each boundary equation and select the appropriate shade option.



```
 Plot1 Plot2 Plot3
▼Y1■.04^(X-3)
◣Y2■5^(2X-6)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Step 2 Graph the system. The x-values of the points in the region where the shadings overlap is the solution set of the original inequality. Using the INTERSECT feature, you can conclude that the solution set is $(3, \infty)$, which is consistent with our algebraic solution set.



```
Intersection
X=3          Y=1
```

[–0.5, 4.5] scl: 0.5 by [–2, 3] scl: 0.5

Exercises

Solve each inequality.

1. $16^x < 8^{x+1}$

2. $32^{5x+2} \geq 16^{5x}$

3. $2^{4x-5} > 0.5^{x-5}$

4. $9^{2x-1} \geq 3^{2x+8}$

5. $343^{x-2} \leq 49$

6. $100^x < 0.01^{3x-4}$

To solve inequalities that involve logarithms, use the following property.

KeyConcept Logarithmic to Exponential Inequality

Words If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
If $b > 1$, $x > 0$, and $\log_b x < y$,
then $0 < x < b^y$.

Example

$\log_2 x > 3$
$x > 2^3$

$\log_3 x < 5$
$0 < x < 3^5$

This property also holds for ≤ and ≥.

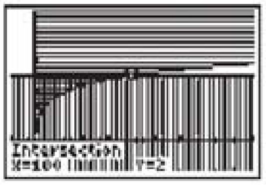
Activity 2 Logarithmic Inequalities

Solve $\log x \leq 2$.

Solve Algebraically	$\log x \leq 2$	Original inequality
	$0 < x \leq 10^2$	Logarithmic to Exponential Inequality
	$0 < x \leq 100$	Simplify.

The solution set is $\{x \mid 0 < x \leq 100, x \in \mathbb{R}\}$ or $(0, 100]$.

Confirm Graphically Graph the system of inequalities $y \leq 2$ and $y > \log x$ (Figure 3.4.1). Using the TRACE and INTERSECT features, you can conclude that the solution set is $(0, 100]$. ✓



[–25, 225] scl: 25 by [–1, 4] scl: 0.5

Figure 3.4.1

To solve inequalities that involve logarithms with the same base on each side, use the following property.

KeyConcept Properties of Inequality for Logarithmic Functions

Words If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Example If $\log_2 x > \log_2 9$, then $x > 9$.

This property also holds for ≤ and ≥.

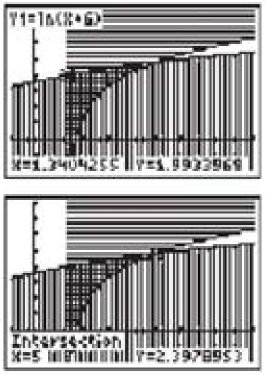
Activity 3 Inequalities with Logarithms on Each Side

Solve $\ln (3x - 4) < \ln (x + 6)$.

Solve Algebraically	$\ln (3x - 4) < \ln (x + 6)$	Original inequality
	$3x - 4 < x + 6$	Property of Inequalities for Logarithmic Functions
	$x < 5$	Division Property of Inequalities

Exclude all values of x such that $3x - 4 \leq 0$ or $x + 6 \leq 0$. Thus, the solution set is $x > 1\frac{1}{3}$ and $x > -6$ and $x < 5$. This compound inequality simplifies to $\left\{x \,\middle|\, 1\frac{1}{3} < x < 5, x \in \mathbb{R}\right\}$ or $\left(1\frac{1}{3}, 5\right)$.

Confirm Graphically Graph the system of inequalities $y < \ln (x + 6)$ and $y > \ln (3x - 4)$ (Figure 3.4.2). Using the TRACE and INTERSECT features, you can conclude that the solution set is $\left(1\frac{1}{3}, 5\right)$. ✓



[–1, 9] scl: 1 by [–1, 4] scl: 0.5

Figure 3.4.2

Exercises

Solve each inequality.

7. $\ln (2x - 1) < 0$

8. $\log (3x - 8) > 2$

9. $\log 2x < -1$

10. $\log (5x + 2) \leq \log (x - 4)$

11. $\ln (3x - 5) > \ln (x + 7)$

12. $\log (x^2 - 6) \geq \log x$

LESSON 3-5 Modeling with Nonlinear Regression

Then

- You modeled data using polynomial functions. (Lesson 2-1)

Now

- 1 Model data using exponential, logarithmic, and logistic functions.

  2 Linearize and analyze data.

Why?

- While exponential growth is not a perfect model for the growth of a human population, government agencies can use estimates from such models to make strategic plans that ensure they will be prepared to meet the future needs of their people.

NewVocabulary
logistic growth function
linearize

1 Exponential, Logarithmic, and Logistic Modeling

In this lesson, we will use the exponential regression features on a graphing calculator, rather than algebraic techniques, to model data exhibiting exponential or logarithmic growth or decay.

Example 1 Exponential Regression

POPULATION Mesa, Arizona, is one of the fastest-growing cities in the United States. Use exponential regression to model the Mesa population data. Then use your model to estimate the population of Mesa in 2020.

Population of Mesa, Arizona (thousands)

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population	0.7	1.6	3.0	3.7	7.2	16.8	33.8	63	152	288	396

Step 1 **Make a scatter plot.**

Let $P(t)$ represent the population in thousands of Mesa t years after 1900. Enter and graph the data on a graphing calculator to create the scatter plot (Figure 3.5.1). Notice that the plot very closely resembles the graph of an exponential growth function.

Step 2 **Find an exponential function to model the data.**

With the diagnostic feature turned on and using ExpReg from the list of regression models, we get the values shown in Figure 3.5.2. The population in 1900 is represented by a and the growth rate, 6.7% per year, is represented by b. Notice that the correlation coefficient $r \approx 0.9968$ is close to 1, indicating a close fit to the data. In the Y= menu, pick up this regression equation by entering VARS, Statistics, EQ, RegEQ.

Step 3 **Graph the regression equation and scatter plot on the same screen.**

Notice that the graph of the regression fits the data fairly well. (Figure 3.5.3).

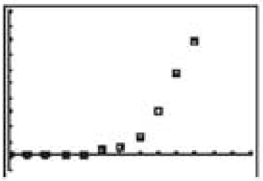
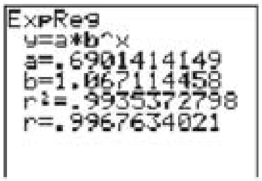

[0, 130] scl: 10 by [−50, 500] scl: 50

Figure 3.5.1



```
ExpReg
 y=a*b^x
 a=.6901414149
 b=1.067114458
 r²=.9935372798
 r=.9967634021
```

Figure 3.5.2

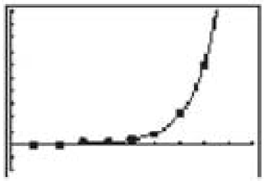

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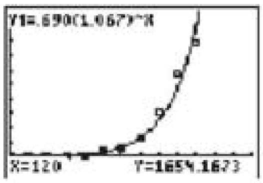
Figure 3.5.3

Step 4 **Use the model to make a prediction.**

To predict the population of Mesa in 2020, 120 years after 1900, use the CALC feature to evaluate the function for $P(120)$ as shown. Based on the model, Mesa will have about 1675 thousand or 1.675 million people in 2020.



Scott E Barbour/The Image Bank/Getty Images

GuidedPractice

1. **INTERNET** The Internet experienced rapid growth in the 1990s. The table shows the number of users in millions for each year during the decade. Use exponential regression to model the data. Then use your model to predict the number of users in 2008. Let x be the number of years after 1990.

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Internet Users	1	1.142	1.429	4.286	5.714	10	21.429	34.286	59.143	70.314

While data exhibiting rapid growth or decay tend to suggest an exponential model, data that grow or decay rapidly at first and then more slowly over time tend to suggest a logarithmic model calculated using natural logarithmic regression.

Example 2 Logarithmic Regression

BIRTHS Use logarithmic regression to model the data in the table about twin births in the United States. Then use your model to predict when the number of twin births in the U.S. will reach 150,000.

Number of Twin Births in the United States							
Year	1995	1997	1998	2000	2002	2004	2005
Births	96,736	104,137	110,670	118,916	125,134	132,219	133,122

Step 1 Let $B(t)$ represent the number of twin births t years after 1990. Then create a scatter plot (Figure 3.5.5). The plot resembles the graph of a logistic growth function.

Step 2 Calculate the regression equation using LnReg. The correlation coefficient $r \approx 0.9949$ indicates a close fit to the data. Rounding each value to three decimal places, a natural logarithm function that models the data is $B(t) = 38{,}428.963 + 35{,}000.168 \ln x$.

```
LnReg
 y=a+blnx
 a=38428.96308
 b=35000.1679
 r²=.9897472487
 r=.9948604167
```

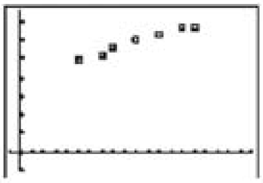
Figure 3.5.4

Step 3 In the Y= menu, pick up this regression equation. Figure 3.5.4 shows the results of the regression $B(t)$. The number of twin births in 1990 is represented by a. The graph of $B(t)$ fits the data fairly well (Figure 3.5.6).

Step 4 To find when the number of twin births will reach 150,000, graph the line $y = 150{,}000$ and the modeling equation on the same screen. Calculating the point of intersection (Figure 3.5.7), we find that according to this model, the number of twin births will reach 150,000 when $t \approx 24$, which is in 1990 + 24 or 2014.

StudyTip

Rounding Remember that the rounded regression equation is not used to make our prediction. A more accurate predication can be obtained by using the *entire* equation.



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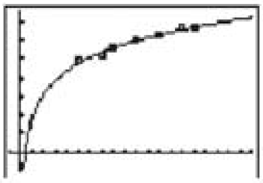
[−20,000; 150,000] scl: 20,000

Figure 3.5.5



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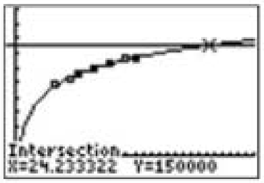
[−20,000; 150,000] scl: 20,000

Figure 3.5.6



[−1, 30] scl: 2 by
[−20,000; 200,000] scl: 20,000

Figure 3.5.7

GuidedPractice

2. **LIFE EXPECTANCY** The table shows average U.S. life expectancies according to birth year. Use logarithmic regression to model the data. Then use the function to predict the life expectancy of a person born in 2020. Let x be the number of years after 1900.

Birth Year	1950	1960	1970	1980	1990	1995	2000	2005
Life Expectancy	68.2	69.7	70.8	73.7	75.4	75.8	77.0	77.8

Exponential and logarithmic growth is unrestricted, increasing at an ever-increasing rate with no upper bound. In many growth situations, however, the amount of growth is limited by factors that sustain the population, such as space, food, and water. Such factors cause growth that was initially exponential to slow down and level out, approaching a horizontal asymptote. A **logistic growth function** models such resource-limited exponential growth.

StudyTip

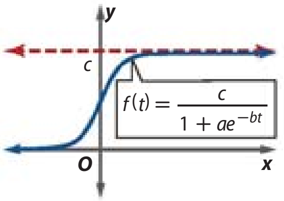
Logistic Decay If $b < 0$, then $f(t) = \dfrac{c}{1 + ae^{-bt}}$ would represent logistic decay. Unless otherwise stated, all logistic models in this text will represent logistic growth.

KeyConcept Logistic Growth Function

A logistic growth function has the form

$$f(t) = \frac{c}{1 + ae^{-bt}},$$

where t is any real number, a, b, and c are positive constants, and c is the limit to growth.



Logistic growth functions are bounded by two horizontal asymptotes, $y = 0$ and $y = c$. The limit to growth c is also called the carrying capacity of the function.

Real-World Example 3 Logistic Regression

BIOLOGY Use logistic regression to find a logistic growth function to model the data in the table about the number of yeast growing in a culture. Then use your model to predict the limit to the growth of the yeast in the culture.

Yeast Population in a Culture

Time (h)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Yeast	10	19	31	45	68	120	172	255	353	445	512	561	597	629	641	653	654	658

Step 1 Let $Y(t)$ represent the number of yeast in the culture after t hours. Then create a scatter plot (Figure 3.5.8). The plot resembles the graph of a logistic growth function.

Step 2 Calculate the regression equation using Logistic (Figure 3.5.9). Rounding each value to three decimal places, a logistic function that models the data is

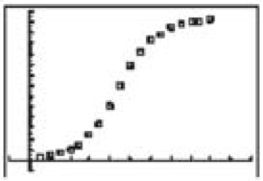
$$Y(t) = \frac{661.565}{1 + 131.178e^{-0.555t}}.$$

Step 3 The graph of $Y(t) = \dfrac{661.565}{1 + 131.178e^{-0.555t}}$ fits the data fairly well (Figure 3.5.10).

Step 4 The limit to growth in the modeling equation is the numerator of the fraction or 661.565. Therefore, according to this model, the population of yeast in the culture will approach, but never reach, 662.

StudyTip

Correlation Coefficients Logistic regressions do not have a corresponding correlation coefficient due to the nature of the models.

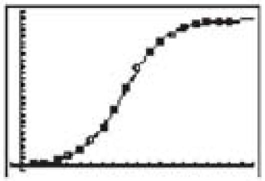


[−2, 22] scl: 2 by [−50, 700] scl: 50

Figure 3.5.8

```
Logistic
 y=c/(1+ae^(-bx))
 a=131.1781232
 b=.5545874884
 c=661.5649456
```

Figure 3.5.9



[−2, 22] scl: 2 by [−50, 700] scl: 50

Figure 3.5.10

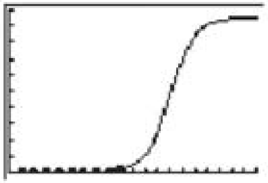
GuidedPractice

3. **FISH** Use logistic regression to model the data in the table about a lake's fish population. Then use your model to predict the limit to the growth of the fish population.

Time (mo)	0	4	8	12	16	20	24
Fish	125	580	2200	5300	7540	8280	8450

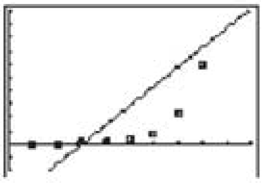
StudyTip

Logistic Regression Notice how the logistic graph at the right represents only the initial part of the graph. Therefore, it is a little more difficult to assess the accuracy of this regression without expanding the domain. The complete graph of the logistic regression is shown below.



While you can use a calculator to find a linear, quadratic, power, exponential, logarithmic, or logistic regression equation for a set of data, it is up to you to determine which model *best* fits the data by looking at the graph and/or by examining the correlation coefficient of the regression. Consider the graphs of each regression model and its correlation coefficient using the same set of data below.

Linear Regression
$r = 0.74$



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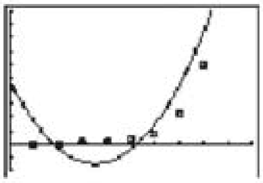
Quadratic Regression
$r = 0.94$



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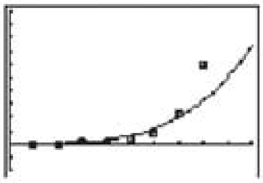
Power Regression
$r = 0.94$



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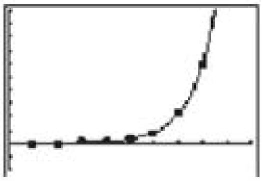
Exponential Regression
$r = 0.99$



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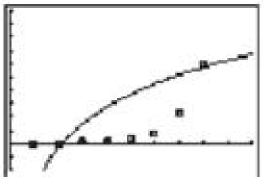
Logarithmic Regression
$r = 0.58$



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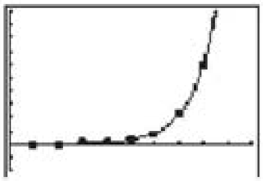
Logistic Regression



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Over the domain displayed, the exponential and logistic regression models appear to most accurately fit the data, with the exponential model having the strongest correlation coefficient.

Example 4 Choose a Regression

EARTHQUAKES **Use the data below to determine a regression equation that best relates the distance a seismic wave can travel from an earthquake's epicenter to the time since the earthquake. Then determine how far from the epicenter the wave will be felt 8.5 minutes after the earthquake.**

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

Step 1 From the shape of the scatter plot, it appears that these data could best be modeled by any of the regression models above except logarithmic. (Figure 3.5.10)

Step 2 Use the LinReg(ax+b), QuadReg, CubicReg, QuartReg, LnReg, ExpReg, PwrReg, and Logistic regression features to find regression equations to fit the data, noting the corresponding correlation coefficients. The regression equation with a correlation coefficient closest to 1 is the quartic regression with equation rounded to $y = 0.702x^4 - 16.961x^3 + 160.826x^2 - 21.045x + 293.022$. Remember to use VARS to transfer the entire equation to the graph.

Step 3 The quartic regression equation does indeed fit the data very well. (Figure 3.5.11)

Step 4 Use the CALC feature to evaluate this regression equation for $x = 8.5$. (Figure 3.5.12) Since $y \approx 4981$ when $x = 8.5$, you would expect the wave to be felt approximately 4981 kilometers away after 8.5 minutes.

StudyTip

Using a Regression Equation Some models are better than others for predicting long-term behavior, while others are a better fit for examining short-term behavior or interpolating data.

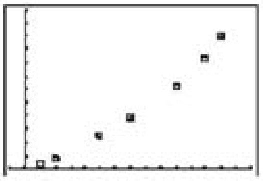

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Figure 3.5.10

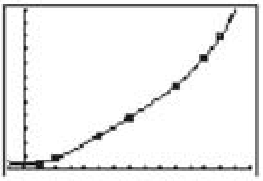

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Figure 3.5.11

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Figure 3.5.12

GuidedPractice

4. **INTERNET** Use the data in the table to determine a regression equation that best relates the cumulative number of domain names that were purchased from an Internet provider each month. Then predict how many domain names will be purchased during the 18th month.

Time (mo)	1	2	3	4	5	6	7	8
Domain Names	211	346	422	468	491	506	522	531

Time (mo)	9	10	11	12	13	14	15	16
Domain Names	540	538	551	542	565	571	588	593

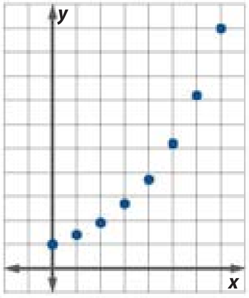
2 Linearizing Data

The correlation coefficient is a measure calculated from a *linear* regression. How then do graphing calculators provide correlation coefficients for *nonlinear* regression? The answer is that the calculators have **linearized** the data, transforming it so that it appears to cluster about a line. The process of transforming nonlinear data so that it appears to be linear is called *linearization*.

To linearize data, a function is applied to one or both of the variables in the data set as shown in the example below.

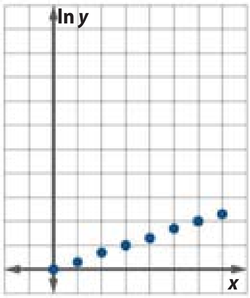
Original Data

x	y
0	1
1	1.4
2	1.9
3	2.7
4	3.7
5	5.2
6	7.2
7	10.0



Linearized Data

x	$\ln y$
0	0
1	0.3
2	0.6
3	1.0
4	1.3
5	1.6
6	2.0
7	2.3



By calculating the equation of the line that best fits the linearized data and then applying inverse functions, a calculator can provide you with an equation that models the original data. The correlation coefficient for this nonlinear regression is actually a measure of how well the calculator was able to fit the *linearized data*.

Data modeled by a quadratic function are linearized by applying a square root function to the y-variable, while data modeled by exponential, power, or logarithmic functions are linearized by applying a logarithmic function to one or both variables.

StudyTip

Linearizing Data Modeled by Other Polynomial Functions

To linearize a cubic function $y = ax^3 + bx^2 + cx + d$, graph $(x, \sqrt[3]{y})$. To linearize a quartic function $y = ax^4 + bx^3 + cx^2 + dx + e$, graph $(x, \sqrt[4]{y})$.

KeyConcept Transformations for Linearizing Data

To linearize data modeled by:

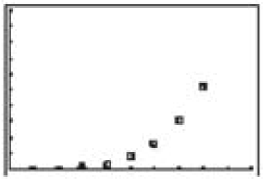
- a quadratic function $y = ax^2 + bx + c$, graph $(x, \sqrt{y})$.
- an exponential function $y = ab^x$, graph $(x, \ln y)$.
- a logarithmic function $y = a \ln x + b$, graph $(\ln x, y)$.
- a power function $y = ax^b$, graph $(\ln x, \ln y)$.

You will justify two of these linear transformations algebraically in Exercises 34 and 35.

Example 5 Linearizing Data

A graph of the data below is shown at the right. Linearize the data assuming a power model. Graph the linearized data, and find the linear regression equation. Then use this linear model to find a model for the original data.

x	0.5	1	1.5	2	2.5	3	3.5	4
y	0.13	2	10.1	32	78.1	162	300.1	512



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StudyTip

Semi-Log and Log-log Data When a logarithmic function is applied to the *x*- or *y*-values of a data set, the new data set is sometimes referred to as the semi-log of the data (*x*, ln *y*) or (ln *x*, *y*). Log-log data refers to data that have been transformed by taking a logarithmic function of both the *x*- and *y*-values, (ln *x*, ln *y*).

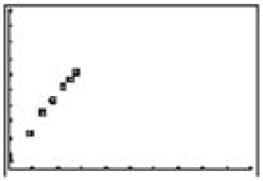
Step 1 **Linearize the data.**

To linearize data that can be modeled by a power function, take the natural log of both the x- and y-values.

ln *x*	−0.7	0	0.4	0.7	0.9	1.1	1.3	1.4
ln *y*	−2	0.7	2.3	3.5	4.4	5.1	5.7	6.2

Step 2 **Graph the linearized data and find the linear regression equation.**

The graph of $(\ln x, \ln y)$ appears to cluster about a line. Let $\hat{x} = \ln x$ and $\hat{y} = \ln y$. Using linear regression, the approximate equation modeling the linearized data is $\hat{y} = 4\hat{x} + 0.7$.



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StudyTip

Comparing Methods Use the power regression feature on a calculator to find an equation that models the data in Example 5. How do the two compare? How does the correlation coefficient from the linear regression in Step 2 compare with the correlation coefficient given by the power regression?

Step 3 **Use the model for the linearized data to find a model for the original data.**

Replace $\hat{x}$ with $\ln x$ and $\hat{y}$ with $\ln y$, and solve for y.

$\hat{y} = 4\hat{x} + 0.7$ Equation for linearized data

$\ln y = 4 \ln x + 0.7$ $\hat{x} = \ln x$ and $\hat{y} = \ln y$

$e^{\ln y} = e^{4 \ln x + 0.7}$ Exponentiate each side.

$y = e^{4 \ln x + 0.7}$ Inverse Property of Logarithms

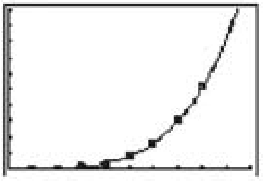
$y = e^{4 \ln x}\, e^{0.7}$ Product Property of Exponents

$y = e^{\ln x^4}\, e^{0.7}$ Power Property of Exponents

$y = x^4 e^{0.7}$ Inverse Property of Logarithms

$y = 2x^4$ $e^{0.7} \approx 2$

Therefore, a power function that models these data is $y = 2x^4$. The graph of this function with the scatter plot of the original data shows that this model is a good fit for the data.



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GuidedPractice

Make a scatter plot of each set of data, and linearize the data according to the given model. Graph the linearized data, and find the linear regression equation. Then use this linear model for the transformed data to find a model for the original data.

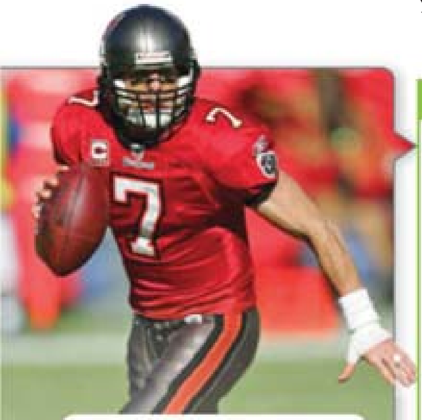
5A. quadratic model

x	0	1	2	3	4	5	6	7
y	0	1	2	9	20	35	54	77

5B. logarithmic model

x	1	2	3	4	5	6	7	8
y	5	7.1	8.3	9.5	9.8	10.4	10.8	11.2

You can linearize data to find models for real-world data without the use of a calculator.



Real-WorldLink

NFL player salaries are regulated by a salary cap, a maximum amount each franchise is allowed to spend on its total roster each season. In 2008, the salary cap per team was $116 million.

Source: NFL

Real-World Example 6 Use Linearization

SPORTS The table shows the average professional football player's salary for several years. Find an exponential model relating these data by linearizing the data and finding the linear regression equation. Then use your model to predict the average salary in 2012.

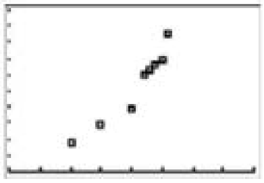
Year	1990	1995	2000	2002	2003	2004	2005	2006
Average Salary ($1000)	354	584	787	1180	1259	1331	1400	1700

Source: NFL Players Association

Step 1 **Make a scatter plot, and linearize the data.**

Let x represent the number of years after 1900 and y the average salary in thousands.

x	90	95	100	102	103	104	105	106
y	354	584	787	1180	1259	1331	1400	1700



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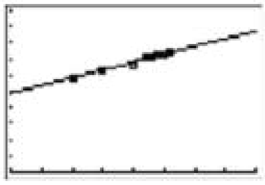
The plot is nonlinear and its shape suggests that the data could be modeled by an exponential function. Linearize the data by finding $(x, \ln y)$.

x	90	95	100	102	103	104	105	106
$\ln y$	5.9	6.4	6.7	7.1	7.1	7.2	7.2	7.4

Step 2 **Graph the linearized data, and find a linear regression equation.**

A plot of the linearized data appears to form a straight line. Letting $\hat{y} = \ln y$, the rounded regression equation is about $\hat{y} = 0.096x - 2.754$.



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Step 3 **Use the model for the linearized data to find a model for the original data.**

Replace $\hat{y}$ with $\ln y$, and solve for y.

$\hat{y} = 0.096x - 2.754$ Equation for linearized data

$\ln y = 0.096x - 2.754$ $\hat{y} = \ln y$

$e^{\ln y} = e^{0.096x - 2.754}$ Exponentiate each side.

$y = e^{0.096x - 2.754}$ Inverse Property of Logarithms

$y = e^{0.096x} e^{-2.754}$ Product Property of Exponents

$y = 0.06e^{0.096x}$ $e^{-2.754} \approx 0.06$

Therefore, an exponential equation that models these data is $y = 0.06e^{0.096x}$.

WatchOut!

Using the Wrong Equation Be careful not to confuse the equation that models the *linearized* data with the equation that models the *original* data.

Step 4 **Use the equation that models the original data to solve the problem.**

To find the average salary in 2012, find y when $x = 2012 - 1900$ or 112. According to this model, the average professional football player's salary in 2012 will be $0.06e^{0.096(112)} \approx$ \$2803 thousand or about \$2.8 million.

GuidedPractice

6. **FALLING OBJECT** Roger drops one of his shoes out of a hovering helicopter. The distance d in feet the shoe has dropped after t seconds is shown in the table.

t	0	1	1.5	2	2.5	3	4	5
d	0	15.7	35.4	63.8	101.4	144.5	258.1	404.8

Find a quadratic model relating these data by linearizing the data and finding the linear regression equation. Then use your model to predict the distance the shoe has traveled after 7 seconds.

Paul Jasienski/Getty Images Sport/Getty Images

Exercises

= Step-by-Step Solutions begin on page R29.

For Exercises 1–3, complete each step.

a. Find an exponential function to model the data.

b. Find the value of each model at $x = 20$. (Example 1)

1.

x	y
1	7
2	11
3	25
4	47
5	96
6	193
7	380

2.

x	y
0	1
1	6
2	23
3	124
4	620
5	3130
6	15,600

3.

x	y
0	25
1	6
2	1.6
3	0.4
4	0.09
5	0.023
6	0.006

4. **GENETICS** *Drosophila melanogaster*, a species of fruit fly, are a common specimen in genetics labs because they reproduce about every 8.5 days, allowing researchers to study several generations. The table shows the population of *drosophila* over a period of days. (Example 1)

Generation	Drosophila	Generation	Drosophila
1	80	5	1180
2	156	6	2314
3	307	7	4512
4	593	8	8843

a. Find an exponential function to model the data.

b. Use the function to predict the population of *drosophila* after 93.5 days.

5. **SHARKS** Sharks have numerous rows of teeth embedded directly into their gums and not connected to their jaws. As a shark loses its teeth, teeth from the next row move forward. The rate of replacement of a row of teeth in days per row increases with the water temperature. (Example 1)

Temp. (°C)	20	21	22	23	24	25	26	27
Days per Row	66	54	44	35	28	22	18	16

a. Find an exponential function to model the data.

b. Use the function to predict the temperature at which sharks lose a row of teeth in 12 days.

6. **WORDS** A word family consists of a base word and all of its derivations. The table shows the percentage of words in an average English text comprised of the most common word families. (Example 2)

Word Families	1000	2000	3000	4000	5000
Percentage of Words	73.1	79.7	84.0	86.7	88.6

a. Find a logarithmic function to model the data.

b. Predict the number of word families that make up 95% of the words in an average English text.

For Exercises 7–9, complete each step.

a. Find a logarithmic function to model the data.

b. Find the value of each model at $x = 15$. (Example 2)

7.

x	y
1	50
2	42
3	37
4	33
5	31
6	28
7	27

8.

x	y
2	8.6
4	7.2
6	6.4
8	5.8
10	5.4
12	5.0
14	4.7

9.

x	y
1	40
2	49.9
3	55.8
4	59.9
5	63.2
6	65.8
7	68.1

10. **CHEMISTRY** A lab received a sample of an isotope of cobalt in 1999. The amount of cobalt in grams remaining per year is shown in the table below. (Example 2)

Year	2000	2001	2002	2003	2004	2005	2006	2007
Cobalt (g)	877	769	674	591	518	454	398	349

a. Find a logarithmic function to model the data. Let $x = 1$ represent 2000.

b. Predict the amount of cobalt remaining in 2020.

For Exercises 11–13, complete each step.

a. Find a logistic function to model the data.

b. Find the value of each model at $x = 25$. (Example 3)

11.

x	y
0	50
2	67
4	80
6	89
8	94
10	97
12	98
14	99

12.

x	y
1	3
2	5
3	7
4	8
5	13
6	16
7	19
8	20

13.

x	y
3	21
6	25
9	28
12	31
15	33
18	34
21	35
24	35

14. **CHEMISTRY** A student is performing a titration in lab. To perform the titration, she uses a burette to add a basic solution of NaOH to a neutral solution. The table shows the pH of the solution as the NaOH is added. (Example 3)

NaOH (mL)	0	1	2	3	5	7.5	10
pH	10	10.4	10.6	11.0	11.3	11.5	11.5

a. Find a logistic function to model the data.

b. Use the model to predict the pH of the solution after 12 milliliters of NaOH have been added.

15 **CENSUS** The table shows the projected population of Maine from the 2000 census. Let *x* be the number of years after 2000. (Example 3)

Year	Population (millions)
2000	1.275
2005	1.319
2010	1.357
2015	1.389
2020	1.409
2025	1.414
2030	1.411

a. Find a logistic function to model the data.

b. Based on the model, at what population does the 2000 census predict Maine's growth to level off?

c. Discuss the effectiveness of the model to predict the population as time increases significantly beyond the domain of the data.

16. **SCUBA DIVING** Scuba divers search for dive locations with good visibility, which can be affected by the murkiness of the water and the penetration of surface light. The table shows the percent of surface light reaching a diver at different depths as the diver descends. (Example 4)

Depth (ft)	Light (%)
15	89.2
30	79.6
45	71.0
60	63.3
75	56.5
90	50.4
105	44.9
120	40.1

a. Use the regression features on a calculator to determine the regression equation that best relates the data.

b. Use the graph of your regression equation to approximate the percent of surface light that reaches the diver at a depth of 83 feet.

17. **EELS** The table shows the average length of female king snake eels at various ages. (Example 4)

Age (yr)	Length (in.)	Age (yr)	Length (in.)
4	8	14	17
6	11	16	18
8	13	18	18
10	15	20	19
12	16		

a. Use the regression features on a calculator to determine if a logarithmic regression is better than a logistic regression. Explain.

b. Use the graph of your regression equation to approximate the length of an eel at 19 years.

For Exercises 18–21, complete each step.

a. Linearize the data according to the given model.

b. Graph the linearized data, and find the linear regression equation.

c. Use the linear model to find a model for the original data. Check its accuracy by graphing. (Examples 5 and 6)

18. exponential

x	*y*
0	11
1	32
2	91
3	268
4	808
5	2400
6	7000
7	22,000

19. quadratic

x	*y*
0	1.0
1	6.6
2	17.0
3	32.2
4	52.2
5	77.0
6	106.6
7	141.0

20. logarithmic

x	*y*
2	80.0
4	83.5
6	85.5
8	87.0
10	88.1
12	89.0
14	90.0
16	90.5

21. power

x	*y*
1	5
2	21
3	44
4	79
5	120
6	180
7	250
8	320

22. **TORNADOES** A tornado with a greater wind speed near the center of its funnel can travel greater distances. The table shows the wind speeds near the centers of tornadoes that have traveled various distances. (Example 6)

Distance (mi)	Wind Speed (mph)
0.50	37
0.75	53
1.00	65
1.25	74
1.50	81
1.75	88
2.00	93
2.25	98
2.50	102
2.75	106

a. Linearize the data assuming a logarithmic model.

b. Graph the linearized data, and find the linear regression equation.

c. Use the linear model to find a model for the original data, and approximate the wind speed of a funnel that has traveled 3.7 miles.

23. **HOUSING** The table shows the appreciation in the value of a house every 3 years since the house was purchased. (Example 6)

Years	0	3	6	9	12	15
Value ($)	78,000	81,576	85,992	90,791	95,859	101,135

a. Linearize the data assuming an exponential model.

b. Graph the linearized data, and find the linear regression equation.

c. Use the linear model to find a model for the original data, and approximate the value of the house 24 years after it is purchased.

24. **COOKING** Cooking times, temperatures, and recipes are often different at high altitudes than at sea level. This is due to the difference in atmospheric pressure, which causes boiling points for various substances, such as water, to be lower at higher altitudes. The table shows the boiling point of water at different elevations above sea level.

Elevation (m)	Boiling Point (°C)
0	100
1000	99.29
2000	98.81
3000	98.43
4000	98.10
5000	97.80
6000	97.53
7000	97.28
8000	97.05
9000	96.83
10,000	96.62

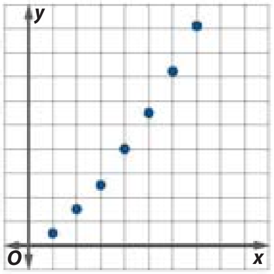
a. Linearize the data for exponential, power, and logarithmic models.

b. Graph the linearized data, and determine which model best represents the data.

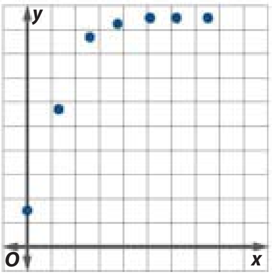
c. Write an equation to model the data based on your analysis of the linearizations.

Determine the model most appropriate for each scatter plot. Explain your reasoning.

25.



26.



Linearize the data in each table. Then determine the most appropriate model.

27.

x	y
2	2.5
4	7.3
6	13.7
8	21.3
10	30.2
12	40.0
14	50.8
16	62.5

28.

x	y
1	6
2	29
3	42
4	52
5	59
6	65
7	70
8	75

29.

x	y
1	37.8
2	17.0
3	7.7
4	3.4
5	1.6
6	0.7
7	0.3
8	0.1

30. **FISH** Several ichthyologists are studying the smallmouth bass population in a lake. The table shows the smallmouth bass population of the lake over time.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Bass	673	891	1453	1889	2542	2967	3018	3011

a. Determine the most appropriate model for the data. Explain your reasoning.

b. Find a function to model the data.

c. Use the function to predict the smallmouth bass population in 2012.

d. Discuss the effectiveness of the model to predict the population of the bass as time increases significantly beyond the domain of the data.

H.O.T. Problems Use Higher-Order Thinking Skills

31. **REASONING** Why are logarithmic regressions invalid when the domain is 0?

32. **CHALLENGE** Show that $y = ab^x$ can be converted to $y = ae^{kx}$.

33. **REASONING** Can the graph of a logistic function ever have any intercepts? Explain your reasoning.

PROOF **Use algebra to verify that data modeled by each type of function can be linearized, or expressed as a function $y = mx + b$ for some values m and b, by replacing (x, y) with the indicated coordinates.**

34. exponential, $(x, \ln y)$

㉟ power, $(\ln x, \ln y)$

36. **REASONING** How is the graph of $g(x) = \frac{5}{1 + e^{-x}} + a$ related to the graph of $f(x) = \frac{5}{1 + e^{-x}}$? Explain.

37. **WRITING IN MATH** Explain how the parameters of an exponential or logarithmic model relate to the data set or situation being modeled.

Spiral Review

Solve each equation. (Lesson 3-4)

38. $3^{4x} = 3^{3-x}$

39. $3^{5x} \cdot 81^{1-x} = 9^{x-3}$

40. $49^{x} = 7^{x^2-15}$

41. $\log_5 (4x - 1) = \log_5 (3x + 2)$

42. $\log_{10} z + \log_{10} (z + 3) = 1$

43. $\log_6 (a^2 + 2) + \log_6 2 = 2$

44. **ENERGY** The energy E, in kilocalories per gram molecule, needed to transport a substance from the outside to the inside of a living cell is given by $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$, where C_1 and C_2 are the concentrations of the substance inside and outside the cell, respectively. (Lesson 3-3)

a. Express the value of E as one logarithm.

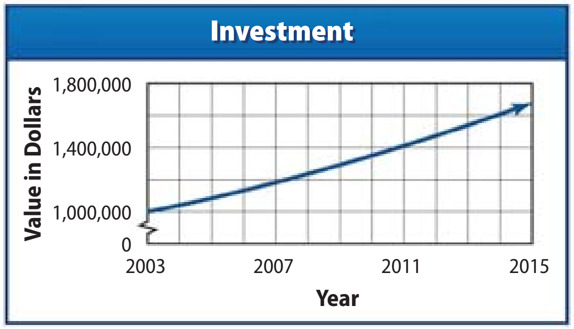
b. Suppose the concentration of a substance inside the cell is twice the concentration outside the cell. How much energy is needed to transport the substance on the outside of the cell to the inside? (Use $\log_{10} 2 \approx 0.3010$.)

c. Suppose the concentration of a substance inside the cell is four times the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside?

45. **FINANCIAL LITERACY** In 2003, Maya inherited \$1,000,000 from her grandmother. She invested all of the money and by 2015, the amount will grow to \$1,678,000. (Lesson 3-1)

a. Write an exponential function that could be used to model the amount of money y. Write the function in terms of x, the number of years since 2003.

b. Assume that the amount of money continues to grow at the same rate. Estimate the amount of money in 2025. Is this estimate reasonable? Explain your reasoning.



Simplify. (Lesson 0-2)

46. $(-2i)(-6i)(4i)$

47. $3i(-5i)^2$

48. i^{13}

49. $(1 - 4i)(2 + i)$

50. $\frac{4i}{3 + i}$

51. $\frac{4}{5 + 3i}$

Skills Review for Standardized Tests

52. **SAT/ACT** A recent study showed that the number of Australian homes with a computer doubles every 8 months. Assuming that the number is increasing continuously, at approximately what monthly rate must the number of Australian computer owners be increasing for this to be true?

A 6.8% **C** 12.5% **E** 2%

B 8.66% **D** 8.0%

53. The data below gives the number of bacteria found in a certain culture. The bacteria are growing exponentially.

Hours	0	1	2	3	4
Bacteria	5	8	15	26	48

Approximately how much time will it take the culture to double after hour 4?

F 1.26 hours **H** 1.68 hours

G 1.35 hours **J** 1.76 hours

54. **FREE RESPONSE** The speed in miles per hour at which a car travels is represented by $v(t) = 60\left(1 - e^{-t^2}\right)$ where t is the time in seconds. Assume the car never needs to stop.

a. Graph $v(t)$ for $0 \le t \le 10$.

b. Describe the domain and range of $v(t)$. Explain your reasoning.

c. What type of function is $v(t)$?

d. What is the end behavior of $v(t)$? What does this mean in the context of the situation?

e. Let $d(t)$ represent the total distance traveled by the car. What type of function does $d(t)$ represent as t approaches infinity? Explain.

f. Let $a(t)$ represent the acceleration of the car. What is the end behavior of $a(t)$? Explain.

CHAPTER 3 Study Guide and Review

Study Guide

KeyConcepts

Exponential Functions (Lesson 3-1)

- Exponential functions are of the form $f(x) = ab^x$, where $a \neq 0$, b is positive and $b \neq 1$. For natural base exponential functions, the base is the constant e.
- If a principal P is invested at an annual interest rate r (in decimal form), then the balance A in the account after t years is given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$, if compounded n times a year or $A = Pe^{rt}$, if compounded continuously.
- If an initial quantity N_0 grows or decays at an exponential rate r or k (as a decimal), then the final amount N after a time t is given by $N = N_0(1 + r)^t$ or $N = N_0 e^{kt}$, where r is the rate of growth per time t and k is the continuous rate of growth at any time t.

Logarithmic Functions (Lesson 3-2)

- The inverse of $f(x) = b^x$, where $b > 0$ and $b \neq 1$, is the logarithmic function with base b, denoted $f^{-1}(x) = \log_b x$.
- If $b > 0$, $b \neq 1$, and $x > 0$, then the exponential form of $\log_b x = y$ is $b^y = x$ and the logarithmic form of $b^y = x$ is $\log_b x = y$. A logarithm is an exponent.
- Common logarithms: $\log_{10} x$ or $\log x$
- Natural logarithms: $\log_e x$ or $\ln x$

Properties of Logarithms (Lesson 3-3)

- Product Property: $\log_b xy = \log_b x + \log_b y$
- Quotient Property: $\log_b \frac{x}{y} = \log_b x - \log_b y$
- Power Property: $\log_b x^p = p \cdot \log_b x$
- Change of Base Formula: $\log_b x = \frac{\log_a x}{\log_a b}$

Exponential and Logarithmic Equations (Lesson 3-4)

- One-to-One Property of Exponents: For $b > 0$ and $b \neq 1$, $b^x = b^y$ if and only if $x = y$.
- One-to-One Property of Logarithms: For $b > 0$ and $b \neq 1$, $\log_b x = \log_b y$ if and only if $x = y$.

Modeling with Nonlinear Regression (Lesson 3-5)

To linearize data modeled by:

- a quadratic function $y = ax^2 + bx + c$, graph $(x, \sqrt{y})$.
- an exponential function $y = ab^x$, graph $(x, \ln y)$.
- a logarithmic function $y = a \ln x + b$, graph $(\ln x, y)$.
- a power function $y = ax^b$, graph $(\ln x, \ln y)$.

KeyVocabulary

algebraic function (p. 158)
common logarithm (p. 173)
continuous compound interest (p. 163)
exponential function (p. 158)
linearize (p. 204)
logarithm (p. 172)
logarithmic function with base *b* (p. 172)
logistic growth function (p. 202)
natural base (p. 160)
natural logarithm (p. 174)
transcendental function (p. 158)

VocabularyCheck

Choose the correct term from the list above to complete each sentence.

1. A logarithmic expression in which no base is indicated uses the ________.

2. ________ are functions in which the variable is the exponent.

3. Two examples of ________ are exponential functions and logarithmic functions.

4. The inverse of $f(x) = b^x$ is called a(n) ________.

5. The graph of a(n) ________ contains two horizontal asymptotes. Such a function is used for growth that has a limiting factor.

6. Many real-world applications use the ________ e, which, like π or $\sqrt{5}$, is an irrational number that requires a decimal approximation.

7. To ________ data, a function is applied to one or both of the variables in the data set, transforming the data so that it appears to cluster about a line.

8. Power, radical, polynomial, and rational functions are examples of ________.

9. ________ occurs when there is no waiting period between interest payments.

10. The ________ is denoted by ln.

CHAPTER 3

Study Guide and Review *Continued*

Lesson-by-Lesson Review

3-1 Exponential Functions (pp. 158–169)

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.

11. $f(x) = 3^x$

12. $f(x) = 0.4^x$

13. $f(x) = \left(\frac{3}{2}\right)^x$

14. $f(x) = \left(\frac{1}{3}\right)^x$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$.

15. $f(x) = 4^x$; $g(x) = 4^x + 2$

16. $f(x) = 0.1^x$; $g(x) = 0.1^{x-3}$

17. $f(x) = 3^x$; $g(x) = 2 \cdot 3^x - 5$

18. $f(x) = \left(\frac{1}{2}\right)^x$; $g(x) = \left(\frac{1}{2}\right)^{x+4} + 2$

Copy and complete the table below to find the value of an investment A for the given principal P, rate r, and time t if the interest is compounded n times annually.

n	1	4	12	365	continuously
A					

19. $P = \$250$, $r = 7\%$, $t = 6$ years

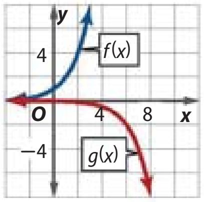
20. $P = \$1000$, $r = 4.5\%$, $t = 3$ years

Example 1

Use the graph of $f(x) = 2^x$ to describe the transformation that results in the graph of $g(x) = -2^{x-5}$. Then sketch the graphs of g and f.

This function is of the form $g(x) = -f(x - 5)$.

So, $g(x)$ is the graph of $f(x) = 2^x$ translated 5 units to the right and reflected in the x-axis.



Example 2

What is the value of $2000 invested at 6.5% after 12 years if the interest is compounded quarterly? continuously?

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound Interest Formula

$= 2000\left(1 + \frac{0.065}{4}\right)^{4(12)}$ $P = 2000$, $r = 0.065$, $n = 4$, $t = 12$

$= \$4335.68$ Simplify.

$A = Pe^{rt}$ Continuous Interest Formula

$= 2000e^{0.065(12)}$ $P = 2000$, $r = 0.065$, $t = 12$

$= \$4362.94$ Simplify.

3-2 Logarithmic Functions (pp. 172–180)

Evaluate each expression.

21. $\log_2 32$

22. $\log_3 \frac{1}{81}$

23. $\log_{25} 5$

24. $\log_{13} 1$

25. $\ln e^{11}$

26. $3^{\log_3 9}$

27. $\log 80$

28. $e^{\ln 12}$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$.

29. $f(x) = \log x$; $g(x) = -\log (x + 4)$

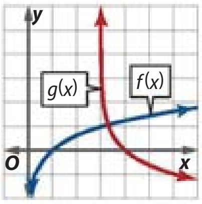
30. $f(x) = \log_2 x$; $g(x) = \log_2 x + 3$

31. $f(x) = \ln x$; $g(x) = \frac{1}{4} \ln x - 2$

Example 3

Use the graph of $f(x) = \ln x$ to describe the transformation that results in the graph of $g(x) = -\ln (x - 3)$. Then sketch the graphs of $g(x)$ and $f(x)$.

This function is of the form $g(x) = -f(x - 3)$. So, $g(x)$ is the graph of $f(x)$ reflected in the x-axis translated 3 units to the right.



3-3 Properties of Logarithms (pp. 181–188)

Expand each expression.

32. $\log_3 9x^3y^3z^6$

33. $\log_5 x^2a^7\sqrt{b}$

34. $\ln \frac{e}{x^2y^3z}$

35. $\log \frac{\sqrt{gj^5k}}{100}$

Condense each expression.

36. $3 \log_3 x - 2 \log_3 y$

37. $\frac{1}{3} \log_2 a + \log_2 (b + 1)$

38. $5 \ln (x + 3) + 3 \ln 2x - 4 \ln (x - 1)$

Example 4

Condense $3 \log_3 x + \log_3 7 - \frac{1}{2} \log_3 x$.

$3 \log_3 x + \log_3 7 - \frac{1}{2} \log_3 x$

$= \log_3 x^3 + \log_3 7 - \log_3 \sqrt{x}$ Power Property

$= \log_3 7x^3 - \log_3 \sqrt{x}$ Product Property

$= \log_3 \frac{7x^3}{\sqrt{x}}$ Quotient Property

3-4 Exponential and Logarithmic Equations (pp. 190–199)

Solve each equation.

39. $3^{x+3} = 27^{x-2}$

40. $25^{3x+2} = 125$

41. $e^{2x} - 8e^x + 15 = 0$

42. $e^x - 4e^{-x} = 0$

43. $\log_2 x + \log_2 3 = \log_2 18$

44. $\log_6 x + \log_6 (x - 5) = 2$

Example 5

Solve $7 \ln 2x = 28$.

$7 \ln 2x = 28$ Original equation

$\ln 2x = 4$ Divide each side by 7.

$e^{\ln 2x} = e^4$ Exponentiate each side.

$2x = e^4$ Inverse Property

$x = 0.5e^4$ or about 27.299 Solve and simplify.

3-5 Modeling With Nonlinear Regression (pp. 200–210)

Complete each step.

- **a.** Linearize the data according to the given model.
- **b.** Graph the linearized data, and find the linear regression equation.
- **c.** Use the linear model to find a model for the original data and graph it.

45. exponential

x	0	1	2	3	4	5	6
y	2	5	17	53	166	517	1614

46. logarithmic

x	1	2	3	4	5	6	7
y	−3	4	8	10	12	14	15

Example 6

Linearize the data shown assuming a logarithmic model, and calculate the equation for the line of best fit. Use this equation to find a logarithmic model for the original data.

x	1	3	5	7	9	10
y	12	−7	−15	−21	−25	−27

Step 1 To linearize $y = a \ln x + b$, graph $(\ln x, y)$.

$\ln x$	0	1.1	1.6	1.9	2.2	2.3
y	12	−7	−15	−21	−25	−27

Step 2 The line of best fit is $y = -16.94x + 11.86$.

Step 3 $y = -16.94 \ln x + 11.86$ $x = \ln x$

Applications and Problem Solving

47. **INFLATION** Prices of consumer goods generally increase each year due to inflation. From 2000 to 2008, the average rate of inflation in the United States was 4.5%. At this rate, the price of milk t years after January 2000 can be modeled with $M(t) = 2.75(1.045)^t$. (Lesson 3-1)

a. What was the price of milk in 2000? 2005?

b. If inflation continues at 4.5%, approximately what will the price of milk be in 2015?

c. In what year did the price of milk reach $4?

48. **CARS** The value of a new vehicle depreciates the moment the car is driven off the dealer's lot. The value of the car will continue to depreciate every year. The value of one car t years after being bought is $f(x) = 18{,}000(0.8)^t$. (Lesson 3-1)

a. What is the rate of depreciation for the car?

b. How many years after the car is bought will it be worth half of its original value?

49. **CHEMISTRY** A radioactive substance has a half-life of 16 years. The number of years t it takes to decay from an initial amount N_0 to N can be determined using $t = \dfrac{16 \log \frac{N}{N_0}}{\log \frac{1}{2}}$. (Lesson 3-2)

a. Approximately how many years will it take 100 grams to decay to 30 grams?

b. Approximately what percentage of 100 grams will there be in 40 years?

50. **EARTHQUAKES** The Richter scale is a number system for determining the strength of earthquakes. The number R is dependent on energy E released by the earthquake in kilowatt-hours. The value of R is determined by $R = 0.67 \cdot \log(0.37E) + 1.46$. (Lesson 3-2)

a. Find R for an earthquake that releases 1,000,000 kilowatt-hours.

b. Estimate the energy released by an earthquake that registers 7.5 on the Richter scale.

51. **BIOLOGY** The time it takes for a species of animal to double is defined as its *generation time* and is given by $G = \dfrac{t}{2.5 \log_b d}$, where b is the initial number of animals, d is the final number of animals, t is the time period, and G is the generation time. If the generation time G of a species is 6 years, how much time t will it take for 5 animals to grow into a population of 3125 animals? (Lesson 3-3)

52. **SOUND** The intensity level of a sound, measured in decibels, can be modeled by $d(w) = 10 \log \frac{w}{w_0}$, where w is the intensity of the sound in watts per square meter and w_0 is the constant 1×10^{-12} watts per square meter. (Lesson 3-4)

a. Determine the intensity of the sound at a concert that reaches 100 decibels.

b. Tory compares the concert with the music she plays at home. She plays her music at 50 decibels, so the intensity of her music is half the intensity of the concert. Is her reasoning correct? Justify your answer mathematically.

c. Soft music is playing with an intensity of 1×10^{-8} watts per square meter. By how much do the decimals increase if the intensity is doubled?

53. **FINANCIAL LITERACY** Delsin has $8000 and wants to put it into an interest-bearing account that compounds continuously. His goal is to have $12,000 in 5 years. (Lesson 3-4)

a. Delsin found a bank that is offering 6% on investments. How long would it take his investment to reach $12,000 at 6%?

b. What rate does Delsin need to invest at to reach his goal of $12,000 after 5 years?

54. **INTERNET** The number of people to visit a popular Web site is given below. (Lesson 3-5)

a. Make a scatterplot of the data. Let 1990 = 0.

b. Linearize the data with a logarithmic model.

c. Graph the linearized data, and find the linear regression equation.

d. Use the linear model to find a model for the original data and graph it.

CHAPTER 3 Practice Test

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.

1. $f(x) = -e^{x+7}$

2. $f(x) = 2\left(\frac{3}{5}\right)^{-x} - 4$

Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$.

3. $f(x) = \left(\frac{1}{2}\right)^{x}$ $\quad g(x) = \left(\frac{1}{2}\right)^{x-3} + 4$

4. $f(x) = 5^{x}$ $\quad g(x) = -5^{-x} - 2$

5. **MULTIPLE CHOICE** For which function is $\lim\limits_{x \to \infty} f(x) = -\infty$?

A $f(x) = -2 \cdot 3^{-x}$

B $f(x) = -\left(\frac{1}{10}\right)^{x}$

C $f(x) = -\log_8 (x - 5)$

D $f(x) = \log_3 (-x) - 6$

Evaluate each expression.

6. $\log_3 \frac{1}{81}$

7. $\log_{32} 2$

8. $\log 10^{12}$

9. $9^{\log_9 5.3}$

Sketch the graph of each function.

10. $f(x) = -\log_4 (x + 3)$

11. $g(x) = \log (-x) + 5$

12. **FINANCIAL LITERACY** You invest $1500 in an account with an interest rate of 8% for 12 years, making no other deposits or withdrawals.

a. What will be your account balance if the interest is compounded monthly?

b. What will be your account balance if the interest is compounded continuously?

c. If your investment is compounded daily, about how long will it take for it to be worth double the initial amount?

Expand each expression.

13. $\log_6 36xy^2$

14. $\log_3 \frac{a\sqrt{b}}{12}$

15. **GEOLOGY** Richter scale magnitude of an earthquake can be calculated using $R = \frac{2}{3} \log \frac{E}{E_0}$, where E is the energy produced and E_0 is a constant.

a. An earthquake with a magnitude of 7.1 hit San Francisco in 1989. Find the scale of an earthquake that produces 10 times the energy of the 1989 earthquake.

b. In 1906, San Francisco had an earthquake registering 8.25. How many times as much energy did the 1906 earthquake produce as the 1989 earthquake?

Condense each expression.

16. $2 \log_4 m + 6 \log_4 n - 3(\log_4 3 + \log_4 j)$

17. $1 + \ln 3 - 4 \ln x$

Solve each equation.

18. $3^{x+8} = 9^{2x}$

19. $e^{2x} - 3e^{x} + 2 = 0$

20. $\log x + \log (x - 3) = 1$

21. $\log_2 (x - 1) + 1 = \log_2 (x + 3)$

22. **MULTIPLE CHOICE** Which equation has no solution?

F $e^{x} = e^{-x}$

G $2^{x-1} = 3^{x+1}$

H $\log_5 x = \log_9 x$

J $\log_2 (x + 1) = \log_2 x$

For Exercises 23 and 24, complete each step.

a. Find an exponential or logarithmic function to model the data.

b. Find the value of each model at $x = 20$.

23.

x	1	3	5	7	9	11	13
y	8	3	0	−2	−3	−4	−5

24.

x	1	3	5	7	9	11	13
y	3	4	5	6	7	9	10

25. **CENSUS** The table gives the U.S. population between 1790 and 1940. Let 1780 = 0.

Year	Population (millions)
1790	4
1820	10
1850	23
1880	50
1910	92
1940	132

a. Linearize the data, assuming a quadratic model. Graph the data, and write an equation for a line of best fit.

b. Use the linear model to find a model for the original data. Is a quadratic model a good representation of population growth? Explain.

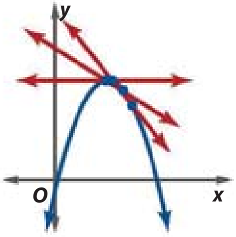
CHAPTER 3

Connect to AP Calculus
Approximating Rates of Change

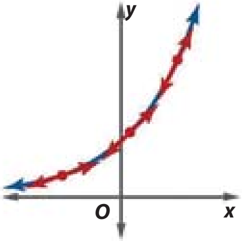
Objective

- Use secant lines and the difference quotient to approximate rates of change.

In Chapter 1, we explored the rate of change of a function at a point using secant lines and the difference quotient. You learned that the rate of change of a function at a point can be represented by the slope of the line tangent to the function at that point. This is called the *instantaneous rate of change* at that point.



The constant *e* is used in applications of continuous growth and decay. This constant also has many applications in differential and integral calculus. The rate of change of $f(x) = e^x$ at any of its points is unique, which makes it a useful function for exploration and application in calculus.

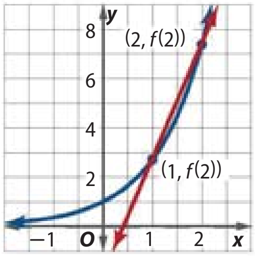


Activity 1 Approximate Rate of Change

Approximate the rate of change of $f(x) = e^x$ at $x = 1$.

Step 1 Graph $f(x) = e^x$, and plot the points $P(1, f(1))$ and $Q(2, f(2))$.

Step 2 Draw a secant line of $f(x)$ through P and Q.



Step 3 Use $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ to calculate the average rate of change m for $f(x)$ using P and Q.

Step 4 Repeat Steps 1–3 two more times. First use $P(1, f(1))$ and $Q(1.5, f(1.5))$ and then use $P(1, f(1))$ and $Q(1.25, f(1.25))$.

Analyze the Results

1. As the x-coordinate of Q approaches 1, what does the average rate of change m appear to approach?
2. Evaluate and describe the overall efficiency and the overall effectiveness of using secant lines to approximate the instantaneous rate of change of a function at a given point.

In Chapter 1, you developed an expression, the *difference quotient*, to calculate the slope of secant lines for different values of *h*.

Difference Quotient

$$m = \frac{f(x + h) - f(x)}{h}$$

As *h* decreases, the secant line moves closer and closer to a line tangent to the function. Substituting decreasing values for *h* into the difference quotient produces secant-line slopes that approach a limit. This limit represents the slope of the tangent line and the instantaneous rate of change of the function at that point.

Activity 2 Approximate Rate of Change

Approximate the rate of change of $f(x) = e^x$ at several points.

Step 1 Substitute $f(x) = e^x$ into the difference quotient. $m = \dfrac{f(x+h) - f(x)}{h}$

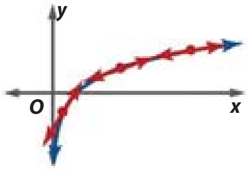
Step 2 Approximate the rate of change of $f(x)$ at $x = 1$ using values of h that approach 0. Let $h = 0.1, 0.01, 0.001$, and 0.0001. $m = \dfrac{e^{x+h} - e^x}{h}$

Step 3 Repeat Steps 1 and 2 for $x = 2$ and for $x = 3$.

Analyze the Results

3. As $h \to 0$, what does the average rate of change appear to approach for each value of x?
4. Write an expression for the rate of change of $f(x) = e^x$ at any point x.
5. Find the rate of change of $g(x) = 3e^x$ at $x = 1$. How did multiplying e^x by a constant affect the rate of change at $x = 1$?
6. Write an expression for the rate of change of $g(x) = ae^x$ at any point x.

In this chapter, you learned that $f(x) = \ln x$ is the inverse of $g(x) = e^x$, and you also learned about some of its uses in exponential growth and decay applications. Similar to e, the rate of change of $f(x) = \ln x$ at any of its points is unique, thus also making it another useful function for calculus applications.



Activity 3 Approximate Rate of Change

Approximate the rate of change of $f(x) = \ln x$ at several points.

Step 1 Substitute $f(x) = \ln x$ into the difference quotient. $m = \dfrac{f(x+h) - f(x)}{h}$

Step 2 Approximate the rate of change of $f(x)$ at $x = 2$ using values of h that approach 0. Let $h = 0.1, 0.01, 0.001$, and 0.0001. $m = \dfrac{\ln(x+h) - \ln x}{h}$

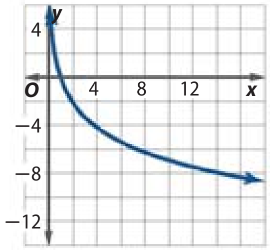
Step 3 Repeat Steps 1 and 2 for $x = 3$ and for $x = 4$.

Analyze the Results

7. As $h \to 0$, what does the average rate of change appear to approach for each value of x?
8. Write an expression for the rate of change of the function $f(x) = \ln x$ at any point x.

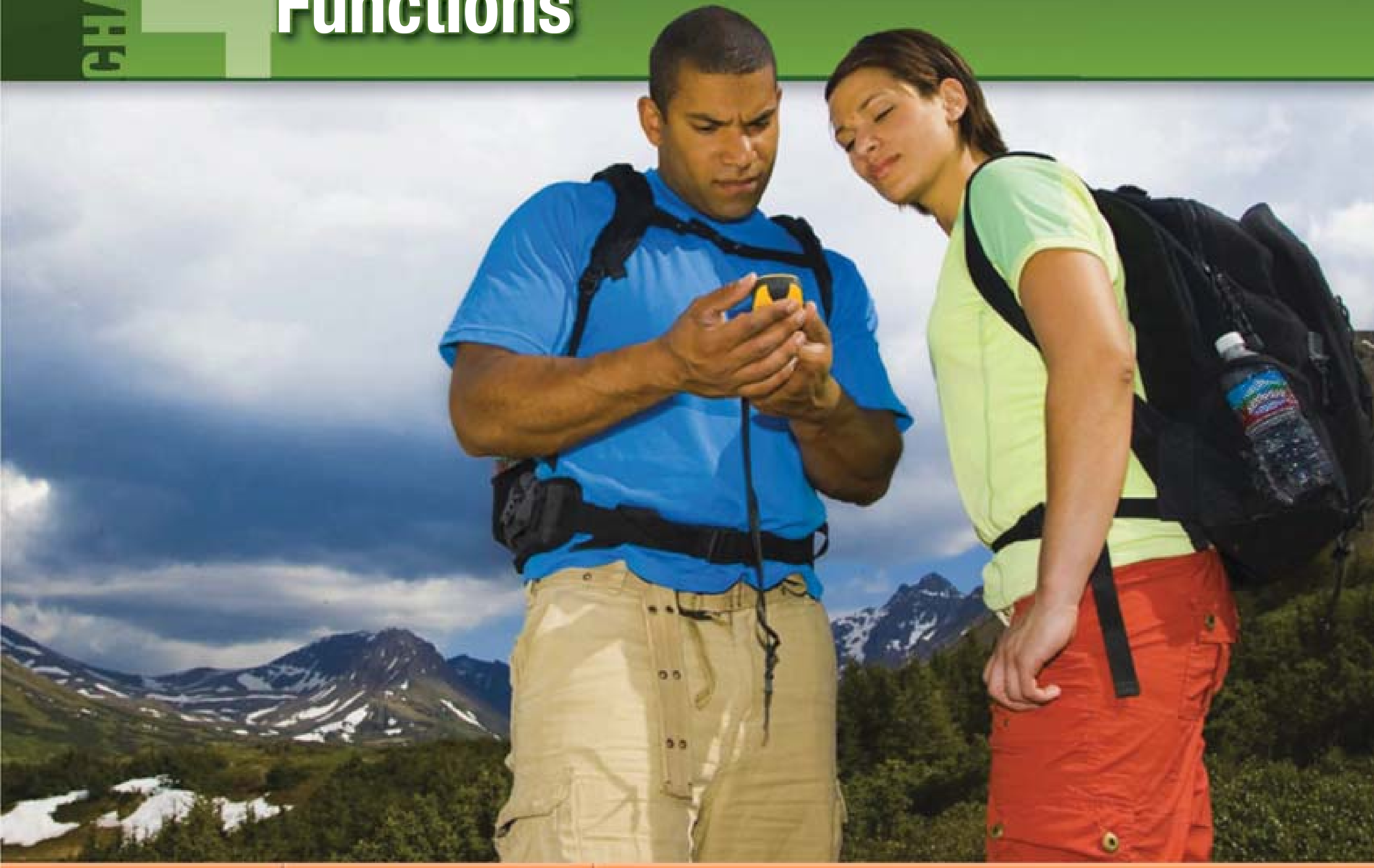
Model and Apply

9. In this problem, you will investigate the rate of change of the function $g(x) = -3 \ln x$ at any point x.
   a. Approximate the rates of change of $g(x)$ at $x = 2$ and then at $x = 3$.
   b. How do these rates of change compare to the rates of change for $f(x) = \ln x$ at these points?
   c. Write an expression for the rate of change of the function $g(x) = a \ln x$ for any point x.



CHAPTER 4

Trigonometric Functions



Michael DeYoung/Photolibrary

Then

In **Chapter 3**, you studied exponential and logarithmic functions, which are two types of transcendental functions.

Now

In **Chapter 4**, you will:

- Use trigonometric functions to solve right triangles.
- Find values of trigonometric functions for any angle.
- Graph trigonometric and inverse trigonometric functions.

Why? ▲

SATELLITE NAVIGATION Satellite navigation systems operate by receiving signals from satellites in orbit, determining the distance to each of the satellites, and then using trigonometry to establish the location on Earth's surface. These techniques are also used when navigating cars, planes, ships, and spacecraft.

PREREAD Use the prereading strategy of previewing to make two or three predictions of what Chapter 4 is about.

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